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NEOCONTINUOUS MIKUSIŃSKI OPERATORS(1)

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ABSTRACT. A class of Mikusiński-type operators in several variables, called *neocontinuous operators*, is studied. These particular operators are closely affiliated with Schwartz distributions on R^k and share certain continuity properties with them. This affiliation is first of all revealed through a common algebraic view of operators and distributions as homomorphic mappings and a new representation theory, and is then characterized in terms of continuity properties of the mappings. The traditional procedures of the operational calculus apply to the class of neocontinuous operators. Moreover, the somewhat vague association of operational and distributional solutions of partial differential equations is replaced by the decisive representation concept, thus illustrating the appropriateness of the study of neocontinuous operators.

1. Introduction. In this paper we investigate a class of Mikusiński-type operators in several variables together with a class of Schwartz distributions. Using an algebraic view [9] of operators and distributions as homomorphic mappings, we consider those operators which, when restricted to suitable domains, are continuous and agree with one or more distributions. These operators are said to represent the distributions and are called *neocontinuous operators*. The concept of representation referred to here is considerably more general than the traditional one used to identify certain operators and distributions having suitably bounded supports, and has been formulated only recently [10] in an attempt to treat partial differential operators by algebraic methods. Thus the class of neocontinuous operators is quite large and can be expected to play a central role in the operational calculus in several variables.

Our attention was first drawn to this class of neocontinuous operators by the familiar proof of the Malgrange-Ehrenpreis theorem on the existence of fundamental functions for (partial) differential operators (with constant coefficients). In this proof, L^2-inequalities for polynomials are used to define a continuous linear form (which inverts the operator) on a subspace of test functions and which is then extended to all test functions by the Hahn-Banach theorem. A version of this technique is employed here to characterize the operators we wish to study and provides for a more intimate affiliation with distributions than heretofore considered in the Mikusiński operational calculus.

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It has been necessary to formulate in some detail the elements of the Mikusiński operational calculus in several variables as there is little standardization except for the one-dimensional case. The individual concepts and proofs are mostly straightforward and rudimentary, and the rather complete development given here is an attempt to provide a general foundation for the operational calculus in several variables.

Following a preliminary section concerning notation and terminology, the algebraic concepts involving distributions as operator homomorphisms, Mikusiński operators and representation are reviewed. Then representation is characterized in terms of continuity and after a few simple examples are given, the main results on neocontinuous operators are presented. The final section relates our concepts to (partial) differential operators, and (together with previous examples) provides concrete illustrations of the various concepts involved.

Recently T. K. Boehme [1] introduced an interesting subalgebra of Mikusiński operators, called regular operators, which have local properties. Some neocontinuous operators (for example, all distributions with left bounded support) are regular and some are not regular and surprisingly enough, the two concepts are rather discordant. Even with regard to one and the same operator they generally concern different attributes of the operator. It seems strange that those attributes of Mikusiński operators which are most closely affiliated with distributions (continuity properties) and those attributes which are most closely affiliated with local properties are so alien. Yet, one of our results implies that these attributes are coincident exactly when the operators are distributions. Boehme also gave a detailed formulation of the Mikusiński operational calculus in several variables with reference to (what we refer to here as) the traditional or classical view. This is a much needed standardization but does not provide for the generalizations considered in this paper. There are, of course, ample reasons for having both formulations and there is very little duplication in the two developments.

2. **Notation and terminology.** Let C denote the field of complex numbers, and for a fixed positive integer k, let R^k denote the real k-space. An element $t \in R^k$ is given by the k-tuple $(t_1, \ldots, t_k)$, where $t_i \in R^1$. We define a right-sided orthant to be a set of the form $R^k_\tau = \{t \in R^k : t_i \geq \tau_i, i = 1, \ldots, k\}$ for some fixed $\tau = (\tau_1, \ldots, \tau_k)$. The traditional development of Mikusiński operators employs the collection of functions on R^k into C which are continuous on the right-sided orthant R^k_0 and zero elsewhere. Since we wish to discuss the relationship between Schwartz distributions and Mikusiński operators, using the algebraic viewpoint [9], we begin our (more general) development by considering the collection $\mathcal{E}$ of infinitely differentiable functions on R^k into C. Except as otherwise noted, $\mathcal{E}$ is endowed with the topology $\mathcal{U}$ of compact convergence of all derivatives. $\mathcal{E}$ is a complex linear space. For a fixed $\tau \in R^k$ and an element $\sigma \in \mathcal{E}$ we denote the τ-translate of σ by σ_τ, where $\sigma_\tau(t) = \sigma(t + \tau)$. A subspace $\mathcal{S}$ of $\mathcal{E}$ is said to be translation-invariant in R^k if $\phi \in \mathcal{S}$ implies that $\phi_\tau \in \mathcal{S}$ for all $\tau \in R^k$. The

support of an element $\psi \in \mathcal{E}$, Support ψ, is the closure of the set on which ψ is not zero.

With suitable $\phi, \psi \in \mathcal{E}$ the ordinary convolution $\sigma = \phi * \psi$ is the function whose value at t is given by

$$\sigma(t) = \phi * \psi(t) = \int_{R^k} \phi(\tau)\psi(t - \tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\tau_1, \ldots, \tau_k)\psi(t_1 - \tau_1, \ldots, t_k - \tau_k)\, d\tau_1 \cdots d\tau_k$$

and belongs to $\mathcal{E}$. We shall single out subspaces of $\mathcal{E}$ which are convolution rings having certain other desirable properties. Three such spaces are dealt with in some detail.

The subspace of $\mathcal{E}$ consisting of those functions with compact support will be denoted by $\mathcal{D}$. This is the space of testing functions for distributions [7], and is endowed with the usual topology $\mathcal{T}$ [3, p. 165] such that $(\mathcal{D}, \mathcal{T})$ is a locally convex topological linear space. Under ordinary pointwise addition and convolution, $\mathcal{D}$ becomes a commutative ring and $\mathcal{E}$ becomes a $\mathcal{D}$-module. For $\phi \in \mathcal{D}$ let $[[\phi]]$ denote the convex hull of the support of ϕ. A theorem of Lions [4, p. 378] states that for nonzero $\phi, \psi \in \mathcal{D}$ we have

$$[[\phi * \psi]] = [[\phi]] + [[\psi]] \tag{2.1}$$

where the right-hand side is the usual vector sum of subsets in R^k. It follows that the ring $\mathcal{D}$ has no divisors of zero.

Our second example of a subspace of $\mathcal{E}$ consists of those functions ϕ in $\mathcal{E}$ whose support is contained in some right-sided orthant depending upon ϕ. This subspace is denoted by $\mathcal{C}_R^\infty$ (R for right-sided) and is well known to be closed under convolution as well as addition. Moreover, a theorem of Mikusiński [6, p. 312] states that for nonzero $\phi, \psi \in \mathcal{C}_R^\infty$ we have

$$C[[\phi * \psi]] = C[[\phi]] + C[[\psi]] \tag{2.2}$$

where $C[[\phi]]$ denotes the right-sided convex cap $C[[\phi]] = \{t: \exists\ \tau \in [[\phi]]$ such that $t_i - \tau_i \geq 0,\ i = 1, \ldots, k\}$, of the convex hull of the support of ϕ. It follows that the ring $\mathcal{C}_R^\infty$ has no divisors of zero.

A more novel subspace of $\mathcal{E}$ for the development of Mikusiński-type operators is the subspace, denoted by *Exp*, consisting of those functions of exponential descent. Each function $\phi \in Exp$ satisfies an inequality of the form $|\phi(t)| \leq ae^{-b|t|}$ ($|t| = |t_1| + \cdots + |t_k|$), for some positive constants a and b depending upon ϕ. It is easy to verify that under addition and convolution *Exp* is a ring. Since the Fourier transform of an element of *Exp* is analytic in a neighborhood of the real line (in each component), it follows that the ring *Exp* also has no divisors of zero.

We wish now to specify in general those subspaces of $\mathcal{E}$ which are to be used in the subsequent development.

