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ALMOST EVERYWHERE CONVERGENCE OF VILENKIN-FOURIER SERIES(1)

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ABSTRACT. It is shown that the partial sums of Vilenkin-Fourier series of functions in $L^q(G)$, $q > 1$, converge almost everywhere, where G is a zero-dimensional, compact abelian group which satisfies the second axiom of countability and for which the dual group X has a certain bounded subgroup structure. This result includes, as special cases, the Walsh-Paley group 2^ω, local rings of integers, and countable products of cyclic groups for which the orders are uniformly bounded.

Introduction. Let X denote the dual group of a compact, abelian, zero-dimensional group G, which satisfies the second axiom of countability. Then X is a discrete, countable, abelian, torsion group. N. Ja. Vilenkin [14] showed X is the union of subgroups $\{X_s\}_{s=0}^{\infty}$, $X_s \subset X_{s+1}$, such that X_{s+1}/X_s is of prime order p_{s+1}. Vilenkin also placed an ordering on X. Such a pair (G, X) is called a Vilenkin system. A Vilenkin system is said to be bounded if $\sup_s p_s < \infty$.

For $f \in L^1(G)$, let $S_n f$ denote the nth partial sum of the Fourier series with respect to X. In this work we prove that $S_n f$ converges to f almost everywhere for each f in $L^q(G)$, $1 < q \leq \infty$. Special cases of this result include the Walsh-Paley series [11], Fourier series on the ring of integers of a local field [8], and countable products of cyclic groups with uniformly bounded orders [10].

In 1966, L. Carleson [3] established the a.e. convergence of the trigonometric Fourier series for $L^2(T)$ where T denotes the circle. This result was extended to $L^q(T)$, $q > 1$, by R. Hunt [6]. The L^2 result for the Walsh-Paley system was first established by P. Billard [1] and later improved by R. Hunt [7]. P. Sjölin [12] then proved the L^q result for the Walsh-Paley system. R. Hunt and M. Taibleson [8] established the result on local rings of integers for L^q, $q > 1$, and certain Orlicz spaces. Recently, R. Moore [10] established the result for $L^q(G)$, $q > 1$, where G is a countable product of discrete cyclic groups Zp_i which satisfies $\sup_i p_i < \infty$. All of these results are based on Carleson's original proof [3] with various modifications and simplifications. A different unpublished proof was recently discovered by C. Fefferman.

The proof given here is also based on Carleson's proof [3]. The simplifications used in the L^2 proof are closely related to those used in [7] while the L^q result is

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based on the proof in [8]. In this proof great use is made of the subgroup structures of X and G.

This work has been divided into four main chapters. In Chapter I the essentials of Vilenkin systems are reviewed. In Chapter II preliminary results are collected. A new proof of Paley's theorem [11], [15] based on the Calderón-Zygmund decomposition [2] is given. In Chapter III the result is proved for $L^2(G)$. Finally, in Chapter IV the main result is extended to $L^q(G)$, $1 < q < 2$.

I. VILENKIN SYSTEMS

The groups G and X. Let G be a zero-dimensional, compact, abelian group which satisfies the second axiom of countability. The dual group of G, X, is a discrete, countable, abelian, torsion group [4, Theorems 24.15 and 24.26]. Vilenkin [14] proved the existence of a sequence of finite subgroups of X, $\{X_s\}_{s=0}^{\infty}$, which satisfy

(1)(i) $X_0 = \{\chi_0\}$, the identity character;

(ii) $X_s \subset X_{s+1}$;

(iii) $X = \bigcup_{s=0}^{\infty} X_s$;

(iv) X_s/X_{s-1} is of prime order p_s;

(v) there exists a sequence $\{\varphi_s\}_{s=0}^{\infty}$ in X such that $\varphi_s \in X_{s+1}\backslash X_s$ and $\varphi_s^{p_{s+1}} \in X_s$.

Such a pair of groups (G, X) as described above is called a Vilenkin system. A Vilenkin system is said to be bounded if $\sup_s p_s = p < \infty$. Throughout this work, we deal solely with a bounded Vilenkin system.

The subgroups G_s. Let G_s denote the annihilator of X_s. That is

$$G_s = \{x \in G : \chi(x) = 1 \text{ for all } \chi \in X_s\}.$$

Then each G_s is a compact, open subgroup of G. In addition, the sequence $\{G_s\}_{s=0}^{\infty}$ satisfies $G_0 = G$, $G_s \supset G_{s+1}$, and $\bigcap_{s=0}^{\infty} G_s = \{e\}$, the identity of G. Vilenkin [14] proved that for each s, there exists $x_s \in G_s \backslash G_{s+1}$ such that $\varphi_s(x_s) = \exp\{2\pi i/p_{s+1}\}$. He also proved that each $x \in G$ has a unique representation of the form $x = \sum_{i=0}^{\infty} b_i x_i$ where $0 \leq b_i < p_{i+1}$. Consequently,

$$(2) \qquad G_s = \left\{x \in G\colon x = \sum_{i=0}^{\infty} b_i x_i \text{ with } b_0 = b_1 = \cdots = b_{s-1} = 0\right\},$$

and each coset of G_s in G has a representation of the form $x + G_s$ with $x = \sum_{i=0}^{s-1} b_i x_i$, $0 \leq b_i < p_{i+1}$.

Each subgroup, G_s, is itself a zero-dimensional, compact, abelian group which satisfies the second axiom of countability. Its dual group can be identified with X/X_s [4, Theorem 24.5]. Thus if (G, X) is a bounded Vilenkin system with bound p, then so is $(G_s, X/X_s)$ for any $s \geq 0$.

The orderings of X and X/X_s. As the choice of the sequence $\{\varphi_s\}_{s=0}^{\infty}$ is not unique, we assume a particular choice has been made. Having done so, the following ordering, introduced by Vilenkin [14], can be placed on X: Let $m_0 = 1$ and let $m_r = \prod_{i=1}^{r} p_i$ for $r \geq 1$. Then each natural number n can be uniquely expressed as $n = \sum_{r=0}^{\infty} \alpha_r m_r$ where $0 \leq \alpha_r < p_{r+1}$, and only finitely many of the α_r's are nonzero. Then we define χ_n by the formula

$$\chi_n = \prod_{r=0}^{\infty} \varphi_r^{\alpha_r}. \tag{3}$$

With this ordering we have

(4)(i) $X_s = \{\chi_n : 0 \leq n < m_s\}$, $s = 0, 1, 2, \ldots$;

(ii) $X/X_s = \{\chi_n \cdot X_s : n$ is of the form $\sum_{r=s}^{\infty} \alpha_r m_r\}$;

(iii) if $n = \alpha_r m_r + k$, $0 \leq k < m_r$, then $\chi_n = (\chi_{m_r})^{\alpha_r} \cdot \chi_k$.

For the sake of brevity, we shall write the dual group of G_s simply as $\{\chi_n : n = \sum_{r=s}^{\infty} \alpha_r m_r\}$. The set $\{\chi_n : n = \sum_{r=s}^{\infty} \alpha_r m_r\}$ has an ordering induced by X. This ordering in turn induces an ordering on X/X_s, which is the one we use.

