TOPOLOGICAL ENTROPY FOR NONCOMPACT SETS

BY

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ABSTRACT. For \( f: X \rightarrow X \) continuous and \( Y \subset X \) a topological entropy \( h(f, Y) \) is defined. For \( X \) compact one obtains results generalizing known theorems about entropy for compact \( Y \) and about Hausdorff dimension for certain \( Y \subset X = S^1 \). A notion of entropy-conjugacy is proposed for homeomorphisms.

The topological entropy of a continuous map on a compact space was defined by Adler, Konheim and McAndrew [1]. In the present paper we will define entropy for subsets of compact spaces in a way which resembles Hausdorff dimension. This will be used to generalize known results about the Hausdorff dimension of the quasiregular points of certain measures and to define a notion of conjugacy that is a cross between the topological and measure theoretic ones.

In [5] we gave a definition of entropy for uniformly continuous maps on metric spaces. That definition was motivated by different examples (linear maps on \( \mathbb{R}^n \) and calculating entropy on \( T^n \)) and it sometimes differs from the definition given here.

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1. The definition. Let \( f: X \rightarrow X \) be continuous and \( Y \subset X \). The topological entropy \( h(f, Y) \) will be defined much like Hausdorff dimension, with the "size" of a set reflecting how \( f \) acts on it rather than its diameter. Let \( \mathcal{A} \) be a finite open cover of \( X \). We write \( E < \mathcal{A} \) if \( E \) is contained in some member of \( \mathcal{A} \) and \( \{ E_i \} < \mathcal{A} \) if every \( E_i < \mathcal{A} \). Let \( n_{f, \mathcal{A}}(E) \) be the biggest nonnegative integer such that

\[
\phi^k E < \mathcal{A} \quad \text{for all} \quad k \in [0, n_{f, \mathcal{A}}(E));
\]

\[
n_{f, \mathcal{A}}(E) = 0 \quad \text{if} \quad E < \mathcal{A} \quad \text{and} \quad n_{f, \mathcal{A}}(E) = +\infty \quad \text{if} \quad \forall k \phi^k E < \mathcal{A}.
\]

Now set

\[
D_{\mathcal{A}}(E) = \exp\left(-n_{f, \mathcal{A}}(E)\right) \quad \text{and} \quad D_{\mathcal{A}}(E, \lambda) = \sum_{i=1}^{\infty} D_{\mathcal{A}}(E_i)^{\lambda}.
\]

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for $E = \{ E_i \}_{i=1}^{\infty}$ and $\lambda \in \mathbb{R}$. We define a measure $m_{\mathcal{Q}, \lambda}$ by

$$m_{\mathcal{Q}, \lambda}(Y) = \lim_{\epsilon \to 0} \inf \left\{ D_{\mathcal{Q}}(\mathcal{E}, \lambda): \bigcup E_i \supset Y \text{ and } D_{\mathcal{Q}}(E_i) < \epsilon \right\}.$$  

Notice that $m_{\mathcal{Q}, \lambda}(Y) < m_{\mathcal{Q}, \lambda'}(Y)$ for $\lambda > \lambda'$ and $m_{\mathcal{Q}, \lambda}(Y) \neq 0, +\infty$ for at most one $\lambda$. Define

$$b_{\mathcal{Q}}(f, Y) = \inf \{ \lambda: m_{\mathcal{Q}, \lambda}(Y) = 0 \}$$

and finally $b(f, Y) = \sup_{\mathcal{Q}} b_{\mathcal{Q}}(f, Y)$ where $\mathcal{Q}$ ranges over all finite open covers of $X$. For $Y = X$ we write $b(f) = b(f, X)$.

**Remark.** The number $b(f, Y) = b_X(f, Y)$ depends very much on which space $X$ we consider the domain of $f$. For instance, $f(x) = x + 1$ defines a homeomorphism of $\mathbb{R}$ which can be extended to a homeomorphism of $S^1$. By Proposition 1 below $b_{S^1}(f, S^1)$ is just the usual entropy of the homeomorphism $f: S^1 \to S^1$ and thus equals $0$ [1, p. 315]; for $Y \subset S^1$ we have $0 \leq b_S(f, Y) \leq b_{S^1}(f, S^1)$ and so $b_{S^1}(f, Y) = 0$. On the other hand suppose $Y = \bigcup_{n=1}^{\infty} (n + A)$ where $A \subset (0, 1)$ is a Cantor set. Since $Y$ is closed in $R$, one can prove $b_Y(f, Y) = b_{R}(f, Y)$. For any homeomorphism $g: A \to A$, $n: Y \to A$ defined by $n(n + a) = g^n(a)$ displays $g$ as a quotient of $f|Y$. From this one can conclude that $b(g) \leq b(f|Y)$; as $b(g)$ can be made large, $b(f|Y) = +\infty$. Then $b_{R}(f, Y) = +\infty$ but $b_{S^1}(f, Y) = 0$. This example was suggested to us by L. Goodwyn.