2.3. **Definition.** A convolution ring $\mathcal{C}$ of functions contained in the $\mathcal{D}$-module $\mathcal{E}$ and containing the ring $\mathcal{D}$ is called an *admissible ring* provided it has no divisors of zero and is a translation-invariant linear space.

The ring $\mathcal{D}$ itself and the rings $\mathcal{C}_R^\infty$ and *Exp* described above are our main examples of admissible rings. Using orthants other than the right-sided ones lead to additional admissible rings [2, p. 470]. (These are isomorphic to $\mathcal{C}_R^\infty$.) We may also obtain additional examples from subspaces of *Exp* in which a finite or infinite number of partial derivatives in some or all variables are required to have decay properties. The three main examples are sufficiently illustrative for our present purposes. However, it would be of interest to further characterize admissible rings. Unless otherwise specified, we shall assume that an arbitrary, but fixed, admissible ring $\mathcal{C}$ has been chosen and consider operators which are associated with $\mathcal{C}$.

3. **Distributions and operator homomorphisms.** The process of regularization of a distribution f by convolution with a function $\phi \in \mathcal{D}$ is well known [3, p. 406] and results in an ordinary function ($f * \phi(t) = \langle f(\tau), \phi(t-\tau)\rangle$) that is infinitely differentiable. Thus under convolution, a distribution f maps the ring ($\mathcal{D}$-module) $\mathcal{D}$ into the $\mathcal{D}$-module $\mathcal{E}$. Moreover, the associative property $f * (\phi * \psi) = (f * \phi) * \psi$ holds for all $\phi, \psi \in \mathcal{D}$. Such mappings, which preserve module multiplication, are called module homomorphisms provided they also preserve addition. Since distributions are linear functionals on $\mathcal{D}$ it follows that the above mapping, $\phi \to f * \phi$, is linear on $\mathcal{D}$. This provides us with an algebraic view of distributions [9] which we formalize in the following definition and subsequent lemma.

3.1. **Definition.** A mapping h: $\mathcal{D} \to \mathcal{E}$ is called an *operator homomorphism* if it satisfies

$$h(\phi * \psi) = h(\phi) * \psi \tag{3.2}$$

for all $\phi, \psi \in \mathcal{D}$.

The linearity of an operator homomorphism readily follows from (3.2) using properties of convolution and the fact that the only function in $\mathcal{E}$ which annihilates all of $\mathcal{D}$ under convolution is the zero function. Of course, each fixed element $\psi \in \mathcal{E}$ defines, and is identifiable as, an operator homomorphism using convolution ($\phi \to \psi * \phi$, for all $\phi \in \mathcal{D}$). In light of the previous discussion, any Schwartz distribution on R^k is similarly identifiable as an operator homomorphism, and we shall see that all operator homomorphisms arise in this particular manner. First we introduce the following device.

3.3. **Definition.** A sequence $\{\phi_n\}$ of elements of $\mathcal{D}$ is called a *delta function sequence* if the supports of the members of the sequence are uniformly bounded and if $\lim \psi * \phi_n = \psi$ (as n tends to ∞) in the $\mathcal{U}$-topology for all $\psi \in \mathcal{E}$.

3.4. **Lemma** [9]. *If h is an operator homomorphism, then there exists a distribution f such that $f * \phi = h(\phi)$ for all $\phi \in \mathcal{D}$.*

Proof. Let h be an operator homomorphism and let $\{\phi_n\}$ be any delta function sequence in $\mathcal{D}$. Because h satisfies (3.2), for any $\phi \in \mathcal{D}$ and any n we have

$$h(\phi) * \phi_n = h(\phi * \phi_n) = h(\phi_n * \phi) = h(\phi_n) * \phi,$$

and so $\lim h(\phi_n) * \phi = h(\phi)$. When the functions $h(\phi_n) * \phi$ and $h(\phi)$ are evaluated at zero, the sequence $\{h(\phi_n)\}$ (considered as a sequence of regular distributions) defines a linear functional f on $\mathcal{D}$ in the usual weak sense for distributions. Such weak limits are known [3, p. 315] to be distributions. Clearly, this limit distribution f satisfies $h(\phi) = f * \phi$ for all $\phi \in \mathcal{D}$.

This lemma allows us to more or less equate operator homomorphisms and distributions on R^k, and we shall freely use well-known properties of distributions [7] in our subsequent discussions of operator homomorphisms.

4. **Operators.** The algebraic view of operators is similar to that of distributions and provides an algebraic framework for representation. We begin with a preliminary definition.

4.1. **Definition.** A mapping p from a nonzero ideal $\mathcal{I}$ of an admissible ring $\mathcal{C}$ into the ring is called a *partial $\mathcal{C}$-homomorphism* if $p(\phi * \psi) = p(\phi) * \psi$ and $p(\phi + \sigma) = p(\phi) + p(\sigma)$ for all $\phi, \sigma \in \mathcal{I}$ and $\psi \in \mathcal{C}$. If p is not the restriction to $\mathcal{I}$ of any partial $\mathcal{C}$-homomorphism defined on a (properly) larger ideal of $\mathcal{C}$, then p is said to be *maximal*. If p is defined on all of $\mathcal{C}$, then p is said to be *total*.

Because $\mathcal{C}$ has no divisors of zero every nonzero partial $\mathcal{C}$-homomorphism is an injective mapping whose image is also a nonzero ideal of $\mathcal{C}$. A partial $\mathcal{C}$-homomorphism p on $\mathcal{I}$ is readily extended to a (unique) maximal one p' by defining $p'(\sigma) = \rho$ whenever $p(\sigma * \phi) = \rho * \phi$ for all $\phi \in \mathcal{I}$. We are ultimately concerned with the maximal extensions of these mappings.

4.2. **Definition.** A maximal partial $\mathcal{C}$-homomorphism m is called a *$\mathcal{C}$-operator*. In particular, a $\mathcal{D}$-operator (maximal partial $\mathcal{D}$-homomorphism) is called a *compact operator*.

In the collection $\mathcal{M}(\mathcal{C})$ of $\mathcal{C}$-operators we can define two internal operations for which $\mathcal{M}(\mathcal{C})$ has a field structure. Let m_1 and m_2 be two $\mathcal{C}$-operators with domains $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. The sum $m_1 + m_2 = m_3$ is then defined to be the maximal extension of the ordinary sum of the restrictions of m_1 and m_2 to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$, and the product $m_1 m_2 = m_4$ is defined to be the maximal extension of the ordinary composition of m_1 and the restriction of m_2 to the ideal $m_2^{-1}(\mathcal{I}_1) \cap \mathcal{I}_2$, where $m_2^{-1}(\mathcal{I}_1) = \{\phi: m_2(\phi) \in \mathcal{I}_1\}$. It is easy to verify that $\mathcal{M}(\mathcal{C})$ is a field under these operations. In particular, the reciprocal $1/m = m^{-1}$ of an operator $m \neq 0$ is the usual inverse mapping which interchanges the roles of the domain and the image ideals. Since each $\psi \in \mathcal{C}$ under convolution defines a (total) $\mathcal{C}$-operator ($\phi \to \psi * \phi$ for all $\phi \in \mathcal{C}$), the field $\mathcal{M}(\mathcal{C})$ contains an isomorphic image of the ring $\mathcal{C}$.

Another view of the field $\mathcal{M}(\mathcal{C})$ is as the field of quotients of the admissible ring $\mathcal{C}$, where a fraction ψ/ϕ determines that unique $\mathcal{C}$-operator m mapping ϕ onto

ψ, i.e., $m(\phi) = \psi$. This approach was adopted by Mikusiński [5] in the original development of operators as elements of the field of quotients of the ring of continuous (complex-valued) functions on $[0, \infty)$ under addition and convolution. The field $\mathcal{M}(\mathcal{C}_R^\infty)$ (for $k = 1$) is isomorphic to this Mikusiński field of operators [8, p. 61]. (The case for $k > 1$ is similar.)