Notation. Throughout this work μ will denote the normalized Haar measure on G. By an interval ω, we shall mean any coset of G_s in G for some $s \geq 0$. If $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s$, then $\mu(\omega) = \mu(G_s) = m_s^{-1}$. If $\omega \in G/G_1$, we define $\omega^* = G$. If $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s$, $s > 1$, we define ω^* as

$$\omega^* = \sum_{i=0}^{s-2} b_i x_i + G_{s-1}. \tag{5}$$

Since there are p_{s-1} intervals ω with the same ω^*, we have

$$\mu(\omega^*) = p_{s-1}\mu(\omega) \leq p\mu(\omega). \tag{6}$$

Let $n = \sum_{r=0}^{\infty} \alpha_r m_r$ and let $\omega \in G/G_s$. Then we define $n(\omega)$ as the integer $\sum_{r=s}^{\infty} \alpha_r m_r$. Then if $x \in \omega \in G/G_s$ is of the form $x = \sum_{i=0}^{s-1} b_i x_i + g_s$, $g_s \in G_s$, we have

$$\chi_n(x) = \left\{\prod_{r=0}^{s-1} \left(\chi_{m_r}\left(\sum_{i=0}^{s-1} b_i x_i\right)\right)^{\alpha_r}\right\}\chi_{n(\omega)}(x).$$

Consequently, $\chi_n(x) = A(\omega)\chi_{n(\omega)}(x)$ as x ranges over ω where $A(\omega)$ is a constant of modulus 1 depending only on ω. We also define

$$c_n(\omega) = c_n(\omega; f) = \mu(\omega)^{-1} \int_\omega f(t)\overline{\chi_{n(\omega)}(t)}\, d\mu(t),$$

and

$$\mathcal{C}_n(\omega^*) = \mathcal{C}_n(\omega^*; f) = \max_{\omega'} |c_{n(\omega')}(\omega'; f)|,$$

where the maximum is taken over all ω' with $\omega'^* = \omega^*$. Throughout this work A will denote a constant, which may vary from line to line, depending only on the bound $p = \sup_s p_s$.

Fourier series and Dirichlet kernels. The Fourier series of a function f in $L^1(G)$ is the series $\sum_{i=0}^{\infty} c_i\chi_i(x)$ where $c_i = \int_G f(t)\overline{\chi_i(t)}\,d\mu(t)$. For the nth partial sums, $S_nf = \sum_{i=0}^{n-1} c_i\chi_i$, we have

$$S_nf(x) = (f * D_n)(x) = \int_G f(t)D_n(x-t)\,d\mu(t),$$

where $D_n(x) = \sum_{i=0}^{n-1}\chi_i(x)$ is the Dirichlet kernel of order n. Vilenkin [14] derived the following formulas:

$$D_{m_s}(x) = m_s I_s(x), \tag{7}$$

where I_s is the characteristic function of G_s. Also if $n = \sum_{s=0}^{\infty}\alpha_s m_s$,

$$D_n(x) = \chi_n(x)\sum_{s=0}^{\infty} D_{m_s}(x)(\chi_{m_s}(x))^{-\alpha_s}\left(\sum_{j=0}^{\alpha_s-1}\chi_{m_s}^j(x)\right), \tag{8}$$

with the appropriate interpretation if $\alpha_s = 0$ or 1. For convenience, we write

$$D_n(x) = \chi_n(x)\sum_{s=0}^{\infty} D_{m_s}(x)\Phi_{m_s,\alpha_s}(x). \tag{9}$$

We define the modified nth partial sum, S_n^*f, by the formula

$$S_n^*f = \chi_n S_n(f\overline{\chi_n}). \tag{10}$$

It follows that $S_n^*f = f * D_n^*$ where

$$D_n^* = \sum_{s=0}^{\infty} D_{m_s}\Phi_{m_s,\alpha_s}. \tag{11}$$

II. PRELIMINARY RESULTS

The modified kernels D_n^*. The modified kernels D_n^* satisfy the following two properties, which will be used in the proof of the main result. Let $n = \sum_{s=0}^{\infty}\alpha_s m_s$. Then

$$D_n^* = \sum_{s=0}^{\infty}\left(\sum_{k=m_{s+1}-\alpha_s m_s}^{m_{s+1}-1}\chi_k\right), \tag{12}$$

where the inner sum is 0 if $\alpha_s = 0$.

(13) If $\omega \in G/G_s$, $s > 0$, and $x \notin \omega$,
$D_n^*(x-t)$ is constant as t ranges over ω.

To prove (12) it suffices to prove

$$D_{m_s}\Phi_{m_s,\alpha_s} = \sum_{k=m_{s+1}-\alpha_s m_s}^{m_{s+1}-1}\chi_k$$

since $D_n^* = \sum_{s=0}^{\infty} D_{m_s}\Phi_{m_s,\alpha_s}$. Using (1)(v), (7), and (4)(iii) we have

$$\begin{aligned}
D_{m_s}\Phi_{m_s,\alpha_s} &= \left(\sum_{\nu=0}^{m_s-1}\chi_\nu\right)(\chi_{m_s})^{-\alpha_s}\left(\sum_{j=0}^{\alpha_s-1}\chi_{m_s}^j\right) \\
&= \left(\sum_{\nu=0}^{m_s-1}\chi_\nu\right)\left(\sum_{j=0}^{\alpha_s-1}(\chi_{m_s})^{j-\alpha_s+p_{s+1}}\right) \\
&= \left(\sum_{\nu=0}^{m_s-1}\chi_\nu\right)\left(\sum_{j=0}^{\alpha_s-1}\chi_{(p_{s+1}+j-\alpha_s)m_s}\right) \\
&= \sum_{\nu=0}^{m_s-1}\sum_{j=0}^{\alpha_s-1}\chi_{\nu+(p_{s+1}+j-\alpha_s)m_s} \\
&= \sum_{k=m_{s+1}-\alpha_s m_s}^{m_{s+1}-1}\chi_k.
\end{aligned}$$

This completes the proof of (12).

To prove (13) we consider an interval $\omega = \sum_{i=0}^{s-1} b_i x_i + G_s$. Each $t \in \omega$ is of the form

$$t = \sum_{i=0}^{s-1} b_i x_i + g_s(t)$$

where $g_s(t) \in G_s$. Let $x \in \sum_{i=0}^{s-1} c_i x_i + G_s$. Then

$$x = \sum_{i=0}^{s-1} c_i x_i + g_s(x)$$

where $g_s(x) \in G_s$. Since $x \notin \omega$, it follows that $b_i \neq c_i$ for some $0 \leq i \leq s-1$. Let ν denote the smallest such i. Then by (2) it follows that $x - t \in G_{\nu-1}\backslash G_\nu$ for all $t \in \omega$. By (7) and (8) we have,

$$\begin{aligned}
D_n^*(x-t) &= \sum_{r=0}^{\infty} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t) \\
&= \sum_{r=0}^{\nu-1} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t) \qquad (14)\\
&= \sum_{r=0}^{\nu-1} m_r(\chi_{m_r}(x-t))^{-\alpha_r}\left(\sum_{j=0}^{\alpha_r-1}\chi_{m_r}^j(x-t)\right).
\end{aligned}$$

For $0 \leq r \leq \nu - 1$, $\chi_{m_r} \in X_{r+1} \subset X_\nu \subset X_s$. Recall that G_s is the annihilator of X_s. Thus for any $t \in \omega$ and $0 \leq r \leq \nu - 1$, we have

$$\begin{aligned}
\chi_{m_r}(x-t) &= \chi_{m_r}\left(\sum_{i=0}^{s-1}(c_i - b_i)x_i\right)\chi_{m_r}(g_s(x))\overline{\chi_{m_r}(g_s(t))} \\
&= \chi_{m_r}\left(\sum_{i=0}^{s-1}(c_i - b_i)x_i\right).
\end{aligned}$$

Hence $\chi_{m_r}(x-t)$ is constant as t ranges over ω for $0 \leq r \leq \nu - 1$. By (14) it follows that $D_n^*(x-t)$ is constant as t ranges over ω. This completes the proof of (13).