**Proposition 1.** If $X$ is compact, then $b(f)$ equals the usual topological entropy.

**Proof.** First let us recall the usual definition of entropy for compact $X$ [1]. Let $\mathcal{A}_{f,n} = \{ A_i_0 \cap f^{-1} A_i_0 \cap \cdots \cap f^{-n+1} A_i_{n-1} : A_i_k \in \mathcal{A} \}$ for an open cover $\mathcal{A}$ of $X$. If $N(\mathcal{B})$ denotes the smallest cardinality of any subcover of the open $\mathcal{B}$, then

$$b(f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{A}_{f,n})$$

exists and the topological entropy is defined by

$$b(f) = \sup_{\mathcal{A}} b(f, \mathcal{A})$$

where $\mathcal{A}$ runs over all finite open covers of $X$. Letting $\mathcal{E}_{f,n}$ be a subcover with $N(\mathcal{E}_{f,n})$ members

$$D_{\mathcal{Q}}(\mathcal{E}_{f,n}, \lambda) \leq N(\mathcal{E}_{f,n}) e^{-n\lambda}$$

and

$$m_{\mathcal{Q}, \lambda}(X) \leq \lim_{n \to \infty} [\exp(-\lambda + n^{-1} \log N(\mathcal{E}_{f,n}))]^n.$$
For \( \lambda > b(f, \tau) \) we get \( m_{\lambda}(X) = 0 \). Hence \( b(f, \tau) \leq b(f, \tau) \).

We prove \( b(f, \tau) \geq b(f, \tau) \) by showing \( b(f, \tau) \leq \lambda \) whenever \( m_{\lambda}(X) = 0 \).

For such a \( \lambda \) there is a countable covering \( \mathcal{D} = \{E_i\} \) of \( X \) so that \( D_{\lambda}(\mathcal{D}, \lambda) < 1 \).

If \( n_{f, \tau}(E_i) < \infty \), we may assume \( E_i \) is open (there is an open \( F_i \supset E_i \) with \( D_{\lambda}(F_i) = D_{\lambda}(E_i) \)). The \( E_i \)'s with \( n_{f, \tau}(E_i) = \infty \) may be replaced by open sets so that \( D_{\lambda}(\mathcal{D}, \lambda) \) is still less than 1 (though it may increase). As \( X \) is compact, the open cover \( \mathcal{D} \) now has a finite subcover \( \mathcal{D} = \{D_1, \ldots, D_m\} \). Then

\[
\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \exp(-\lambda n_{f, \tau}(D_{i_1}, \ldots, D_{i_s})) = \sum_{s=1}^{\infty} D_{\lambda}(\mathcal{D}, \lambda)^{t_s} \times \infty
\]

where \( n_{f, \tau}(D_{i_1}, \ldots, D_{i_s}) = \sum_{r=1}^{s} n_{f, \tau}(D_{i_r}) \).

Let

\[
C(D_{i_1}, \ldots, D_{i_s}) = \{x \in X: f^r x \in D_{i_r} \text{ for each } r \in [1, s]\}
\]

where \( t_r = n_{f, \tau}(D_{i_r}) + \cdots + n_{f, \tau}(D_{i_{r-1}}) \).

Then \( C(D_{i_1}, \ldots, D_{i_s}) < \mathcal{D}_{f, \tau} \) for \( n \leq n_{f, \tau}(D_{i_1}, \ldots, D_{i_s}) \). If \( M = \max_i n_{f, \tau}(D_{i_1}) \), then \( C(D_{i_1}, \ldots, D_{i_s}) : s \geq 1, n_{f, \tau}(D_{i_1}, \ldots, D_{i_s}) / n \in [n, n + M] \) is a cover of \( X \) subordinate to \( \mathcal{D}_{f, \tau} \).

Hence

\[
N(\mathcal{D}_{f, \tau}) \leq e^{MA} \sum_{\mathcal{D}_{f, \tau}} \exp(-\lambda n_{f, \tau}(D_{i_1}, \ldots, D_{i_s})); n_{f, \tau}(D_{i_1}, \ldots, D_{i_s}) / n \in [n, n + M] \).
\]

As the right side is bounded in \( n \), \( b(f, \tau) \leq \lambda \).

This proof is almost identical with Furstenberg [10, Proposition III.1] and resembles the proof of a well-known theorem of information theory [19].