Since $\mathcal{D} \subset \mathcal{C}$, the field of compact operators $\mathcal{M}(\mathcal{D})$ can be imbedded in a natural way in the field $\mathcal{M}(\mathcal{C})$. In particular $\mathcal{M}(\mathcal{D})$ can be imbedded in $\mathcal{M}(\mathcal{C}_R^\infty)$. On the other hand, some $\mathcal{C}_R^\infty$-operators (such as the one defined by the mapping $\phi \to \sum \phi_{n^2}$ $(n = 0, 1, \ldots,))$ do not agree as mappings on $\mathcal{D}$ with any nonzero compact operator, and thus under this natural imbedding $\mathcal{M}(\mathcal{D})$ is a proper subfield of $\mathcal{M}(\mathcal{C}_R^\infty)$.

An admissible ring $\mathcal{C}$ is a complex linear space and scalar multiplication commutes with convolution, thus for each $c \in C$, the mapping $\phi \to c\phi$, for all $\phi \in \mathcal{C}$, defines a (total) $\mathcal{C}$-operator. It follows that $\mathcal{M}(\mathcal{C})$ is also a complex linear space. Moreover, because of the maximal property of operators, the domain and the image of each operator are linear spaces and an operator is a linear mapping. Similarly, since an admissible ring is translation-invariant and translation in R^k commutes with convolution, it follows that the domain and image of an operator are each translation-invariant.

In the subsequent material we shall often omit the $\mathcal{C}$-prefix (as above) and just refer to operators with the understanding that a particular admissible ring has been selected. The remaining lemmas of this section are concerned with additional properties of operators (considered as mappings) which shall be needed in later sections.

4.3. **Lemma.** *An operator m, mapping $\mathcal{J}$ into $\mathcal{C}$, commutes with translation in R^k.*

Proof. Since $\mathcal{J}$ and $\mathcal{C}$ are translation-invariant and translation in R^k commutes with convolution, we have

$$m(\phi_\tau) * \psi = m(\phi_\tau * \psi) = m(\phi * \psi_\tau) = m(\phi) * \psi_\tau = (m(\phi))_\tau * \psi,$$

for all $\phi \in \mathcal{J}$ and $\psi \in \mathcal{C}$. Since $\mathcal{C}$ has no divisors of zero we conclude that $m(\phi_\tau) = (m(\phi))_\tau$.

The next result follows directly from the linearity of an operator and of its domain.

4.4. **Lemma.** *Let m be an operator with domain $\mathcal{J}$. Then the (evaluation) map $\phi \to m(\phi)(0)$ is linear on $\mathcal{J}$.*

4.5. **Lemma.** *Let m be a nonzero compact operator. Then for any nonzero ϕ and ψ in the domain of m we have*

$$\text{Support } (m(\phi)) \subset [[m(\psi)]] - [[\psi]] + [[\phi]].$$

Proof. Applying Lions' theorem (2.1) to both sides of the equality $m(\phi) * \psi = m(\psi) * \phi$ we obtain

$$[[m(\phi)]] + [[\psi]] = [[m(\psi)]] + [[\phi]].$$

Since Support$(m(\phi)) \subset [[m(\phi)]]$, the conclusion follows.

In the same manner, using Mikusiński's theorem (2.2), we obtain the following result.

4.6. **Lemma.** *Let m be a nonzero $\mathcal{C}_R^\infty$-operator. Then for any nonzero ϕ and ψ in the domain of m we have*

$$\text{Support } (m(\phi)) \subset C[[m(\psi)]] - C[[\psi]] + C[[\phi]].$$

5. **Representation.** One of our main objectives in this work is to relate operators and operator homomorphisms (distributions) in a comprehensive and a decisive, algebraic manner. We are able to do this because we define operators on ideals of an admissible ring $\mathcal{C}$ and operator homomorphisms on the ring $\mathcal{D}$ which is a subring of $\mathcal{C}$. We begin by considering the domains of certain operators and then by considering suitable restrictions of these operators.

5.1. **Definition.** An operator is said to be *semicompact* if its domain contains a nonzero function ϕ in $\mathcal{D}$.

Traditionally such an operator is viewed as the quotient of two ring elements with the denominator function having compact (nonempty) support.

If m is a semicompact $\mathcal{C}$-operator, then the intersection of its domain $\mathcal{I}$ with $\mathcal{D}$ is a nonzero $\mathcal{D}$-ideal $\mathcal{I}_{\mathcal{D}} = \mathcal{I} \cap \mathcal{D}$, and the restriction of m to $\mathcal{I}_{\mathcal{D}}$ uniquely identifies m as a $\mathcal{C}$-operator. Moreover, any restriction of m to a nonzero $\mathcal{D}$-ideal uniquely identifies m as a $\mathcal{C}$-operator, and thus for each fixed ϕ $(\neq 0) \in \mathcal{D}$ and each fixed $\psi \in \mathcal{C}$, the mapping which sends ϕ to ψ determines a unique $\mathcal{C}$-operator. On the other hand such restrictions of semicompact operators might possibly coincide with restrictions of certain operator homomorphisms. This leads us to the following.

5.2. **Definition.** An operator m is said to *represent* an operator homomorphism h if m and h agree, as mappings, on some nonzero $\mathcal{D}$-ideal, i.e., if m and h have a common restriction to some nonzero $\mathcal{D}$-ideal. The largest $\mathcal{D}$-ideal on which m and h agree is called the *domain of representation* (of h by m), and if the domain of representation is the full domain (in $\mathcal{D}$) $\mathcal{I}_{\mathcal{D}} = \mathcal{I} \cap \mathcal{D}$ ($\mathcal{I}$ = domain of m) of the operator m, then m is said to *fully* represent the operator homomorphism h. If the domain of representation is all of $\mathcal{D}$, then m is called a *distributional operator*.

5.3. **Remark.** In the latter case, the restriction of a distributional operator m to $\mathcal{D}$ is, in fact, a distribution. This terminology corresponds to the usual identification of an operator as a distribution [12].

According to this definition, a $\mathcal{C}$-operator can represent more than one

operator homomorphism, but an operator homomorphism can be represented by at most one $\mathcal{C}$-operator. Whenever an operator homomorphism h maps some particular ϕ $(\neq 0) \in \mathcal{D}$ to an element in an admissible ring $\mathcal{C}$, then the $\mathcal{C}$-operator m determined by the fraction $h(\phi)/\phi$ represents h on the $\mathcal{D}$-ideal $\phi * \mathcal{D}$. The domain of representation, however, may be a larger ideal. We wish to characterize those operators which represent operator homomorphisms (distributions) and hereafter we shall assume that all operators referred to are semicompact.

5.4. **Remark.** For a given admissible ring $\mathcal{C}$ and a particular operator $m \in \mathcal{M}(\mathcal{C})$ there may be operators associated with other admissible rings agreeing with a restriction of m on some nonzero $\mathcal{D}$-ideal. These other operators would seem, somehow, to be "equivalent" to m and, of course, they represent any operator homomorphism that m does. Thus we are led rather naturally to the following auxiliary definition.

5.5. **Definition.** Two operators (associated with different admissible rings) are said to *represent* each other if they agree, as mappings, on a nonzero $\mathcal{D}$-ideal; i.e., if they have a common restriction to a nonzero $\mathcal{D}$-ideal.

This definition establishes an equivalence relationship in the collection of all semicompact operators which correspond to the traditional identification of operators as quotients (fractions). In particular, this corresponds to the natural imbedding of the field $\mathcal{M}(\mathcal{D})$ of compact operators in each field $\mathcal{M}(\mathcal{C})$, where a $\mathcal{C}$-operator m with domain $\mathcal{J}$ represents a compact operator if and only if $m(\phi) \in \mathcal{D}$ for some ϕ $(\neq 0) \in \mathcal{D} \cap \mathcal{J}$.

In order to discuss certain continuity properties associated with representation we shall need to consider ideals of $\mathcal{D}$ which are also translation-invariant linear spaces. These are singled out in the following manner.