Plancherel's formula. In this section we deal with the completeness of the system X on G and X/X_s on G_s by using probabilistic methods. Let B denote the class of Borel sets, that is, the sigma-algebra generated by the compact sets in G. Let F_s denote the sigma-algebra generated by the cosets of G_s in G. If F denotes the sigma-algebra generated by $\bigcup_{s=0}^{\infty} F_s$, then $F = B$ [9, Lemma 3.2]. Let $x \in \omega = \sum_{i=0}^{s-1} b_i x_i + G_s$. Then

$$\begin{aligned} S_{m_s} f(x) &= \int_G f(t) D_{m_s}(x - t)\, d\mu(t) \\ &= m_s \int_{x+G_s} f(t)\, d\mu(t) \\ &= \mu(\omega)^{-1} \int_\omega f(t)\, d\mu(t). \end{aligned}$$

It follows that

$$S_{m_s} f = E(f \mid F_s) \tag{15}$$

where $E(f \mid K)$ denotes the conditional expectation of f with respect to the sigma-algebra K [13, p. 90]. Since $F = B$, the martingale convergence theorem [8, Theorem 3.1] implies $S_{m_s} f \to f$ a.e. as $s \to \infty$. The completeness of X on G now follows since any function, $f \in L^1(G)$, which has all vanishing coefficients, must satisfy $f(x) = 0$ a.e.

The completeness of X/X_s on G_s follows by an identical argument and normalization of the Haar measure on G_s. A simple translation argument shows that X/X_s is a complete orthonormal system on any coset of G_s in G with respect to the normalized measure $m_s \mu$.

We now have the following version of Plancherel's formula: Let $f \in L^2(G)$ and let ω be any interval. Then

$$\sum_{n\mu(\omega)^{-1}=0}^{\infty} |c_{n(\omega)}(\omega)|^2 = \mu(\omega)^{-1} \int_\omega |f(t)|^2 d\mu(t). \tag{16}$$

The martingale maximal function. In place of the Hardy-Littlewood maximal function, we use a probabilistic analogue, the martingale maximal function. Let $f \in L^1(G)$ and define

$$E * f(x) = \sup_{s \geq 0} |E(f \mid F_s)(x)| = \sup_{s \geq 0} |S_{m_s} f(x)|.$$

Then the martingale maximal theorem states that if $1 < q \leq \infty$,

$$\|E * f\|_q \leq A_q \|f\|_q, \tag{17}$$

where A_q depends only on q [11, Theorem 6, p. 91]. Furthermore, we have $A_q = O(q/(q-1)) = O(1)$ as $q \to \infty$ [11, Lemma 2, p. 93].

Paley's theorem. The result proved in this section, Paley's theorem, states that the nth partial sum operators are bounded, uniformly in n, from $L^q(G)$ into itself for $1 < q < \infty$. That is, there exists a constant A_q depending only on q such that for $n \geq 1$ and $f \in L^q(G)$, $1 < q < \infty$,

$$\|S_n f\|_q \leq A_q \|f\|_q. \tag{18}$$

We begin the proof by making several reductions. By considering f^+ and f^- separately, we may assume f is nonnegative. Since $S_n f = \chi_n S_n^*(f\overline{\chi_n})$, it suffices to prove the result for S_n^*. Since $S_n^* = \overline{\chi_n} S_n(f\chi_n)$, we have

$$\|S_n^*\|_2 \leq \|f\|_2. \tag{19}$$

To obtain the result for $1 < q \leq 2$, it suffices, by the Marcinkiewicz interpolation theorem [16, p. 112, vol. 2], to prove S_n^* has weak type (1, 1) independent of n. That is, for any $\lambda > 0$,

$$\mu\{x \in G : |S_n^* f(x)| > \lambda\} \leq A\lambda^{-1}\|f\|_1. \tag{20}$$

A standard duality argument, which we delay until the end of this section, then yields the result for $q > 2$.

To prove (20), we use a Calderón-Zygmund decomposition [2]. Let $\lambda > 0$ be fixed. We may assume $\|f\|_1 < \lambda$. Let

$$\Omega_1 = \left\{\omega : \omega = b_0 x_0 + G_1, \mu(\omega)^{-1} \int_\omega f(t)\, d\mu(t) > \lambda\right\},$$

$$\Omega_2 = \left\{\omega : \omega = \sum_{i=0}^{1} b_i x_i + G_2, \omega \not\subset \Omega_1, \mu(\omega)^{-1} \int_\omega f(t)\, d\mu(t) > \lambda\right\}.$$

In general, let

$$\Omega_j = \left\{\omega : \omega = \sum_{i=0}^{j-1} b_i x_i + G_j, \omega \not\subset \bigcup_{i=1}^{j-1} \Omega_i, \mu(\omega)^{-1} \int_\omega f(t)\, d\mu(t) > \lambda\right\}.$$

We obtain a sequence $\{\Omega_j\}_{j=1}^\infty$ and set $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Define

$$\begin{aligned} g(x) &= \mu(\omega)^{-1} \int_\omega f(t)\, d\mu(t) && \text{if } x \in \omega, \omega \in \Omega, \\ &= f(x) && \text{if } x \notin \omega, \omega \in \Omega, \end{aligned}$$

and let $b = f - g$. Then

$$\begin{aligned} \mu\{x \in G: |S^* f(x)| > \lambda\} \leq\ & \mu\{x \in G: |S_n^* g(x)| > \lambda/2\} \\ & + \mu\{x \in G: |S_n^* b(x)| > \lambda/2\}. \end{aligned}$$

We show that each of these expressions is dominated by $A\lambda^{-1}\|f\|_1$. We begin with the estimate for g which readily follows from the inequality $\|g\|_2^2 \leq A\lambda\|f\|_1$. We note that this estimate relies heavily on the bound of the p_s's. It follows from the martingale convergence theorem that $g(t) \leq \lambda$ for almost all t outside Ω. We have

$$\begin{aligned}\int_G (g(t))^2\,d\mu(t) &= \sum_{\omega\notin\Omega}\int_\omega (g(t))^2\,d\mu(t) + \sum_{\omega\in\Omega}\int_\omega (g(t))^2\,d\mu(t)\\ &\leq \sum_{\omega\notin\Omega}\lambda\int_\omega f(t)\,d\mu(t) + \sum_{\omega\in\Omega}\int_\omega (g(t))^2\,d\mu(t).\end{aligned}$$

Using (6), we obtain

$$\begin{aligned}\sum_{\omega\in\Omega}\int_\omega (g(t))^2\,d\mu(t) &= \sum_{\omega\in\Omega}\int_\omega g(t)\,(\mu(\omega)^{-1}\int_\omega f(s)\,d\mu(s))\,d\mu(t)\\ &\leq \sum_{\omega\in\Omega}\int_\omega g(t)\left(\frac{\mu(\omega^*)}{\mu(\omega)}\right)\left(\mu(\omega^*)^{-1}\int_{\omega^*} f(s)\,d\mu(s)\right)d\mu(t)\\ &\leq p\lambda\sum_{\omega\in\Omega}\int_\omega g(t)\,d\mu(t)\\ &= p\lambda\sum_{\omega\in\Omega}\int_\omega f(t)\,d\mu(t).\end{aligned}$$