We now state (without proof) some basic facts.

**Proposition 2.** (a) If \( f_1: X_1 \to X_1 \) and \( f_2: X_2 \to X_2 \) are topologically conjugate (i.e., there is a homeomorphism \( \pi: X_1 \to X_2 \) with \( f_1 \pi = f_2 \pi \)), then

\[
b(f_1, Y_1) = b(f_2, \pi(Y_1)) \quad \text{for } Y_1 \subset X_1.
\]

(b) \( b(f, f(Y)) = b(f, Y) \).

(c) \( b(f, \bigcup_{i=1}^{\infty} Y_i) = \sup_i b(f, Y_i) \).

(d) \( b(f^m, Y) = mb(f, Y) \) for \( m > 0 \).

We now give an example which motivated this paper. Define \( f: S^1 \to S^1 \) by \( f(z) = z^n \). If \( Y \subset S^1 \) is closed and \( f(Y) \subset Y \) then the Hausdorff dimension of \( Y \) satisfies \( bd(Y) = b(f, |Y|)/\log n \). This was proved by Furstenberg [10, Proposition III.1]. For an ergodic \( f \)-invariant probability measure \( \mu \) on \( S^1 \), it is known that

\[
bd(G(\mu)) = b(\mu, f) / \log n \quad \text{where } G(\mu) \text{ denotes}
\]
the set of generic points of \( \mu \). The above two formulas suggest that one might have \( b_{\mu}(f) = b(f,G(\mu)) \) if the right side is correctly defined for the noncompact set \( G(\mu) \). The intermediate Hausdorff dimension of course motivated our definition of entropy; Theorem 3 shows that the hoped for formula holds for any continuous map on a compact metric space. We mention that another aspect of Colebrook's paper [7] has been generalized by K. Sigmund [20].

2. Goodwyn's theorem. In this section we will generalize a theorem of Goodwyn [13]. For a continuous map \( f: X \rightarrow X \) let \( M(f) \) be the set of all \( f \)-invariant Borel probability measures on \( X \). We refer the reader to [4] or [14] for the definition of \( b_{\mu}(f) \).

**Theorem 1.** Let \( f: X \rightarrow X \) be a continuous map of a compact metric space and \( \mu \in M(f) \). If \( Y \subseteq X \) and \( \mu(Y) = 1 \), then \( b_{\mu}(f) \leq b(f,Y) \).

**Lemma 1.** Let \( \alpha \) be a finite Borel partition of \( X \) such that every \( x \in X \) is in the closures of at most \( M \) sets of \( \alpha \). Then

\[
b_{\mu}(f,\alpha) \leq b(f,Y) + \log M.
\]

**Proof.** For each \( x \in X \) let \( l_n(x) = -\log \mu(A) \) where \( A \in \alpha_{n,n} \) contains \( x \). The Shannon-McMillan-Breiman theorem [14] says that for some \( \mu \)-integrable function \( l(x) \) one has \( l_n(x)/n \rightarrow l(x) \) a.e. and \( a = \int l(x)d\mu = b_{\mu}(f,\alpha) \). For \( \delta > 0 \) the set \( Y_{\delta} = \{ y \in Y: l(y) \geq a - \delta \} \) has positive measure. By Egorov's theorem there is an \( N \) so that

\[
Y_{\delta,N} = \{ y \in Y_{\delta}: l_n(y)/n \geq a - 2\delta, \forall n \geq N \}
\]

has positive measure.

Let \( \mathcal{B} \) be a finite open cover of \( X \) each member of which intersects at most \( M \) members of \( \alpha \). Suppose \( \mathcal{E} = \{ E_i \} \) covers \( Y \) and \( D_{\mathcal{E}}(E_i) \leq e^{-N} \). If \( \beta \in \alpha_{n,\mathcal{E}}(E_i) \) intersects \( Y_{\delta,N} \), then \( \mu(\beta) \leq \exp((- a + 2\delta)n_{\mathcal{E},\beta}(E_i)) \). Since \( E_i \cap Y_{\delta,N} \) is covered by at most \( M^{n_{\mathcal{E},\beta}(E_i)} \) such \( \beta \)'s,

\[
\mu(E_i \cap Y_{\delta,N}) \leq \exp(n_{\mathcal{E},\beta}(E_i)(\log M - a + 2\delta)).
\]

For \( \lambda = - \log M + a - 2\delta \) we have

\[
D_{\mathcal{E}}(\mathcal{E},\lambda) = \sum_i \exp(-\lambda n_{\mathcal{E},\beta}(E_i)) \geq \sum_i \mu(E_i \cap Y_{\delta,N}) \geq \mu(Y_{\delta,N}).
\]

Letting \( \mathcal{E} \) vary, \( n_{\mathcal{E},\lambda}(Y) \geq \mu(Y_{\delta,N}) > 0 \). Hence \( b(f,Y) \geq b_{\mathcal{E}}(f,Y) \geq \lambda = - \log M + a - 2\delta \). Letting \( \delta \to 0 \) we have our result.