5.6. **Definition.** A nonzero ideal of $\mathcal{D}$ which is also a translation-invariant linear space is called an *effective ideal.*

Of course $\mathcal{D}$ itself is an effective ideal. If $\mathcal{J}$ is any effective ideal, then so is $\phi * \mathcal{J} = \{\phi * \psi : \psi \in \mathcal{J}\}$ for a fixed ϕ $(\neq 0) \in \mathcal{D}$. For any two effective ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ their intersection $\mathcal{J}_1 \cap \mathcal{J}_2$ is also an effective ideal. Since an operator m is a linear mapping and commutes with translation in R^k, if m is semicompact then its domain in $\mathcal{D}$ is an effective ideal, and if m represents a compact operator then its image in $\mathcal{D}$ is also an effective ideal. An example of a noneffective ideal is given by the smallest $\mathcal{D}$-ideal containing a fixed ϕ $(\neq 0) \in \mathcal{D}$, $\mathcal{J} = N\phi + \phi * \mathcal{D} = \{n\phi + \phi * \psi : \psi \in \mathcal{D}, n \text{ an integer}\}$.

Since all operator homomorphisms and operators are linear mappings and commute with translation in R^k, it readily follows that a domain of representation is always a translation-invariant linear space. We formulate this observation as a lemma.

5.7. **Lemma.** *The domain of representation $\mathcal{J}$ of an operator homomorphism h by an operator m is an effective ideal.*

We are now ready to give a necessary and sufficient condition for an operator to represent an operator homomorphism. Whenever an operator represents an operator homomorphism it necessarily inherits certain continuity properties. One such property is illustrated in the following proposition.

5.8. **Proposition.** *Let m be an operator which represents an operator homomorphism h, and let $\mathcal{J}$ be the domain of representation. Then the evaluation map $\phi \to m(\phi)(0)$ on $\mathcal{J}$ is continuous with respect to the (relative) $\mathcal{T}$-topology of $\mathcal{D}$.*

Proof. By hypothesis we have $m(\phi) = h(\phi)$ whenever $\phi \in \mathcal{J}$, and by 3.4 there exists a distribution f such that $h(\phi) = f * \phi$ for all $\phi \in \mathcal{D}$. Hence $m(\phi)(0) = f * \phi(0) = \langle f(t), \phi(-t)\rangle$ for $\phi \in \mathcal{J}$, and the conclusion follows since distributions are $\mathcal{T}$-continuous linear functionals on $\mathcal{D}$ [11, p. 22].

The next proposition shows that this particular continuity property is, in fact, characteristic of the operators which represent operator homomorphisms.

5.9. **Proposition.** *Let m be an operator and assume that for some effective ideal $\mathcal{J}$ in the domain of m the map $\phi \to m(\phi)(0)$ on $\mathcal{J}$ is continuous with respect to the (relative) $\mathcal{T}$-topology of $\mathcal{D}$. Then m represents an operator homomorphism.*

Proof. Let us define a functional f on the effective ideal $\mathcal{J}$ by

$$\phi \to \langle f(t), \phi(t)\rangle = m(\phi)(0). \tag{5.10}$$

By 4.4, f is linear and by hypothesis, f is $\mathcal{T}$-continuous. Thus using the Hahn-Banach theorem (for locally convex topological linear spaces [3, p. 180]) we conclude that f can be extended to the whole of $\mathcal{D}$ so as to be linear and $\mathcal{T}$-continuous, i.e., f can be extended to a distribution. For convenience, let us use the same symbol f to denote one such extension. We can then define an operator homomorphism h by the equation $h(\phi) = f^{\vee} * \phi$ for all $\phi \in \mathcal{D}$, where $f^{\vee}(t) = f(-t)$ is also a distribution. Now if $\phi \in \mathcal{J}$, we have (using (5.10) and 4.3)

$$h(\phi)(\tau) = \langle f(-t), \phi(\tau - t)\rangle = \langle f(t), \phi(\tau + t)\rangle = m(\phi_\tau)(0) = m(\phi)(\tau),$$

and so m represents h on $\mathcal{J}$.

5.11. **Remark.** If an operator m satisfies the hypothesis of this last proposition, then by 4.3, for *each* $\tau \in R^k$ the (evaluation) map $\phi \to m(\phi)(\tau)$ on $\mathcal{J}$ is $\mathcal{T}$-continuous. Thus if we introduce the topology $\mathcal{P}$ of pointwise convergence on R^k, the operator m manifests itself to be a $(\mathcal{T}, \mathcal{P})$-continuous mapping from $\mathcal{J}$ into $\mathcal{E}$. For our purposes, the latter is a somewhat more appropriate interpretation of this continuity property.

6. **Examples.** A few elementary examples are now presented to illustrate the above concepts and to delimit representations among operators and between operators and operator homomorphisms (distributions). Additional illustrations and applications to the important area of partial differential operators is postponed until §8. It seems appropriate to first consider the identity mapping.

6.1. In the usual manner, let $\delta = \delta(t)$ denote the (Dirac) delta function considered as a distribution. It may be equated with the identity mapping on $\mathcal{D}$ as an operator homomorphism. This latter mapping also defines a compact operator in $\mathcal{M}(\mathcal{D})$ (as does any operator homomorphism with range in $\mathcal{D}$). Moreover, for any admissible ring $\mathcal{C}$, it is the restriction to $\mathcal{D}$ of the identity mapping on $\mathcal{C}$, and so the identity $\mathcal{C}$-operator is always a distributional operator and is equated with the distribution δ. For a fixed $\tau \in R^k$ we shall denote the τ-translate of δ by δ_τ, where $\delta_\tau(t) = \delta(t + \tau)$ and thus $\delta_\tau * \phi(t) = \phi(t + \tau) = \phi_\tau(t)$ for any $\phi \in \mathcal{D}$.

The second example illustrates especially the fact that an operator may fully represent more than one operator homomorphism.

6.2. Let $\tau = (\tau_1, \ldots, \tau_k)$ $(\neq 0) \in R^k$ be fixed and consider the compact operator m defined by the mapping m: $\phi \to \phi - \phi_\tau$ on $\mathcal{D}$. It is a distributional operator in $\mathcal{M}(\mathcal{D})$ which represents the operator homomorphism h identified with the distribution $\delta - \delta_\tau$. However, the compact operator $1/m$ is not a distributional operator since its domain $\mathcal{J} = \{\phi - \phi_\tau : \phi \in \mathcal{D}\}$ is a proper ideal of $\mathcal{D}$. (Note that the support of $\phi - \phi_\tau$, ϕ $(\neq 0) \in \mathcal{D}$, has a length of at least $|\tau_i|$ in the ith component.) On the other hand, the compact operator $1/m$ does *represent* an operator homomorphism (i.e., a distribution). In fact, it represents the two (distinct) operator homomorphisms h_f and h_g identified with the distributions $f = \sum \delta_{n\tau}$ $(n \geq 0)$ and $g = -\sum \delta_{n\tau}$ $(n \leq -1)$, respectively. Indeed for all $\phi \in \mathcal{D}$ we have $(1/m)(\phi - \phi_\tau) = \phi$ and

$$\begin{aligned} h_f(\phi - \phi_\tau) &= \sum_{n \geq 0} \delta_{n\tau} * (\phi - \phi_\tau) = \phi \\ &= -\sum_{n \leq -1} \delta_{n\tau} * (\phi - \phi_\tau) = h_g(\phi - \phi_\tau). \end{aligned}$$

Thus $1/m$ fully represents both h_f and h_g. For any other admissible ring $\mathcal{C}$, the $\mathcal{C}$-operator n: $\psi \to \psi - \psi_\tau$ also represents the operator homomorphism h on all of $\mathcal{D}$, and is a distributional operator. The $\mathcal{C}$-operator $1/n$ represents the two operator homomorphisms h_f and h_g as well as the compact operator $1/m$. In the case $\mathcal{C} = \mathcal{C}_R^\infty$, with each $\tau_i \geq 0$, the $\mathcal{C}_R^\infty$-operator $1/n$ is a distributional operator since it agrees with h_f on all of $\mathcal{D}$.