Hence

$$\begin{aligned}\int_G (g(t))^2\,d\mu(t) &\leq \lambda\sum_{\omega\in\Omega}\int_\omega f(t)\,d\mu(t) + p\lambda\sum_{\omega\in\Omega}\int_\omega f(t)\,d\mu(t)\\ &\leq p\lambda\int_G f(t)\,d\mu(t).\end{aligned}$$

The estimate for g now follows:

$$\mu\{x \in G\colon |S_n^* g(x)| > \lambda/2\} \leq 4\lambda^{-2}\|g\|_2^2 \leq (4\lambda^{-2})(p\lambda\|f\|_1) = 4p\lambda^{-1}\|f\|_1.$$

To prove $\mu\{x \in G\colon |S_n^* b(x)| > \lambda/2\} \leq A\lambda^{-1}\|f\|_1$, we write

$$\begin{aligned}&\mu\{x \in G\colon |S_n^* b(x)| > \lambda/2\}\\ &\quad\leq \mu\{x \in G\colon |S_n^* b(x)| > \lambda/2, x \notin \omega \in \Omega\}\\ &\qquad+ \mu\{x \in G\colon |S_n^* b(x)| > \lambda/2, x \in \omega \in \Omega\}\\ &\quad\leq \mu\{x \in G\colon |S_n^* b(x)| > \lambda/2, x \notin \omega \in \Omega\} + \sum_{\omega\in\Omega}\mu(\omega).\end{aligned}$$

It suffices to prove

$$x \notin \omega \in \Omega \quad \text{implies} \quad S_n^* b(x) = 0 \tag{21}$$

and

$$\sum_{\omega\in\Omega}\mu(\omega) \leq A\lambda^{-1}\|f\|_1. \tag{22}$$

To prove (21), we note that $\int_\omega b(t)\,d\mu(t) = 0$ for each $\omega \in \Omega$. We write

$$S_n^* b(x) = \int_G b(t) D_n^*(x-t)\,d\mu(t)$$
$$= \sum_{\omega\in\Omega} \int_\omega b(t) D_n^*(x-t)\,d\mu(t).$$

For $x \notin \omega$, (13) implies $D_n^*(x-t)$ is constant as t ranges over ω. Since b has a vanishing integral on each ω, it follows that $S_n^* b(x) = 0$ for $x \notin \omega \in \Omega$, and (21) is proved. To prove (22), recall that each $\omega \in \Omega$ satisfies $\mu(\omega)^{-1} \int_\omega f(t)\,d\mu(t) > \lambda$. Thus

$$\sum_{\omega\in\Omega} \mu(\omega) < \lambda^{-1} \sum_{\omega\in\Omega} \int_\omega f(t)\,d\mu(t) \le \lambda^{-1} \|f\|_1,$$

and (22) is proved.

We finally extend the result for $q > 2$ by a duality argument. At the same time, we shall obtain an estimate of the operator norm, $\|S_n^*\|_q$, from L^q into itself, as q tends to infinity. By the Marcinkiewicz interpolation theorem [16, p. 112, vol. 2] there exists a constant A independent of n such that, for $1 < q < 2$, $\|S_n^*\|_q \le A(q/(q-1))$. Let $q' > 2$ satisfy $q^{-1} + q'^{-1} = 1$. Then

$$\begin{aligned}
\|S_n^* f\|_{q'} &= \sup_{h\in L^q(G);\|h\|_q\le 1} \int_G S_n^* f(x)\overline{h(x)}\,d\mu(x)\\
&= \sup_{h\in L^q(G);\|h\|_q\le 1} \int_G f(x)\overline{S_n^* h(x)}\,d\mu(x)\\
&\le \sup_{h\in L^q(G);\|h\|_q\le A(q/(q-1))} \int_G f(x)\overline{h(x)}\,d\mu(x)\\
&\le A(q/(q-1))\|f\|_{q'}\\
&= Aq'\|f\|_{q'}.
\end{aligned}$$

Hence $\|S_n^*\|_{q'} \le Aq'$. That is

$$(23)\qquad \|S_n^*\|_{q'} = O(q') \quad \text{as } q' \to \infty$$

with bound independent of n. This completes the proof of Paley's theorem.

III. THE L^2 RESULT

Introduction and basic results. The main result of this work is the following

Theorem. *Let $f \in L^2(G)$. Then $S_n f$ converges to f almost everywhere as n tends to infinity.*

As in the case of Paley's theorem, we make several reductions of the proof. Let Mf be defined by $Mf(x) = \sup_{n\ge 0}|S_n f(x)|$ for $x \in G$. Then it suffices to prove that, for every $\lambda > 0$,

$$(24)\qquad \mu\{x \in G : Mf(x) > \lambda\} \le A\lambda^{-2}\|f\|_2^2,$$

where A is independent of f and λ. To see this, let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a positive sequence decreasing to zero, and let $\{P_k\}_{k=1}^{\infty}$ be a sequence of finite linear combinations of characters such that $\|f - P_k\|_2^2 \leq \varepsilon_k^3$. Then assuming (24), we have

$$\begin{aligned}
\mu&\left\{x \in G\colon \limsup_{n\to\infty} |S_n f(x) - f(x)| > \varepsilon_k\right\} \\
&\leq \mu\left\{x \in G\colon \limsup_{n\to\infty} |S_n(f - P_k)(x)| > \varepsilon_k/3\right\} \\
&\quad + \mu\left\{x \in G\colon \limsup_{n\to\infty} |S_n P_k(x) - P_k(x)| > \varepsilon_k/3\right\} \\
&\quad + \mu\{x \in G\colon |P_k(x) - f(x)| > \varepsilon_k/3\} \\
&\leq \mu\{x \in G\colon M(f - P_k)(x) > \varepsilon_k/3\} \\
&\quad + \mu\{x \in G\colon |P_k(x) - f(x)| > \varepsilon_k/3\} \\
&\leq 3A\varepsilon_k^{-2}\|f - P_k\|_2^2 + 3\varepsilon_k^{-2}\|f - P_k\|_2^2 \\
&\leq A\varepsilon_k .
\end{aligned}$$

For each positive integer N let $M_N f(x) = \max_{1\leq n\leq m_N} |S_n f(x)|$. For each $\lambda > 0$, we define an exceptional set $E(\lambda, N, f)$ such that

$$\mu(E(\lambda, N, f)) \leq A_1 \lambda^{-2} \|f\|_2^2, \tag{25}$$

and

$$x \notin E(\lambda, N, f) \quad \text{implies} \quad M_N f(x) \leq A_2 \lambda, \tag{26}$$

where A_1 and A_2 are two positive constants which do not depend on N, λ, or f. Since

$$\begin{aligned}
\{x \in G\colon Mf(x) > \lambda\} &= \{x \in G\colon M(A_2 f) > A_2\lambda\} \\
&\subset \bigcup_{N=1}^{\infty} E(\lambda, N, A_2 f),
\end{aligned}$$

(25) and (26) imply

$$\begin{aligned}
\mu\{x \in G\colon Mf(x) > \lambda\} &\leq \mu\left\{\bigcup_{N=1}^{\infty} E(\lambda, N, A_2 f)\right\} \\
&= \lim_{N\to\infty} \mu\{E(\lambda, N, A_2 f)\} \\
&\leq A_1 \lambda^{-2} \|A_2 f\|_2^2 \\
&= A_1 A_2^2 \lambda^{-2} \|f\|_2^2 .
\end{aligned}$$

Thus it suffices to prove (25) and (26) for λ, N, and f fixed. From this point on we shall write $E(\lambda, N, f)$ simply as E. We may also assume $\|f\|_2 < \lambda$.