**Lemma 2.** Let \( \mathcal{G} \) be a finite open cover of \( X \). For each \( n > 0 \) there is a finite Borel partition \( \alpha_n \) of \( X \) such that \( f^k \alpha_n < \mathcal{G} \) for all \( k \in [0,n) \) and at most \( n \) card \( \mathcal{G} \) sets in \( \alpha_n \) can have a point in all their closures.
Proof. This idea for this lemma is from Goodwyn [13] and the statement as above is in [15]. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $g_1, \ldots, g_m$ be a partition of unity subordinate to $\mathcal{A}$. Then $G = (g_1, \ldots, g_m): X \to s_{m-1} \subset \mathbb{R}^m$ where $s_{m-1}$ is an $m - 1$ dimensional simplex. Now $\{U_1, \ldots, U_m\}$ is an open cover of $s_{m-1}$ where $U_i = \{x \in s_{m-1}: x_i > 0\}$ and $G^{-1}U_i \subset A_i$. As $(s_{m-1})^n$ is $nm - n$ dimensional, there is a finite Borel partition $\alpha^*_n$ of $s_{m-1}$ with at most $nm$ elements having a point in all their closures and such that each member of $\alpha^*_n$ lies in some $U_{i_1} \times \cdots \times U_{i_n}$. Then $\alpha_n = L^{-1}\alpha^*_n$ works where $L = (G, G \circ f, \ldots, G \circ f^{n-1}): X \to s_{m-1}$.

Lemma 3. Given a finite Borel partition $\beta$ and $\epsilon > 0$ there is an open cover $\mathcal{A}$ so that $H_\mu(\beta | \alpha) < \epsilon$ whenever $\alpha$ is a finite Borel partition with $\alpha < \mathcal{A}$.

Proof. Let $\beta = \{B_1, \ldots, B_m\}$. There is a $\delta > 0$ so that the following is true:

$$H_\mu(\beta | \alpha) < \epsilon$$

if there is a Borel partition $\{C_1, \ldots, C_m\}$

with each $C_i$ a union of members of $\alpha$ and $\sum_{i \neq j} P(B_i \cap C_j) < \delta$

(see [4, Theorem 6.2]). Choose compact sets $K_i \subset B_i$ so that $\mu(B_i \setminus K_i) < \delta/m$. Let $\mathcal{A}$ be an open cover each member of which intersects at most one $K_i$. For $\alpha < \mathcal{A}$ put $A \in \alpha$ in $C_i$ if $A \cap K_i \neq \emptyset$, and in any $C_j$ if $A \cap \bigcup_j K_j = \emptyset$. Then $C_i \cap K_i = \emptyset$ for $i \neq j$ and so

$$\sum_{i \neq j} P(B_i \cap C_j) \leq \sum_i P(B_i \setminus K_i) < \delta.$$

Proof of Theorem 1. Let $\beta$ be a finite Borel partition of $X$ and $\epsilon > 0$. Let $\mathcal{A}$ be as in Lemma 3 and $\alpha_n$ as in Lemma 2. Then

$$b_\mu(f, \beta) = n^{-1}b_\mu(m, \beta_{f,n}) \leq n^{-1}b_\mu(f^n, \alpha_n) + n^{-1}H_\mu(\beta_{f,n} | \alpha_n)$$

$$\leq n^{-1}[b(f^n, Y) + \log(n \text{ card } \mathcal{A})] + n^{-1} \sum_{k=0}^{n-1} H_\mu(f^{-k}\beta | \alpha_n)$$

$$\leq b(f, Y) + n^{-1} \log(n \text{ card } \mathcal{A}) + n^{-1} \sum_{k=0}^{n-1} H_\mu(\beta | f^k \alpha_n)$$

$$\leq b(f, Y) + n^{-1} \log(n \text{ card } \mathcal{A}) + \epsilon.$$

Here we used Lemmas 1 and 2 and some general facts:
Proofs of these are in [4] and [14]. Finally, let \( n \to \infty \) and then let \( \epsilon \to 0 \). The proof is finished.