It is well known [5, p. 356–357] that not every distribution is (in the traditional sense) an operator. The next example demonstrates the more general fact that not every distribution (operator homomorphism) can be represented by an operator.

6.3. For simplicity let $k = 1$, and consider the mapping on $\mathcal{D}$ into $\mathcal{E}$ defined by

$$\phi(t) \to \psi(t) = \sum_{-\infty}^{\infty} \phi(t + n^3) = \sum_{-\infty}^{\infty} \phi_{n^3}(t)$$

for all $\phi \in \mathcal{D}$. This mapping is an operator homomorphism, but for each (nonzero) image function ψ the convolution $\psi * \psi$ does not exist. To see this, let ϕ $(\neq 0) \in \mathcal{D}$ be given and let $\tau \in R^1$ be such that $\phi * \phi(\tau) \neq 0$. Then for each

n we have $\phi_{n^3} * \phi_{(-n)^3}(\tau) = \phi * \phi(\tau) \neq 0$. However, since $\phi * \phi$ has compact support, for all but finitely many pairs of integers m and n with $m \neq -n$, we have $\phi_{n^3} * \phi_{m^3}(\tau) = \phi * \phi(\tau + (n + m)(n^2 + m^2 - nm)) = 0$, and the conclusion follows. Since the image of an operator is contained in a convolution ring, the above operator homomorphism cannot be represented by any operator.

Of course not every operator is a distributional operator. The following example shows that not every operator can *fully* represent an operator homomorphism (see also 8.5).

6.4. Let ϕ $(\neq 0) \in \mathcal{D}$ be considered as the operator in $\mathcal{M}(\mathcal{D})$ which maps ψ to $\phi * \psi$, for all $\psi \in \mathcal{D}$, and let $m = 1/\phi$ denote the inverse operator of ϕ. Then m is not $(\mathcal{T}, \mathcal{P})$-continuous on its domain $\phi * \mathcal{D}$ (an effective ideal) since we can select a sequence $\{\sigma_n\}$ in $\mathcal{D}$ so that $\mathcal{T}$-lim $\phi * \sigma_n = 0$ but $\mathcal{P}$-lim $\sigma_n \neq 0$ (as n tends to ∞). For example, we can require that the support of σ_n be contained in a k-sphere of radius $1/n$ while $\max|\sigma_n(t)| = 1$. Then since $m(\phi * \sigma_n) = \sigma_n$, it follows that m is not $(\mathcal{T}, \mathcal{P})$-continuous on $\phi * \mathcal{D}$. In 7.18 we show further that $m = 1/\phi$ is not $(\mathcal{T}, \mathcal{P})$-continuous on any effective ideal, and hence does not represent any operator homomorphism.

7. **Neocontinuous operators and their calculus.** In this section we shall study the class of operators which possess the continuity property of §5. We paraphrase this property according to 5.11 as follows.

7.1. **Definition.** An operator is said to be *neocontinuous* if for some effective ideal $\mathcal{J}$ in the domain of m the restriction of m to $\mathcal{J}$ is $(\mathcal{T}, \mathcal{P})$-continuous. For each admissible ring $\mathcal{C}$, the collection of neocontinuous $\mathcal{C}$-operators is denoted by $\mathcal{N}(\mathcal{C})$. When $\mathcal{C}$ is understood, the collection of neocontinuous operators is sometimes denoted simply by $\mathcal{N}$.

This definition is now used to restate the results 5.8 and 5.9.

7.2. **Theorem.** *An operator m is neocontinuous if and only if m represents an operator homomorphism.*

The next few theorems provide for and demonstrate an effective operational calculus of neocontinuous operators.

7.3. **Theorem.** *The collection $\mathcal{N}(\mathcal{C})$ of neocontinuous $\mathcal{C}$-operators is a (complex) linear subspace of the linear space $\mathcal{M}(\mathcal{C})$ of Mikusiński operators.*

Proof. Let m be a neocontinuous operator and let c $(\neq 0) \in C$. Then, since m is a linear mapping and the domain of m is a linear space, the domain of cm is the same as the domain of m and so cm is clearly neocontinuous. Moreover, if the operators m_1 and m_2 are $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively, then the operator $m_1 + m_2$ is $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideal $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$, since, restricted to $\mathcal{J}$, the operator $m_1 + m_2$ reduces to ordinary addition of functions.

We consider now the three classical transformations of the operational calculus. We first define these on the linear space $\mathcal{E}$, and then extend them to operators.

The *exponential shift* $\mathbf{T}_\lambda$ is defined for each complex k-vector $\lambda = (\lambda_1, \ldots, \lambda_k)$ by

$$\mathbf{T}_\lambda(\psi)(t) = e^{\lambda \cdot t}\psi(t) \tag{7.4}$$

for all $\psi \in \mathcal{E}$, where $\lambda \cdot t = \lambda_1 t_1 + \cdots + \lambda_k t_k$. For each λ, $\mathbf{T}_\lambda$ is a $\mathcal{U}$-homeomorphism and a $\mathcal{P}$-homeomorphism on $\mathcal{E}$, as well as a $\mathcal{T}$-homeomorphism and a ring automorphism when restricted to $\mathcal{D}$. Under composition, where $\mathbf{T}_\lambda \circ \mathbf{T}_\mu = \mathbf{T}_{\lambda+\mu}$, the collection of all $\mathbf{T}_\lambda$ forms a commutative group.

For any $a = (a_1, \ldots, a_k) \in R^k$ with each $a_i \neq 0$, the *dilatation* $\mathbf{U}_a$ is defined by

$$\mathbf{U}_a(\psi)(t) = \mathbf{X}a\psi(at) \tag{7.5}$$

for all $\psi \in \mathcal{E}$, where $\mathbf{X}a = |a_1 a_2 \cdots a_k|$ and $at = (a_1 t_1, \ldots, a_k t_k)$. Each $\mathbf{U}_a$ has the above stated properties for an exponential shift, and under composition, where $\mathbf{U}_a \circ \mathbf{U}_b = \mathbf{U}_{ab}$, the collection of all $\mathbf{U}_a$ also forms a commutative group. We shall employ the suggestive notation $\mathbf{U}_{1/a}$ with $1/a = (1/a_1, \ldots, 1/a_k)$ for the inverse transformation of $\mathbf{U}_a$.

The *algebraic* (partial) *derivative* $\mathbf{D}_i$ is defined for each $i = 1, \ldots, k$, by

$$\mathbf{D}_i(\psi)(t) = -\pi_i(t)\psi(t) \tag{7.6}$$

for all $\psi \in \mathcal{E}$, where $\pi_i(t) = t_i$. These transformations are $(\mathcal{U}, \mathcal{U})$-continuous and $(\mathcal{P}, \mathcal{P})$-continuous on $\mathcal{E}$ and $(\mathcal{T}, \mathcal{T})$-continuous when restricted to $\mathcal{D}$. Moreover, on $\mathcal{D}$, they have the algebraic properties of a derivation.

7.7. **Remark.** If $\mathcal{J}$ is an effective ideal, then it is easy to verify that $\mathbf{T}_\lambda \mathcal{J} = \{\mathbf{T}_\lambda(\phi) : \phi \in \mathcal{J}\}$ and $\mathbf{U}_a \mathcal{J} = \{\mathbf{U}_a(\phi) : \phi \in \mathcal{J}\}$ are also effective ideals. In particular, since $\mathbf{T}_\lambda$ and $\mathbf{U}_a$ are automorphisms of the ring $\mathcal{D}$, we have $\mathbf{T}_\lambda(\phi * \mathcal{D}) = \mathbf{T}_\lambda(\phi) * \mathcal{D}$ and $\mathbf{U}_a(\phi * \mathcal{D}) = \mathbf{U}_a(\phi) * \mathcal{D}$, for any $\phi \in \mathcal{D}$. For $\phi, \psi \in \mathcal{J}$ we also have $\mathbf{D}_i(\phi * \psi) = \mathbf{D}_i(\phi) * \psi + \phi * \mathbf{D}_i(\psi) \in \mathbf{D}_i \mathcal{J} \cap \mathcal{J}$, where $\mathbf{D}_i \mathcal{J} = \{\mathbf{D}_i(\sigma) : \sigma \in \mathcal{J}\}$.