The exceptional set E will consist of two basic parts, E_1 and E_2. E_1 will be made up of certain intervals ω, and it will be easy to show

$$\mu(E_1) \leq A\lambda^{-2}\|f\|_2^2. \tag{27}$$

E_2 will be more complicated. We shall define a sequence, $\{\Lambda_j^*\}_{j=1}^{\infty}$, of collections of pairs $(n(\omega^*), \omega^*)$, where n is a positive integer. For each pair $(n(\omega^*), \omega^*) \in \Lambda_j^*$, we define an exceptional subset $V(n(\omega^*), \omega^*, j)$ such that

$$\mu\{V(n(\omega^*), \omega^*, j)\} \leq p^{-3j}\mu(\omega^*). \tag{28}$$

By using Plancherel's formula (16), we shall prove that

$$\sum_{\Lambda_j^*} \mu(\omega^*) \leq Ap^{2j}\lambda^{-2}\|f\|_2^2, \tag{29}$$

where the sum is taken over all pairs $(n(\omega^*), \omega^*) \in \Lambda_j^*$. Setting

$$E_2 = \bigcup_{j=1}^{\infty} \bigcup_{\Lambda_j^*} V(n(\omega^*), \omega^*, j),$$

(28) and (29) imply

$$\mu(E_2) \leq A\left(\sum_{j=1}^{\infty} p^{-j}\right)\lambda^{-2}\|f\|_2^2. \tag{30}$$

Combining (27) and (30), we have

$$\mu(E) \leq A\lambda^{-2}\|f\|_2^2. \tag{31}$$

For certain pairs $(n(\omega^*), \omega^*) \in \Lambda_j^*$, we define a partition of ω^*, $\Pi(n(\omega^*), \omega^*, j)$, where the elements of the partition are intervals. If $x \notin E$ and $\bar{\omega}^*$ denotes the partition element which contains x, we obtain the estimate

$$|S_{n(\omega^*)}f(x) - S_{n(\bar{\omega}^*)}f(x)| \leq p^{-j/2}\lambda. \tag{32}$$

If the partition $\Pi(n(\bar{\omega}^*), \bar{\omega}^*, j')$ were defined for some $j' < j$, we could repeat the above argument and find $\bar{\bar{\omega}}^*$ such that $x \in \bar{\bar{\omega}}^*$ and

$$|S_{n(\bar{\omega}^*)}f(x) - S_{n(\bar{\bar{\omega}}^*)}f(x)| \leq p^{-j'/2}\lambda.$$

Summing over all such estimates would show that for $x \notin E$, $|S_nf(x)| \leq (\sum_{j=1}^{\infty} p^{-j/2})\lambda$, and we would be done. However, since $\Pi(n(\bar{\omega}^*), \bar{\omega}^*, j')$ may not be defined, we must change from $(n(\bar{\omega}^*), \omega^*)$ to a new pair $(\tilde{n}(\tilde{\omega}^*), \tilde{\omega}^*)$ and make the appropriate estimates. After this modification, we shall be able to prove that if $x \notin E$, $|S_nf(x)| \leq A\lambda$ where A is a constant which depends only on p.

Selected pairs Λ_j and Λ_j^*. Let $\omega \in G/G_s$, $1 \leq s \leq N$, and consider the collection of pairs $\{(n(\omega), \omega) : 1 \leq n \leq m_N\}$. For each pair set

$$\Delta(n(\omega), \omega) = \max\{|c_{\bar{n}(\bar{\omega})}(\bar{\omega})| : \bar{\omega} \supset \omega^*, \bar{n}(\omega) = n(\omega)\}. \tag{33}$$

Let Λ_j denote the collection of pairs $(n(\omega), \omega)$ which satisfy

$$|c_{n(\omega)}(\omega)| \geq p^{-j}\lambda, \tag{34}$$

and for which one of the following conditions holds:

$$\omega^* = G \quad \text{and} \quad |c_{n(\omega)}(\omega)| < p^{-j+1}\lambda, \tag{35}$$

$$\omega^* \neq G \quad \text{and} \quad \Delta(n(\omega), \omega) < p^{-j}\lambda. \tag{36}$$

To estimate $\sum \mu(\omega)$, where the sum is taken over all pairs $(n(\omega), \omega) \in \Lambda_j$, we use a collection of "polynomials", $P_j(x; \omega)$. Let

$$P_j(x; \omega) = \sum_{(n(\overline{\omega}), \overline{\omega}) \in \Lambda_j; \overline{\omega} \supset \omega} c_{n(\overline{\omega})}(\overline{\omega}) \chi_{n(\overline{\omega})}(x). \tag{37}$$

Suppose $\omega \in G/G_s$, $s > 1$. Then

$$\begin{aligned}
&\int_\omega |f(t) - P_j(t; \omega)|^2 \, d\mu(t) \\
&\quad = \int_\omega \Big| f(t) - P_j(t; \omega^*) - \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega) \chi_{n(\omega)}(t) \Big|^2 \, d\mu(t) \\
&\quad = \int_\omega |f(t) - P_j(t; \omega^*)|^2 \, d\mu(t) \\
&\qquad - 2 \operatorname{Re} \Big\{ \int_\omega f(t) \sum_{(n(\omega), \omega) \in \Lambda_j} \overline{c_{n(\omega)}(\omega) \chi_{n(\omega)}(t)} \, d\mu(t) \Big\} \\
&\qquad + 2 \operatorname{Re} \Big\{ \int_\omega P_j(t; \omega^*) \sum_{(n(\omega), \omega) \in \Lambda_j} \overline{c_{n(\omega)}(\omega) \chi_{n(\omega)}(t)} \, d\mu(t) \Big\} \\
&\qquad + \int_\omega \Big| \sum_{(n(\omega), \omega) \in \Lambda_j} c_{n(\omega)}(\omega) \chi_{n(\omega)}(t) \Big|^2 \, d\mu(t).
\end{aligned} \tag{38}$$

To see that the third integral in (38) is zero, consider a single term of the product, $c_{\overline{n}(\overline{\omega})}(\overline{\omega}) \overline{c_{n(\omega)}(\omega)} \chi_{\overline{n}(\overline{\omega})}(t) \overline{\chi_{n(\omega)}(t)}$. By (37) we have $\overline{\omega} \supset \omega^*$ and $(\overline{n}(\overline{\omega}), \overline{\omega}) \in \Lambda_j$. Hence, (34) implies

$$|c_{\overline{n}(\overline{\omega})}(\overline{\omega})| \geq p^{-j}\lambda. \tag{39}$$

By the ordering on X/X_s, $\chi_{\overline{n}(\overline{\omega})}$ and $\chi_{n(\omega)}$ are orthogonal on ω unless

$$\overline{n}(\omega) = n(\omega). \tag{40}$$

Consequently, (33), (39), and (40) imply

$$\Delta(n(\omega), \omega) \geq p^{-j}\lambda. \tag{41}$$

But $(n(\omega), \omega) \in \Lambda_j$, and $\omega^* \neq G$ since $\omega \in G/G_s$, $s > 1$, By (36),

$$\Delta(n(\omega), \omega) < p^{-j}\lambda. \tag{42}$$

(41) and (42) are a contradiction, and so the third integral of (38) vanishes. Applying Plancherel's formula (16) to the last integral of (38), we have

$$\int_\omega \Big| \sum_{(n(\omega),\omega)\in\Lambda_j} c_{n(\omega)}(\omega)\chi_{n(\omega)}(t)\Big|^2 d\mu(t) = \mu(\omega) \sum_{(n(\omega),\omega)\in\Lambda_j} |c_{n(\omega)}(\omega)|^2. \tag{43}$$