3. Generic points. For \( X \) a compact metric space, the set \( M(X) \) of all Borel probability measures on \( X \) with the weak topology is a compact metrizable space [18]. \( \mu_n \to \mu \) implies that for \( V \supset K \) with \( V \) open and \( K \) compact one has
\[
\liminf \mu_n(V) \geq \mu(K).
\]
For \( x \in X \) let \( \mu_x \) denote the unit measure concentrated on \( x \). If a continuous \( f: X \to X \) is given, define
\[
\mu_{x,n} = n^{-1}(\mu_x + \mu_{fx} + \cdots + \mu_{f^{n-1}x}).
\]
Let \( V_f(x) \) be the set of all limit points in \( M(X) \) of the sequence \( \mu_{x,n} \). Then \( V_f(x) \neq \emptyset \) and one checks that \( V_f(x) \subset M(f) \). \( x \) is a generic point for \( \mu \) if
\[
V_f(x) = \{\mu\}.
\]
Our main result is that \( b(f, G(\mu)) = b_f(\mu) \) for \( \mu \) ergodic where \( G(\mu) \) is the set of generic points for \( \mu \).

\[ p = (p_1, \ldots, p_N) \] is an \( N \)-distribution if \( \sum_i p_i = 1 \) and \( p_i > 0 \); we set \( H(p) = -\sum_i p_i \log p_i \). If \( a = (a_1, \ldots, a_m) \in \{1, \ldots, N\}^m \), then dist \( a = (p_1, \ldots, p_N) \) where \( p_i = m^{-1} \) (number of \( j \) with \( a_j = i \)). If \( p \) and \( q \) are \( N \)-distributions, then \( |p - q| = \max_i |p_i - q_i| \).

Lemma 4. Let
\[
R(N, m, t) = \{a \in \{1, \ldots, N\}^m : \text{dist } a \leq t\}.
\]
Then, fixing \( N \) and \( t \),
\[
\limsup_{m \to \infty} \frac{1}{m} \log \text{card } R(N, m, t) \leq t.
\]

Proof. For an \( N \)-distribution \( q \) and \( \alpha \in (0, 1) \) consider \( R_m(q) = \{a \in \{1, \ldots, N\}^m : |q - \text{dist } a| < \alpha\} \). Let \( \mu \) be the measure on \( \Sigma_N = \{1, \ldots, N\}^Z \) for the Bernoulli shift with distribution \( q' = (1 - \alpha)q + \alpha(1/N, \ldots, 1/N) \). Each \( a \in R_m(q) \) corresponds to a cylinder set \( C_a \subset \Sigma_N \). Since \( |q - \text{dist } a| < \alpha \), the number of \( i \)'s occurring in \( a \) is at most \( (q_i + \alpha)m \). As the symbol \( i \) has probability \( q_i' = (1 - \alpha)q_i + \alpha/N \), \( \mu(C_a) \geq \prod_{i=1}^N q_i'^{(q_i + \alpha)m} \).
Since the \( C_i \)'s are disjoint and have total \( \mu \)-measure at most 1,

\[
1 \geq \text{card } \bigcap_i R_{m}(q) \prod q_i^{(q_i + \alpha)m}.
\]

Taking logarithms we get

\[
\frac{1}{m} \log \text{card } R_{m}(q) \leq \sum_i - (q_i + \alpha) \log q_i' 
\leq H(q') + \sum_i (|q_i' - q_i| + \alpha) \log q_i'.
\]

As \( q_i' \geq \alpha/N, \) \( |\log q_i'| \leq \log N - \log \alpha; \) also \( |q_i' - q_i| = (\alpha/N - \alpha q_i) \leq 2\alpha. \) So

\[
m^{-1} \log \text{card } R_{m}(q) \leq H(q') + 3\alpha N (\log N - \log \alpha).
\]

Now \( H(q) \) is uniformly continuous in \( q \) and \( \log \alpha \rightarrow 0 \) as \( \alpha \rightarrow 0. \) Hence given any \( \epsilon > 0, \) for small \( \alpha \) one has

\[
m^{-1} \log \text{card } R_{m}(q) \leq H(q) + \epsilon
\]

for all \( m \) and \( q. \)

Once an \( \alpha \) is chosen one can find a finite set \( Q \) of \( N \)-distributions so that

(a) \( H(q) \leq t \) for \( q \in Q \) and

(b) if \( H(q*) \leq t, \) then \( |q* - q| < \alpha \) for some \( q \in Q. \)

Then \( R(N, m, t) \subset \bigcup_{q \in Q} R_{m}(q). \)

\[
m^{-1} \log \text{card } R(N, m, t) \leq m^{-1} \log \text{card } Q + (t + \epsilon).
\]

Letting \( m \rightarrow \infty \) and then \( \epsilon \rightarrow 0 \) we get our result.