7.8. **Remark.** In order to consider the transformations $\mathbf{T}_\lambda$, $\mathbf{U}_a$ and $\mathbf{D}_i$ as mappings from an admissible ring $\mathcal{C}$ into itself we require that the functions $e^{\lambda \cdot t}\phi(t)$, $\mathbf{X}a\phi(at)$ and $\pi_i \phi(t)$ belong to $\mathcal{C}$ whenever $\phi \in \mathcal{C}$. Thus it sometimes becomes necessary to impose additional restrictions on the parameters λ and a, depending upon $\mathcal{C}$. For example, if $\mathcal{C} = Exp$, then we shall require that $\operatorname{Re} \lambda_i = 0$, for each i, and if $\mathcal{C} = \mathcal{C}_R^\infty$, we shall require that $a_i > 0$ for each i.

7.9. **Definition.** For a $\mathcal{C}$-operator m and for each suitable (see 7.8) $\lambda \in C^k$ and $a \in R^k$, we define the $\mathcal{C}$-operators $\mathbf{T}_\lambda m$, $\mathbf{U}_a m$ and $\mathbf{D}_i m$ as the maximal extensions of partial $\mathcal{C}$-homomorphisms satisfying

$$\mathbf{T}_\lambda m(\phi) = \mathbf{T}_\lambda \circ m \circ \mathbf{T}_{-\lambda}(\phi), \tag{7.10}$$

$$\mathbf{U}_a m(\phi) = \mathbf{U}_a \circ m \circ \mathbf{U}_{1/a}(\phi), \tag{7.11}$$

$$\mathbf{D}_i m(\phi) = \mathbf{D}_i \circ m(\phi) - m \circ \mathbf{D}_i(\phi), \qquad i = 1, \ldots, k, \tag{7.12}$$

whenever $\mathbf{T}_{-\lambda}(\phi)$, $\mathbf{U}_{1/a}(\phi)$, and the pair ϕ, $\mathbf{D}_i(\phi)$, respectively, are in the domain of m.

It is easy to verify that the compositions (7.10)–(7.12) define partial $\mathcal{C}$-homomorphisms and that through these definitions each $\mathbf{T}_\lambda$, $\mathbf{U}_a$ and $\mathbf{D}_i$ is extended to a linear transformation of the linear space $\mathcal{M}(\mathcal{C})$ into itself. Moreover each $\mathbf{T}_\lambda$ and $\mathbf{U}_a$ is an automorphism of the field $\mathcal{M}(\mathcal{C})$, while each $\mathbf{D}_i$ is a derivation of the field $\mathcal{M}(\mathcal{C})$. We shall now see that these transformations also preserve neocontinuity.

7.13. **Theorem.** *If $m \in \mathcal{N}(\mathcal{C})$, then $\mathbf{T}_\lambda m$, $\mathbf{U}_a m$ and $\mathbf{D}_i m$ belong to $\mathcal{N}(\mathcal{C})$, for all suitable λ and a and for all $i = 1, \ldots, k$.*

Proof. Suppose that m is $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideal $\mathcal{J}$. Then for each ψ in the effective ideal $\mathbf{T}_\lambda \mathcal{J}$ we have, by (7.10),

$$\mathbf{T}_\lambda m(\psi) = \mathbf{T}_\lambda(m(\mathbf{T}_{-\lambda}(\psi))). \tag{7.14}$$

Since $\mathbf{T}_{-\lambda}$ is a $\mathcal{T}$-homeomorphism on $\mathcal{D}$, m is $(\mathcal{T}, \mathcal{P})$-continuous on $\mathcal{J}$ and $\mathbf{T}_\lambda$ is a $\mathcal{P}$-homeomorphism on $\mathcal{E}$, it follows from (7.14) that $\mathbf{T}_\lambda m$ is $(\mathcal{T}, \mathcal{P})$-continuous on $\mathbf{T}_\lambda \mathcal{J}$. A similar argument can be applied to show that the operator $\mathbf{U}_a m$ is $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideal $\mathbf{U}_a \mathcal{J}$. Now let ϕ $(\neq 0) \in \mathcal{J}$; then for each $\psi \in \mathcal{J}$ we have $\mathbf{D}_i(\phi * \psi) \in \mathcal{J}$, as in 7.7. Thus for each $\sigma = \phi * \psi$ in the effective ideal $\phi * \mathcal{J}$ we have, by (7.12),

$$\mathbf{D}_i m(\sigma) = \mathbf{D}_i(m(\sigma)) - m(\mathbf{D}_i(\sigma)). \tag{7.15}$$

Since $\mathbf{D}_i$ is $(\mathcal{T}, \mathcal{T})$-continuous on $\mathcal{D}$ and $(\mathcal{P}, \mathcal{P})$-continuous on $\mathcal{E}$, and m is $(\mathcal{T}, \mathcal{P})$-continuous on $\mathcal{J}$, it follows from (7.15) that $\mathbf{D}_i m$ is $(\mathcal{T}, \mathcal{P})$-continuous on $\phi * \mathcal{J}$.

An operator homomorphism (distribution under convolution) is known to be $(\mathcal{T}, \mathcal{U})$-continuous on $\mathcal{D}$, and thus we can expect neocontinuous operators to possess an even stronger continuity property than in 7.1.

7.16. **Proposition.** *Let $m \in \mathcal{N}$ and suppose m is $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideal $\mathcal{J}$. Then m is $(\mathcal{T}, \mathcal{U})$-continuous on $\mathcal{J}$.*

Proof. By (the proof of) 5.9, there exists an operator homomorphism h such that $m(\phi) = h(\phi)$ whenever $\phi \in \mathcal{J}$. By 3.4 there exists a distribution f such that $h(\phi) = f * \phi$ for all $\phi \in \mathcal{D}$. Hence $m(\phi) = f * \phi$ for $\phi \in \mathcal{J}$, and the conclusion follows from well-known continuity properties for distributions under convolution [11, p. 41].

Since $\mathcal{U}$ is finer than $\mathcal{P}$ we have the following immediate consequence of the above proposition.

7.17. **Corollary.** *An operator m is neocontinuous if and only if for some effective ideal $\mathcal{J}$ in the domain of m, the restriction of m to $\mathcal{J}$ is $(\mathcal{T},\mathcal{U})$-continuous.*

7.18. **Example.** Using this corollary we can now demonstrate that a compact operator of the form $m = 1/\phi$ $(0 \neq \phi \in \mathcal{D})$ fails to be neocontinuous (see 6.4). For suppose that m is $(\mathcal{T},\mathcal{U})$-continuous on an effective ideal $\mathcal{J}$ and that $\phi * \sigma \in \mathcal{J}$ with $0 \neq \sigma \in \mathcal{D}$. (Recall that the domain of m is $\phi * \mathcal{D}$.) Then $\phi * \sigma * \mathcal{D} \subset \mathcal{J}$ and $m(\phi * \sigma * \rho) = \sigma * \rho$ for each $\rho \in \mathcal{D}$. For simplicity we assume the dimension is one and then choose a sequence $\{\rho_n\}$ in $\mathcal{D}$ satisfying (for this fixed σ)

$$\text{(a)} \qquad \left|\sigma * \rho_n(t) - \frac{\sigma^{(n)}(t)}{\sup|\sigma^{(n)}|}\right| < 1$$

and

$$\text{(b)} \qquad \left|\sigma^{(1)} * \rho_n(t) - \frac{\sigma^{(n+1)}(t)}{\sup|\sigma^{(n)}|}\right| < 1$$