Dropping the third integral in (38) and using (43) we obtain

$$\begin{aligned}
&\int_\omega |f(t) - P_j(t;\omega)|^2\, d\mu(t)\\
&\quad= \int_\omega |f(t) - P_j(t;\omega^*)|^2\, d\mu(t)\\
&\qquad - 2\,\mathrm{Re}\Big\{\sum_{(n(\omega),\omega)\in\Lambda_j} \overline{c_{n(\omega)}(\omega)} \int_\omega f(t)\overline{\chi_{n(\omega)}(t)}\, d\mu(t)\Big\}\\
&\qquad + \mu(\omega) \sum_{(n(\omega),\omega)\in\Lambda_j} |c_{n(\omega)}(\omega)|^2\\
&\quad= \int_\omega |f(t) - P_j(t;\omega^*)|^2\, d\mu(t)\\
&\qquad - 2\mu(\omega) \sum_{(n(\omega),\omega)\in\Lambda_j} |c_{n(\omega)}(\omega)|^2 + \mu(\omega) \sum_{(n(\omega),\omega)\in\Lambda_j} |c_{n(\omega)}(\omega)|^2\\
&\quad= \int_\omega |f(t) - P_j(t;\omega^*)|^2\, d\mu(t) - \mu(\omega) \sum_{(n(\omega),\omega)\in\Lambda_j} |c_{n(\omega)}(\omega)|^2.
\end{aligned} \tag{44}$$

Summing (44) over all $\omega \in G/G_s$, we obtain

$$\begin{aligned}
&\sum_{\omega\in G/G_s} \int_\omega |f(t) - P_j(t;\omega)|^2\, d\mu(t)\\
&\quad= \sum_{\omega\in G/G_s} \int_\omega |f(t) - P_j(t;\omega^*)|^2\, d\mu(t)\\
&\qquad - \sum_{\omega\in G/G_s} \sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega)|c_{n(\omega)}(\omega)|^2\\
&\quad= \sum_{\omega\in G/G_{s-1}} \int_\omega |f(t) - P_j(t;\omega)|^2\, d\mu(t)\\
&\qquad - \sum_{\omega\in G/G_s} \sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega)|c_{n(\omega)}(\omega)|^2.
\end{aligned} \tag{45}$$

We repeat the above argument, beginning with (38), to the first term on the right-hand side of (45). We continue this procedure until we obtain after a finite number of steps

$$\begin{aligned}
&\sum_{\omega\in G/G_s} \int_\omega |f(t) - P_j(t;\omega)|^2\, d\mu(t)\\
&\quad= \sum_{\omega\in G/G_1} \int_\omega |f(t) - P_j(t;\omega^*)|^2\, d\mu(t)\\
&\qquad - \sum_{r=1}^{s} \sum_{\omega\in G/G_r} \sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega)\,|c_{n(\omega)}(\omega)|^2.
\end{aligned} \tag{46}$$

If $\omega \in G/G_1$, $P_j(t;\omega^*) = 0$ for all t. Setting $s = N$ in (46), we obtain

$$0 \leq \sum_{\omega \in G/G_N} \int_\omega |f(t) - P_j(t;\omega)|^2 \, d\mu(t)$$
$$= \|f\|_2^2 - \sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega)|c_{n(\omega)}(\omega)|^2.$$

Consequently

$$\sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega)|c_{n(\omega)}(\omega)|^2 \leq \|f\|_2^2. \tag{47}$$

If $(n(\omega), \omega) \in \Lambda_j$, we have, by (34), $|c_{n(\omega)}(\omega)| \geq p^{-j}\lambda$. Therefore (47) implies

$$\sum_{(n(\omega),\omega)\in\Lambda_j} \mu(\omega) \leq p^{2j}\lambda^{-2}\|f\|_2^2. \tag{48}$$

We now define Λ_j^* as the collection of pairs $\{(n(\omega^*), \omega^*) : (n(\omega), \omega) \in \Lambda_j\}$. Note that for each pair in Λ_j, there are at most p pairs in Λ_j^*. This fact, (6) and (48) imply

$$\sum_{\Lambda_j^*} \mu(\omega^*) \leq p^2 \sum_{\Lambda_j} \mu(\omega) \leq p^{2j+2}\lambda^{-2}\|f\|_2^2, \tag{49}$$

where the sums in (49) are taken over all pairs in Λ_j^* and Λ_j respectively. Estimate (49) will be used later to estimate $\mu(E_2)$.

The set E_1. At this point we must define E_1, the first part of the exceptional set E. Let

$$\overline{E_1} = \left\{\omega : \mu(\omega)^{-1} \int_\omega |f(t)|^2 d\mu(t) > \lambda^2\right\} \quad \text{and}$$
$$E_1 = \bigcup_{\omega \in \overline{E_1}} \{x \in G : x \in \omega^*\}. \tag{50}$$

Using (6) and (50) we obtain

$$\mu(E_1) \leq p \sum_{\omega\in\overline{E_1}} \mu(\omega) < p\lambda^{-2} \sum_{\omega\in\overline{E_1}} \int_\omega |f(t)|^2 \, d\mu(t)$$
$$\leq A\lambda^{-2}\|f\|_2^2. \tag{51}$$

Now suppose $\omega^* \not\subset E_1$. Then if ϖ is such that $\varpi^* = \omega^*$, we have $\varpi \not\subset \overline{E_1}$. Consequently, for any n, we have

$$\begin{aligned} |c_{n(\varpi)}(\varpi)| &= \mu(\varpi)^{-1} \left| \int_\varpi f(t)\overline{\chi_{n(\varpi)}(t)}\, d\mu(t) \right| \\ &\leq \mu(\varpi)^{-1} \int_\varpi |f(t)|\, d\mu(t) \\ &\leq \mu(\varpi)^{-1} \left(\int_\varpi |f(t)|^2 d\mu(t) \right)^{1/2} (\mu(\varpi))^{1/2} \\ &< \mu(\varpi)^{-1/2}(\lambda^2\mu(\varpi))^{1/2} = \lambda. \end{aligned} \tag{52}$$

It follows that if $\omega^* \not\subset E_1$,

$$\mathcal{C}_{n(\omega^*)}(\omega^*) < \lambda \tag{53}$$

for all n.

The partitions $\Pi(n(\omega^*), \omega^*, j)$. In this section we define a partition $\Pi(n(\omega^*), \omega^*, j)$ for each pair $(n(\omega^*), \omega^*) \in \Lambda_j^*$ such that $\omega^* \not\subset E_1$. If $\omega^* \in G/G_s$, $0 \leq s < N$, the elements of the partition $\Pi(n(\omega^*), \omega^*, j)$ will be cosets in G/G_r where $s < r \leq N$.

At this point we must make a small technical adjustment. If $\omega^* = G$, by (35) a pair $(n(\omega^*), \omega^*) = (n, \omega^*)$ may belong to more than one Λ_j^*. If this is so, we delete (n, ω^*) from all Λ_j^* except the one with minimal j.