Now suppose \( \beta = \{B_1, \ldots, B_N\} \) is a cover of \( X. \) An \( n \)-choice for \( x \) (with respect to \( \beta \) and \( f \)) is a \( \overline{B} = (B_{i_0}, \ldots, B_{i_{n-1}}) \in \beta^n \) with \( f^k(x) \in B_{i_k} \) for \( k \in [0, n). \) An \( n \)-choice gives an \( N \)-distribution \( q(\overline{B}) = \text{dist}(i_0, \ldots, i_{n-1}). \) The set of such distributions for the various \( n \)-choices for \( x \) we denote by \( \text{Dist}_\beta(x, n). \)

Lemma 5. Suppose \( f: X \rightarrow X \) is a continuous map of a topological space, \( \beta \) an open cover of \( X, \) \( \beta \) a finite cover of \( X \) and \( M \) a positive integer so that \( f^k \beta < \beta \) for all \( k \in [0, M). \) For \( t > 0 \) define

\[
Q(t, \beta) = \left\{ x \in X: \liminf_{n \rightarrow \infty} (\inf_{q \in \text{Dist}_\beta(x, n)} H(q) : q \in \text{Dist}_\beta(x, n)) \leq t \right\}
\]

Then \( h(f, Q(t, \beta)) \leq t/M. \)

Proof. Let \( N = \text{card } \beta \) and \( \epsilon > 0. \) By Lemma 4 there is an \( m_\epsilon \) so that

\[
\text{card } R(N, m, t + \epsilon) \leq e^{m(t + 2\epsilon)}
\]

for all \( m \geq m_\epsilon. \) As \( \text{Dist}_\beta(x, n) \) depends only slightly on the last few \( f^k(x) \) when \( n \) is large and \( H(q) \) is continuous in \( q, \) one has
\[ \lim \inf_{m \to \infty} \left( \inf \{ H(q) : q \in \text{Dist}_\beta(x, m\mathbb{M}) \} \right) \leq t \]

for \( x \in Q(t, \beta) \). Let \( B_n(x) = (B_i, \ldots, B_{i+n-1}) \) be an \( n \)-choice with distribution \( q(x, n) \) minimizing \( H(q) \) over \( \text{Dist}_\beta(x, n) \). For \( k \in [0, M) \) define

\[ q_k(x, m) = \text{dist} \{ i_k+r\mathbb{M} : r \in [0, m) \}. \]

Then \( q(x, m\mathbb{M}) = (1/M) \sum_k q_k(x, m) \). By the concavity of \( H(q) \) in \( q \) one has \( H(q_k(x, m)) \leq H(q(x, m\mathbb{M})) \) for some \( k \) (depending on \( x \) and \( m \)).

Fix now any \( m_0 > m_\varepsilon \). For \( m \geq m_0 \) and \( k \in [0, M) \) define

\[ S(m, k) = \{ x \in X : H(q_k(x, m)) \leq t + \varepsilon \}. \]

Then \( Q(t, \beta) \subseteq \bigcup S(m, k) \). Let \( S(m, k) = \{ x \in X : m \geq m_0, k \in [0, M) \} \). Assume \( x \in S(m, k) \). Define

\[ A_k(x, m) = \{ y \in X : f^r y \in B_i \text{ for } r \in [0, k) \text{ and } f^r y \in B_{i+r\mathbb{M}} \text{ for } r \in [0, m) \}. \]

Now \( f^r A_k(x, m) \) is contained in some member of \( \beta \) for each \( j \in [0, m\mathbb{M}) \). Hence \( D_{\beta} A_k(x, m) \leq e^{-m\mathbb{M}}. \) Let \( \bar{E}(m_0) = \{ A_k(x, m) : x \in S(m, k), m \geq m_0, k \in [0, M) \} \).

Then \( \bar{E}(m_0) \) covers \( Q(t, \beta) \). Since there are at most \( (\text{card } \beta)^k \) different \( A_k(x, m) \) with \( x \in S(m, k) \),

\[ D_{\beta}(\bar{E}(m_0), (t + 3\varepsilon)/M) \leq \sum_{k \in [0, M)} (\text{card } \beta)^k \text{ card } R(N, m, t + \varepsilon) e^{-m(t + 3\varepsilon)} \]

\[ \leq (\text{card } \beta)^{M-1} \sum_{m \geq m_0} e^{-m\varepsilon}. \]

As this quantity approaches 0 as \( m_0 \to \infty \), \( m_{\beta, (t + 3\varepsilon)/M}(Q(t, \beta)) = 0 \) and \( b_\beta(f, Q(t, \beta)) \leq (t + 3\varepsilon)/M \). Now let \( \varepsilon \to 0 \).