for all $t \in R^1$, where $\sigma^{(n)}(t)$ is the ordinary nth derivative of σ and $\sup|\sigma^{(n)}| = \sup\{|\sigma^{(n)}(t)|: t \in R^1\}$. This can be accomplished by choosing ρ_n, with support in $[-1,1]$, to sufficiently approximate the distribution $\delta^{(n)}(t)/\sup|\sigma^{(n)}|$. Now $|\sigma^{(n)}(t)/\sup|\sigma^{(n)}| \leq 1$ for all t and n, while the sequence of numbers $\{\sup|\sigma^{(n+1)}|/\sup|\sigma^{(n)}|\}$ is unbounded since the function $\sigma(t)$ fails to be analytic for some values of t. Hence we can choose a sequence $\{\alpha_n\}$ of positive numbers satisfying $\lim \alpha_n = 0$ (as n tends to ∞) while at the same time satisfying $\limsup \alpha_n|\sup \sigma^{(n+1)}|/\sup|\sigma^{(n)}| = \infty$. Then, by (a), $\lim \alpha_n\sigma * \rho_n(t) = 0$ uniformly in t and so $\lim \alpha_n\phi * \sigma * \rho_n = 0$ in the $\mathcal{T}$-topology on $\mathcal{D}$. But the corresponding image sequence $\{m(\alpha_n\phi * \sigma * \rho_n)\}$ fails to converge in the $\mathcal{U}$-topology on $\mathcal{E}$ since $m(\alpha_n\phi * \sigma * \rho_n) = \alpha_n\sigma * \rho_n$ for each n and, by (b), $\limsup|(\alpha_n\sigma * \rho_n)^{(1)}| = \limsup|\alpha_n\sigma^{(1)} * \rho_n| = \infty$. Thus m is not $(\mathcal{T},\mathcal{U})$-continuous on $\mathcal{J}$ as originally supposed. A similar argument can be applied in the higher dimensional cases.

For compact distributional operators, 7.16 leads to the following:

7.19. **Proposition.** *Let m be a compact distributional operator. Then the mapping $\phi \to m(\phi)$ from $\mathcal{D}$ into $\mathcal{D}$ is $(\mathcal{T},\mathcal{T})$-continuous.*

Proof. We may assume $m \neq 0$ and choose a nonzero $\sigma \in \mathcal{D}$. From 4.5 we have, for each nonzero $\phi \in \mathcal{D}$,

$$\text{(7.20)} \qquad \text{Support } m(\phi) \subset [[m(\sigma)]] - [[\sigma]] + [[\phi]].$$

For any compact $K \subset R^k$, let $\mathcal{D}(K) = \{\phi \in \mathcal{D}: \text{Support } \phi \subset K\}$. Then with $K_1 = [[m(\sigma)]] - [[\sigma]]$, (7.20) implies that m maps $\mathcal{D}(K)$ into the subspace

$\mathcal{D}(K_1 + K)$. But for $\mathcal{D}(K_1 + K)$ the relative $\mathcal{T}$-topology and $\mathcal{U}$-topology are identical [3, pp. 89, 165], and so, by 7.16, m restricted to $\mathcal{D}(K)$ is $(\mathcal{T}, \mathcal{T})$-continuous. Since $\mathcal{T}$ is the final topology for $\mathcal{D} = \bigcup\{\mathcal{D}(K)$: compact $K \subset R^k\}$ [3, p. 165], it follows that m is $(\mathcal{T}, \mathcal{T})$-continuous.

7.21. **Remark.** It is well known that a distribution f has compact support if and only if $f * \phi \in \mathcal{D}$ for each $\phi \in \mathcal{D}$. Thus any compact distributional operator m fully represents an operator homomorphism which, in turn, corresponds to a distribution with compact support, since $m(\phi) \in \mathcal{D}$ for each $\phi \in \mathcal{D}$ by 4.5.

The product of two operators in the field $\mathcal{M}(\mathcal{C})$ has been defined in §4. In essence it is composition of the mappings, and we can make effective use of 7.19 when one factor is a compact distributional operator.

7.22. **Theorem.** *Let $m_1, m_2 \in \mathcal{N}$ and assume that m_2 is a compact distributional operator. Then the product $m_1 m_2 \in \mathcal{N}$.*

Proof. Let m_1 be $(\mathcal{T}, \mathcal{P})$-continuous on the effective ideal $\mathcal{J}_1$. Then $\mathcal{J} = m_2^{-1}(\mathcal{J}_1)$ is an effective ideal and for each $\phi \in \mathcal{J}$ we have

$$m_1 m_2(\phi) = m_1(m_2(\phi)) \tag{7.23}$$

where $m_2(\phi) \in \mathcal{J}_1$. Since m_2 restricted to $\mathcal{J}$ is $(\mathcal{T}, \mathcal{T})$-continuous and m_1 restricted to $\mathcal{J}_1$ is $(\mathcal{T}, \mathcal{P})$-continuous, (7.23) implies that the product $m_1 m_2$ restricted to $\mathcal{J}$ is $(\mathcal{T}, \mathcal{P})$-continuous.

7.24. **Remark.** This result corresponds to the fact that the convolution of a distribution with compact support and an arbitrary distribution is again a distribution.

7.25. **Theorem.** *Let $m \in \mathcal{N}(\mathcal{C})$ and assume that the restriction of m to the effective ideal $\mathcal{J}$ is $(\mathcal{T}, \mathcal{P})$-continuous. If $\mathcal{J}$ contains a delta function sequence $\{\phi_n\}$, then there is a unique operator homomorphism h represented by m on $\mathcal{J}$. Moreover h is fully represented by m. If, in addition, $\mathcal{C}$ is one of $\mathcal{D}$ or $\mathcal{C}_R^\infty$, then $\mathcal{D} \subset$ domain m and m is $(\mathcal{T}, \mathcal{P})$-continuous on all of $\mathcal{D}$, i.e., m is a distributional operator.*

Proof. We consider the last result first. When $\mathcal{C}$ is either $\mathcal{D}$ or $\mathcal{C}_R^\infty$ we can use support properties and the maximal property of operators to show that $\mathcal{D} \subset$ domain m. To this end, let $\phi \in \mathcal{D}$, then since $\{\phi_n\}$ is a delta function sequence in $\mathcal{J}$, we have $\lim \phi * \phi_n = \phi$ (as n tends to ∞) in the $\mathcal{T}$-topology on $\mathcal{D}$. Since m is $(\mathcal{T}, \mathcal{P})$-continuous on $\mathcal{J}$ there exists a distribution f satisfying $m(\phi * \phi_n) = f * (\phi * \phi_n)$ for all n. Hence we also have the existence of $\lim m(\phi * \phi_n)$ in the $\mathcal{U}$-topology on $\mathcal{E}$. Define $\sigma \in \mathcal{E}$ by

$$\sigma = \lim_{n\to\infty} m(\phi * \phi_n). \tag{7.26}$$

If $\mathcal{C} = \mathcal{D}$, $\phi \neq 0$ and $m \neq 0$, we have from 4.5,

$$\text{Support } m(\phi * \phi_n) \subset [[m(\phi * \phi_1)]] - [[\phi * \phi_1]] + [[\phi * \phi_n]].$$

Then since the supports of the sequence $\{\phi_n\}$, and hence also the supports of the sequence $\{\phi * \phi_n\}$, are uniformly bounded, it follows that the supports of the sequence $\{m(\phi * \phi_n)\}$ are uniformly bounded. This implies that the sequence $\{m(\phi * \phi_n)\}$ is not only $\mathcal{U}$-convergent, but is also $\mathcal{T}$-convergent and that $\sigma \in \mathcal{D}$. Similarly, if $\mathcal{C} = \mathcal{C}_R^\infty$, $\phi \neq 0$ and $m \neq 0$, then from 4.6 we have

$$\text{Support } m(\phi * \phi_n) \subset C[[m(\phi * \phi_1)]] - C[[\phi * \phi_1]] + C[[\phi * \phi_n]].$$