Suppose $\omega^* \not\subset E_1$ and $(n(\omega^*), \omega^*) \in \Lambda_j^*$. Then we show

$$\mathcal{C}_{n(\omega^*)}(\omega^*) < p^{-j+1}\lambda. \tag{54}$$

Consider $\overline{\omega}$ such that $\overline{\omega}^* = \omega^*$ and $|c_{n(\overline{\omega})}(\overline{\omega})| > 0$. Since $\omega^* \not\subset E_1$, we have $|c_{n(\overline{\omega})}(\overline{\omega})| < \lambda$ so there exists $\tilde{j} \geq 1$ such that $p^{-\tilde{j}}\lambda \leq |c_{n(\overline{\omega})}(\overline{\omega})| < p^{-\tilde{j}+1}\lambda$. If $\omega^* = G$, (34) and (35) imply $(n(\overline{\omega},)\overline{\omega}) \in \Lambda_{\tilde{j}}$. By the above deletion it follows that $\tilde{j} \geq j$. Therefore

$$\begin{aligned}\mathcal{C}_{n(\omega^*)}(\omega^*) &= \max_{\overline{\omega}^* = \omega^*} |c_{n(\overline{\omega})}(\overline{\omega})| \\ &< p^{-\tilde{j}+1}\lambda \leq p^{-j+1}\lambda\end{aligned}$$

and (54) is true. If $\omega^* \neq G$ and $\overline{\omega} = \sum_{i=0}^{s-1} b_i x_i + G_s$, $s > 1$, we have by (4)(iii) and (7) applied to $(X/X_{s-1}, G_{s-1})$,

$$\begin{aligned}|c_{n(\overline{\omega})}(\overline{\omega})| &= \mu(\overline{\omega})^{-1} \left| \int_{\overline{\omega}} f(t)\overline{\chi_{n(\overline{\omega})}(t)}\, d\mu(t) \right| \\ &= \mu(\overline{\omega})^{-1} p_s^{-1} \left| \int_{\omega^*} f(t)\overline{\chi_{n(\overline{\omega})}(t)} \sum_{\nu=0}^{p_s-1} \chi_{m_{s-1}}^{\nu}\left(\sum_{i=0}^{s-1} b_i x_i - t \right) d\mu(t) \right| \\ &= \mu(\omega^*)^{-1} \left| \sum_{\nu=0}^{p_s-1} \int_{\omega^*} f(t)\overline{\chi_{n(\overline{\omega})}(t)}\, \overline{\chi_{m_{s-1}}^{\nu}(t)}\, d\mu(t) \right| \\ &= \mu(\omega^*)^{-1} \left| \sum_{\nu=0}^{p_s-1} \int_{\omega^*} f(t)\overline{\chi_{\nu m_{s-1}+n(\overline{\omega})}(t)}\, d\mu(t) \right| \\ &\leq \sum_{\nu=0}^{p_s-1} \mu(\omega^*)^{-1} \left| \int_{\omega^*} f(t)\overline{\chi_{\nu m_{s-1}+n(\overline{\omega})}(t)}\, d\mu(t) \right|.\end{aligned} \tag{55}$$

Since $(n(\omega^*), \omega^*) \in \Lambda_j^*$, there exists $\tilde{\omega}$ with $\tilde{\omega}^* = \omega^*$ and $(n(\tilde{\omega}), \tilde{\omega}) \in \Lambda_j$. If $n_\nu(\omega^*) = \nu m_{s-1} + n(\overline{\omega})$, $\nu = 0, 1, \ldots, p_{s-1}$, we have $n_\nu(\tilde{\omega}) = n(\overline{\omega}) = n(\tilde{\omega})$. Since $\omega^* \neq G$, (36) implies

$$\mu(\omega^*)^{-1} \left| \int_{\omega^*} f(t)\overline{\chi_{n_\nu(\omega^*)}(t)}\, d\mu(t) \right| \leq \Delta(n(\tilde{\omega}), \tilde{\omega}) < p^{-j}\lambda. \tag{56}$$

Combining (55) and (56), we obtain

(57)
$$|c_{n(\varpi)}(\varpi)| < \sum_{\nu=0}^{p_s-1} p^{-j}\lambda \le p^{-j+1}\lambda.$$

Since ϖ was any interval with $\varpi^* = \omega^*$, (57) implies

$$\mathcal{C}_{n(\omega^*)}(\omega^*) = \max_{\varpi^*=\omega^*} |c_{n(\varpi)}(\varpi)| < p^{-j+1}\lambda$$

and (54) is true if $\omega^* \neq G$. This establishes (54).

Let $(n(\omega^*), \omega^*) \in \Lambda_j^*$, $\omega^* \not\subset E_1$, and $\omega^* \in G/G_s$. We define the partition $\Pi(n(\omega^*), \omega^*, j)$ as follows: Let

$$\Omega_1(n(\omega^*), \omega^*, j) = \{\omega \in G/G_{s+1} \colon \omega \subset \omega^*, \mathcal{C}_{n(\omega^*)}(\omega) \ge p^{-j+1}\lambda\},$$

$$\Omega_2(n(\omega^*), \omega^*, j) = \{\omega \in G/G_{s+2} \colon \omega \subset \omega^* \backslash \Omega_1(n(\omega^*), \omega^*, j), \mathcal{C}_{n(\omega^*)}(\omega) \ge p^{-j+1}\lambda\}.$$

In general, if $1 \le i < N - s$, let

$$\Omega_i n(\omega^*), \omega^*, j)$$
$$= \left\{\omega \in G/G_{s+i} \colon \omega \subset \omega^* \backslash \bigcup_{r=1}^{i-1} \Omega_r(n(\omega^*), \omega^*, j), \mathcal{C}_{n(\omega^*)}(\omega) \ge p^{-j+1}\lambda\right\}.$$

Finally, let

$$\Omega_{N-s}(n(\omega^*), \omega^*, j) = \left\{\omega \in G/G_N \colon \omega \subset \omega^* \backslash \bigcup_{r=1}^{N-s-1} \Omega_r(n(\omega^*), \omega^*, j)\right\}.$$

Then $\bigcup_{r=1}^{N-s} \Omega_r(n(\omega^*), \omega^*, j)$ forms a partition of ω^*, $\Pi(n(\omega^*), \omega^*, j)$ with the following properties:

(58)(i) $\varpi \subsetneq \omega^*$ for each $\varpi \in \Pi(n(\omega^*), \omega^*, j)$;

(ii) if $\varpi \subset \tilde{\omega} \subsetneq \omega^*$ and $\varpi \in \Pi(n(\omega^*), \omega^*, j)$, $|c_{n(\omega^*)}(\tilde{\omega})| < p^{-j+1}\lambda$:

(iii) if $\varpi \in \Pi(n(\omega^*), \omega^*, j)$ and $\varpi \in G/G_s$, $s < N$, then $|c_{n(\omega^*)}(\tilde{\omega})| \ge p^{-j+1}\lambda$ for at least one $\tilde{\omega}$ such that $\tilde{\omega}^* = \varpi$.

To see (58)(i) note that each $\varpi \in \Pi(n(\omega^*), \omega^*, j)$ must satisfy $\mathcal{C}_{n(\omega^*)}(\varpi) \ge p^{-j+1}\lambda$, and by (54) this cannot be satisfied if $\varpi = \omega^*$. To see (58)(ii) we note that if $|c_{n(\omega^*)}(\tilde{\omega})| \ge p^{-j+1}\lambda$, $\mathcal{C}_{n(\omega^*)}(\tilde{\omega}^*) \ge p^{-j+1}\lambda$ and so there exists a largest interval $\hat{\omega}^*$ such that $\mathcal{C}_{n(\omega^*)}(\hat{\omega}^*) \ge p^{-j+1}\lambda$, $\tilde{\omega}^* \subset \hat{\omega}^* \subset \omega^*$. Then $\hat{\omega}^* \in \Pi(n(\omega^*), \omega^*, j)$. But then $\varpi \subset \hat{\omega}^*$ which is impossible since $\varpi \in \Pi(n(\omega^*), \omega^*, j)$. Thus (58)(ii) holds. (58)(iii) is clear from the construction of $\Pi(n(\omega^*), \omega^*, j)$.