**Theorem 2.** Let \( f : X \to X \) be a continuous map on a compact metric space. Set

\[ Q_R(t) = \{ x \in X : 3\mu \in V_f(x) \text{ with } b_\mu(f) \leq t \}. \]

Then \( b(f, Q_R(t)) \leq t \).

**Proof.** Let \( \beta \) be a finite open cover of \( X \) and \( \alpha \) a Borel partition of \( X \) with the closures of members of \( \alpha \) contained in members of \( \beta \). Fix \( \varepsilon > 0 \) and let
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$W_\epsilon(M) = \{ x \in X : \exists \mu \in V_f(x) \text{ with } (1/M)H_\mu(\alpha_{f,M}) < t + \epsilon \}$. 

If $b_\mu(f) \leq t$, then 

$$\lim_{M \to \infty} \frac{1}{M} H_\mu(\alpha_{f,M}) = b_\mu(f) \leq b_\mu(f)$$

implies that $(1/M)H_\mu(\alpha_{f,M}) < t + \epsilon$ for some $M$. Hence $QR(t) \subseteq \bigcup_M W_\epsilon(M)$. 

Now fix an $M$ and let $\alpha_{f,M} = \{ E_1, \ldots, E_N \}$. Pick $U_i \supseteq E_i$ open so that $/kU_i < B$ for $k \in [0, M]$; set $\beta = \{ U_1, \ldots, U_N \}$. We will show $W_\epsilon(M) \subseteq Q(M(t + 2\epsilon), \beta)$. Consider $x \in W_\epsilon(M)$ and $\mu \in V_f(x)$ with $(1/M)H(\alpha_{f,M}) < t + \epsilon$. 

Let $q' = (\mu(E_1), \ldots, \mu(E_N))$ and pick $\delta > 0$ so that 

$$|q - q'| \leq \delta \implies H(q) \leq M(t + 2\epsilon).$$

Now choose compact $K_i \subseteq E_i$ so that $\mu(E_i \setminus K_i) < \delta/2N$ and disjoint open $V_i$'s with $U_i \supseteq V_i \supseteq K_i$. Let $B_n(x) \in \beta^n$ be an $n$-choice for $x$ so that $B_{i,k} = U_i$ whenever $x \in V_i$. Since $\mu \in V_f(x)$, $\mu_{x,n} \to \mu$ for some $n \to \infty$. For large $j$ one has 

$$\mu_{x,n_j}(V_i) \geq \mu(K_i) - \delta/2N$$

for all $i$. If $q_j = \text{dist } B_n(x) = (q_1, \ldots, q_N)$, it follows that $q_j \geq \mu(K_i) - \delta/2N \geq \mu(E_i) - \delta/N$. We get $|q_j - q'| \leq \delta$ and $H(q_j) \leq M(t + 2\epsilon)$. Hence $x \in Q(M(t + 2\epsilon), \beta)$. 

Lemma 5 now gives us $b_\beta(f, W_\epsilon(M)) \leq t + 2\epsilon$. By Proposition 2(d) we get $b_\beta(f, QR(t)) \leq t + 2\epsilon$. Letting $\epsilon \to 0$ and varying $\beta$ we are done.

Corollary. Let $f : X \to X$ be a continuous map of a compact metric space. Then 

$$b(f) = \sup_{\mu \in M(f)} b_\mu(f).$$

Proof. Let $t = \sup_{\mu} b_\mu(f)$. As $V_f(x) \neq \emptyset$ for $x \in X$, $X \subseteq QR(t)$ and $b(f) = b(f, X) \leq t$. On the other hand $b(f) \geq t$ by Goodwyn's theorem (Theorem 1).

Remark. This result is already known; see [8] for the finite dimensional metric case and [12] for compact Hausdorff spaces.

Theorem 3. Let $f$ be a continuous map on a compact metric space and $\mu \in M(f)$ be ergodic. Let $G(\mu)$ be the set of generic points of $\mu$, i.e., 

$$G(\mu) = \{ x : V_f(x) = \{ \mu \} \}.$$ 

Then $b(f, G(\mu)) = b_\mu(f)$. 

Proof. By the ergodic theorem, one has $\mu(G(\mu)) = 1$. Theorem 1 then gives $H(f, G(\mu)) \geq b_\mu(f)$. As $G(\mu) \subseteq QR(b_\mu(f))$, Theorem 2 gives the reverse inequality.
4. A type of conjugacy. We will call two homeomorphisms \( f: X \to X \) and \( g: Y \to Y \) *entropy conjugate* if there are \( X' \subset X \) and \( Y' \subset Y \) such that

(i) \( X' \) and \( Y' \) are Borel sets,
(ii) \( f(X') \subset X' \), \( g(Y') \subset Y' \),
(iii) \( b(f, X') < b(f), b(g, Y') < b(g) \), and
(iv) \( f|_{X'} \) and \( g|_{Y'} \) are topologically conjugate.