Here all of the subsets $C[[\phi * \phi_n]]$ are contained in a fixed right-sided orthant, and thus the supports of the sequence $\{m(\phi * \phi_n)\}$ are all contained in a fixed right-sided orthant. Therefore the sequence $\{m(\phi * \phi_n)\}$ converges to a function in $\mathcal{C}_R^\infty$. Now for either $\mathcal{C} = \mathcal{D}$ or $\mathcal{C} = \mathcal{C}_R^\infty$ and for $\psi \in \mathcal{J} =$ domain of m, we have from (7.26) and from the continuity properties of convolution in $\mathcal{C}$,

$$\begin{aligned}\sigma * \psi &= (\lim m(\phi * \phi_n)) * \psi = \lim(m(\sigma * \sigma_n) * \psi)\\ &= \lim(m(\psi) * \phi * \phi_n) = m(\psi) * \lim(\phi * \phi_n)\\ &= m(\psi) * \phi.\end{aligned}$$

Thus the maximal property of the operator m implies that $\phi \in \mathcal{J}$, with $m(\phi) = \sigma$. Hence $\mathcal{D} \subset \mathcal{J}$. Since the restriction of m to $\mathcal{D}$ satisfies (3.2), m represents an operator homomorphism on all of $\mathcal{D}$, and so is $(\mathcal{T}, \mathcal{P})$-continuous on $\mathcal{D}$, i.e., m is a distributional operator. Now for the first two results (with general $\mathcal{C}$) we simply define the operator homomorphism, using (7.26), by $h(\phi) = \sigma = \lim m(\phi * \phi_n)$ for each $\phi \in \mathcal{D}$. Then it is easy to verify that h satisfies (3.2) and agrees with m for all $\phi \in \mathcal{D} \cap \mathcal{J}$, i.e., the operator homomorphism h is fully represented by the operator m. Clearly, h is uniquely determined by the values of the operator m on $\mathcal{J}$.

7.27. **Remark.** T. K. Boehme [1] calls a $\mathcal{C}_R^\infty$-operator *regular* if its domain contains a delta function sequence of a special type, which he calls an "approximate identity". Thus all distributional operators are regular. Moreover, according to 7.25, if a neocontinuous $\mathcal{C}_R^\infty$-operator m is regular, with an approximate identity contained in the domain of continuity of m, then m is a distributional operator. However, there are significant examples (see 8.5) of neocontinuous operators which are regular but yet are not distributional operators.

8. **Differential operators.** Finally, we illustrate the algebraic view of operators and distributions and the concepts of representation in the important area of the theory of partial differential equations.

8.1. **Definition.** For $j = 1, \ldots, k$, $[s_j]$ denotes the operator obtained as the maximal extension of the partial $\mathcal{C}$-homomorphism satisfying $[s_j](\phi) = \partial\phi/\partial t_j$ whenever $\phi \in \mathcal{D}$. More generally, for $\alpha = (\alpha_1, \ldots, \alpha_k)$ a k-tuple of nonnegative integers, $P[s] = P[s_1, \ldots, s_k] = \sum c_\alpha [s]^\alpha$ $(\alpha_1 + \cdots + \alpha_k = \|\alpha\| \leq p)$, where the

coefficients c_α of the polynomial P are complex numbers, denotes the operator obtained as the maximal extension of the partial $\mathcal{C}$-homomorphism satisfying

$$P[s](\phi) = \sum_{\|\alpha\| \leq p} c_\alpha \frac{\partial^{\alpha_1 + \cdots + \alpha_k} \phi}{\partial t_1^{\alpha_1} \cdots \partial t_k^{\alpha_k}}$$

whenever $\phi \in \mathcal{D}$. These are called *differential operators.*

Clearly the domain of a differential operator $P[s]$ contains all of $\mathcal{D}$ and $P[s](\mathcal{D}) \subset \mathcal{D}$. This (together with 7.22) gives rise to the following:

8.2. **Proposition.** *A differential operator $P[s]$ is a compact distributional operator and $P[s]m \in \mathcal{N}$ whenever $m \in \mathcal{N}$.*

The next result is an operational version of the Malgrange-Ehrenpreis theorem on the existence of fundamental functions for differential operators (with constant coefficients). A proof in which (3.2) is established directly on all of $\mathcal{D}$ for the fundamental functions is given in [10].

8.3. **Theorem.** *If $P[s]$ is a (nonzero) differential operator, then the inverse operator $m = 1/P[s]$ represents an operator homomorphism, and thus is neocontinuous.*

8.4. **Remark.** In [10, Corollary 1] it is further shown that when $m = 1/P[s]$ is considered a compact operator it fully represents all operator homomorphisms associated with (distributional) fundamental functions of $P[s]$. The domain of representation is $\mathcal{J} = P[s]\mathcal{D}$ and m maps $P[s]\phi \in \mathcal{J}$ to $\phi \in \mathcal{D}$. However, nontrivial operator homomorphism solutions (distributional solutions in R^k) of homogeneous equations are not represented by operators, since $P[s]m = 0$ in any field $\mathcal{M}$ requires either $P[s] = 0$ or $m = 0$.

8.5. **Example.** Boehme [1] has shown that for the diffusion (heat) operator $P[s] = [s_1]^2 - [s_2]$ in $\mathcal{M}(\mathcal{C}_R^\infty)$, the inverse operator $m = 1/P[s]$ is regular (see 7.27). However, it is not distributional and hence by 7.25 an approximate identity cannot lie in the domain of continuity of m. This $\mathcal{C}_R^\infty$-operator is an example of a neocontinuous operator which does not *fully* represent an operator homomorphism.

8.6. **Theorem.** *Let $P[s]$ be a (nonzero) differential operator and let m be a neocontinuous operator. Then the quotient operator $m_1 = m/P[s]$ is neocontinuous and represents certain operator homomorphisms. In particular, if m represents the operator homomorphism h, then m_1 represents any operator homomorphism h_1 satisfying $P[s]h_1 = h$ on $\mathcal{D}$.*

Proof. If m represents h on the effective ideal $\mathcal{J}$, then the domain of m_1 contains the effective ideal $\mathcal{J}_1 = P[s]\mathcal{J}$ and m_1 maps $P[s]\phi \in \mathcal{J}_1$ to $m(\phi) = h(\phi)$ for each $\phi \in \mathcal{J}$. Thus m_1 represents (on $\mathcal{J}_1$) any operator homomorphism h_1 which, in turn, satisfies $P[s]h_1 = h$ on $\mathcal{D}$ for if $\phi \in \mathcal{J}$, then $h(\phi) = P[s]h_1(\phi) = h_1(P[s]\phi)$ while $m(\phi) = (m/P[s])(P[s]\phi)$ and so h_1 and $m_1 = m/P[s]$ agree on $\mathcal{J}_1 = P[s]\mathcal{J}$.

That such h_1 always exist follows from the corresponding existence theorem [11, p. 298] for distribution solutions in R^k of inhomogeneous partial differential equations.

8.7. **Remark.** This last result is mainly an operational representation theorem for solutions (in the sense of distributions) of inhomogeneous partial differential equations. For example, if in the theorem, m is also a differential operator $Q[s]$ then the quotient $R[s] = Q[s]/P[s]$ represents all distributions f (operator homomorphisms) satisfying $P(\partial/\partial t)f(t) = Q(\partial/\partial t)\delta(t)$ in R^k, where the support of the right-hand member is concentrated at the origin.

8.8. **Remark.** The linear transformations $\mathbf{T}_\lambda$, $\mathbf{U}_a$ and $\mathbf{D}_i$ defined in 7.9 for operators are applied to quotients of differential operators as follows:

$$\mathbf{T}_\lambda R[s] = R([s] - \lambda) = R([s_1] - \lambda_1, \ldots, [s_k] - \lambda_k),$$

$$\mathbf{U}_a R[s] = R((1/a) \cdot [s]) = R([s_1]/a_1, \ldots, [s_k]/a_k), \quad \text{and}$$

$$\mathbf{D}_i R[s] = \partial R[s]/\partial s_i, \qquad i = 1, \ldots, k.$$

These expressions account for the names given to these particular linear transformations and allow for certain formal procedures in the treatment of partial differential equations in R^k.

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