Unfortunately this does not seem to be an equivalence relation.

**Proposition 3.** If \( f \) and \( g \) are entropy-conjugate homeomorphisms of compact metric spaces, then \( b(f) = b(g) \).

**Proof.** Suppose \( \mu \in \mathcal{M}(f) \) and \( b_\mu(f) > b(f, X' \setminus X') \). Since \( \mu \) is \( f \)-invariant and \( f(X') \subset X' \), one can find \( B \subset X \setminus X' \) with \( \mu(B) = \mu(X \setminus X') \) and \( f(B) = B \). By Theorem 1, \( \mu(B) < 1 \). Define \( \mu_{X'}(E) = \mu(E \cap X') / \mu(X') \). Then \( \mu_{X'} = \mu \) (if \( \mu(X') = 1 \)) or \( \mu = \mu(X') \mu_{X'} + \mu(B) \mu_B \). In the second case \( \mu_{X'}\mu_B \in \mathcal{M}(f) \) and

\[
\mu(f) = \mu(X') \mu_{X'}(f) + \mu(B) \mu_B(f).
\]

By Theorem 1 we have \( b_{\mu_B}(f) \leq b(f, X' \setminus X') < b_\mu(f) \) and so \( b_{\mu_{X'}}(f) \geq b_\mu(f) \). If \( \mu_{X'} = \mu \), we of course also have \( b_{\mu_{X'}}(f) \geq b_\mu(f) \). Since \( \mu_{X'}(X') = 1 \), the topological conjugacy of \( f|_{X'} \) and \( g|_{Y'} \) gives us a measure \( \nu \) on \( Y' \) with \( (g, \nu) \) conjugate to \( (f, \mu_{X'}) \); in particular

\[
b_\nu(g) = b_{\mu_{X'}}(f) \geq b_\mu(f).
\]

By Goodwyn’s theorem \( b(g) \geq b_\mu(f) \). Using the Dinaburg-Goodman theorem (corollary to Theorem 2) one can make \( b_{\mu}(f) \) arbitrarily close to \( b(f) \) (and so satisfy \( b_{\mu}(f) > b(f, X' \setminus X') \)). One gets \( b(g) \geq b(f) \). By symmetry one likewise has \( b(g) \leq b(f) \).

There is a natural class of homeomorphisms for which the converse of Proposition 3 may hold. Let \( \Sigma_n = \Pi_Z \{1, \ldots, n\} \) and define the shift \( \sigma_n : \Sigma_n \to \Sigma_n \) by

\[
(\sigma_n x)_i = x_{i+1} \quad \text{for} \quad x = (x_i).
\]

\( \sigma_n \) is a homeomorphism of the compact metrizable space \( \Sigma_n \). For \( A \) an \( n \times n \) matrix of 0’s and 1’s define

\[
\Sigma_n(A) = \{(X_i) \in \Sigma_n : A_{X_i, X_{i+1}} = 1 \quad \forall i\}.
\]

Then \( \sigma_n | \Sigma_n(A) \) is a homeomorphism of a compact space.

**Conjecture.** Suppose \( \sigma_n | \Sigma_n(A) \) and \( \sigma_m | \Sigma_m(B) \) are topologically mixing and have the same topological entropy. Then they are entropy conjugate.

This conjecture is related to the symbolic dynamics of diffeomorphisms [2],
From [6] it follows that the nonwandering set of an Axiom A diffeomorphism is entropy conjugate to some $\Sigma_n(A)$ (called a subshift of finite type). The codings of [2] show that the conjecture is true for the subshifts of finite type that arise from hyperbolic automorphisms of $T^2$. The codings were used in [2] to prove that entropy classifies such maps on $T^2$ up to measure theoretic conjugacy; Friedman and Ornstein [11] now supplant these codes for this purpose. The notion of entropy conjugacy attempts to clarify the topological content of the Adler-Weiss codings (see problem 3 of [21]).

**Proposition 4.** Suppose $f$ and $g$ are entropy-conjugate homeomorphisms of compact metric spaces. Then $f$ is intrinsically ergodic iff $g$ is.

**Proof.** Intrinsic ergodicity [17] means there is a unique $\mu \in M(f)$ with $h_{\mu}(f) = h(f)$. The proof is like that of Proposition 3.

**REFERENCES**