The basic estimate. Let $(n(\omega^*), \omega^*) \in \Lambda_j^*$, $\omega^* \in G/G_s$, and $\omega^* \not\subset E_1$. Then the partition $\Pi(n(\omega^*), \omega^*, j)$ is defined. Let $\tilde{\omega}$ satisfy $\varpi \subset \tilde{\omega} \subset \omega^*$ where ϖ is any element of $\Pi(n(\omega^*), \omega^*, j)$. Then $\tilde{\omega}$ is a union of elements $\omega' \in \Pi(n(\omega^*), \omega^*, j)$. This follows from the fact that given any two cosets, either they are disjoint or one contains the other. Our aim is to estimate $S_{n(\omega^*)}f(x) - S_{n(\tilde{\omega})}f(x)$ where $x \in \tilde{\omega}$. We define

$$
\begin{aligned}
h(t) &= 0 && \text{if } t \notin \omega^*, \\
&= \mu(\varpi)^{-1} \int_{\varpi} f(t)\overline{\chi_{n(\omega^*)}(t)}\,d\mu(t) && \text{if } t \in \varpi \in \Pi(n(\omega^*), \omega^*, j).
\end{aligned} \tag{59}
$$

Note that if $t \in \varpi \in \Pi(n(\omega^*), \omega^*, j)$, $h(t) = c_{n(\omega^*)}(\varpi)$. Consequently, by (58)(ii), we have

$$\|h\|_\infty \leq p^{-j+1}\lambda. \tag{60}$$

If $\tilde{\omega} \in G/G_{s'}$, $s' \geq s$, we have by (9)

$$
\begin{aligned}
S_{n(\omega^*)}f(x) - S_{n(\tilde{\omega})}f(x) &= \int_G f(t)\{D_{n(\omega^*)}(x-t) - D_{n(\tilde{\omega})}(x-t)\}\,d\mu(t) \\
&= \int_G f(t)\left\{\chi_{n(\omega^*)}(x-t)\left(\sum_{r=s}^{\infty} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t)\right)\right. \\
&\qquad \left. - \chi_{n(\tilde{\omega})}(x-t)\left(\sum_{r=s'}^{\infty} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t)\right)\right\} d\mu(t).
\end{aligned} \tag{61}
$$

By (7) both sums vanish if $x - t \notin G_s$ or equivalently if $t \notin \omega^*$. Now

$$\chi_{n(\omega^*)} = \left(\prod_{r=s}^{s'-1} \varphi_{m_r}^{\alpha_r}\right)\chi_{n(\tilde{\omega})},$$

where by (1)(v), $\varphi_{m_r} \in X_{s'}$ for $s \leq r \leq s' - 1$. By (7) the second sum vanishes unless $x - t \in G_{s'}$. Consequently, $\chi_{n(\omega^*)}$ and $\chi_{n(\tilde{\omega})}$ agree whenever the second sum does not vanish. Using the facts, we write (61) as

$$
\begin{aligned}
&S_{n(\omega^*)}f(x) - S_{n(\tilde{\omega})}f(x) \\
&\quad = \int_{\omega^*} f(t)\chi_{n(\omega^*)}(x-t)\left(\sum_{r=s}^{s'-1} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t)\right) d\mu(t) \\
&\quad = \chi_{n(\omega^*)}(x) \sum_{\omega' \in \Pi} \int_{\omega'} f(t)\overline{\chi_{n(\omega^*)}(t)}\left(\sum_{r=s}^{s'-1} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t)\right) d\mu(t)
\end{aligned} \tag{62}
$$

where the sum is taken over all $\omega' \in \Pi(n(\omega^*), \omega^*, j)$. For each ω',

$$\sum_{r=s}^{s'-1} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t)$$

is constant as t ranges over ω'. To see this, first consider the case $x \notin \omega'$. The result follows by applying (13) with $n' = \sum_{r=s}^{s'-1} \alpha_r m_r$. In the case $x \in \omega'$, we have $x - t \in G_{s'}$ as t ranges over ω'. With $n' = \sum_{r=s}^{s'-1} \alpha_r m_r$, we have by (12),

$$\sum_{r=s}^{s'-1} D_{m_r}\Phi_{m_r,a_r} = \sum_{r=s}^{s'-1} \sum_{k=m_{r+1}-a_r m_r}^{m_{r+1}-1} \chi_k .$$

In particular, $\sum_{r=s}^{s'-1} D_{m_r}\Phi_{m_r,a_r}$ is a sum of characters $\{\chi_k\}$ with $k < m_{s'}$. Hence by (4)(i) $\sum_{r=s}^{s'-1} D_{m_r}\Phi_{m_r,a_r}$ is a sum of characters from $X_{s'}$. Since $x - t \in G_{s'}$ as t ranges over ω', the result holds. By (59) it follows that for each $\omega' \in \Pi(n(\omega^*), \omega^*, j)$, we may replace $f(t)\overline{\chi_{n(\omega^*)}(t)}$ by $h(t)$ in (62). Using this fact, (12), and (59), we obtain

$$\begin{aligned}
&S_{n(\omega^*)}f(x) - S_{n(\tilde{\omega})}f(x)\\
&\quad = \chi_{n(\omega^*)}(x) \sum_{\omega' \in \Pi} \int_{\omega'} h(t)\left(\sum_{r=s}^{s'-1} D_{m_r}(x-t)\Phi_{m_r,a_r}(x-t)\right) d\mu(t)\\
&\quad = \chi_{n(\omega^*)}(x) \sum_{r=s}^{s'-1} \int_{\omega^*} h(t) D_{m_r}(x-t)\Phi_{m_r,a_r}(x-t)\, d\mu(t)\\
&\quad = \chi_{n(\omega^*)}(x) \sum_{r=s}^{s'-1} \int_G h(t)\left(\sum_{k=m_{r+1}-a_r m_r}^{m_{r+1}-1} \chi_k(x-t)\right) d\mu(t)\\
&\quad = \chi_{n(\omega^*)}(x) \sum_{r=s}^{s'-1} \sum_{k=m_{r+1}-a_r m_r}^{m_{r+1}-1} \chi_k(x) \int_G h(t)\overline{\chi_k(t)}\, d\mu(t)\\
&\quad = \chi_{n(\omega^*)}(x) S_{m_{s'}}(S^*_{n(\omega^*)}h)(x).
\end{aligned} \tag{63}$$

The last equality follows from (12). It now follows from (15) that

$$|S_{n(\omega^*)}f(x) - S_{n(\tilde{\omega})}f(x)| \le E^*(S^*_{n(\omega^*)}h)(x) \tag{64}$$

where E^* denotes the martingale maximal function.

The set E_2. We are now in position to define E_2. For each pair $(n(\omega^*), \omega^*) \in \Lambda_j^*$ with $\omega^* \not\subset E_1$, we define the subset

$$V(n(\omega^*), \omega^*, j) = \{x \in \omega^* : E^*(S^*_{n(\omega^*)}h)(x) > p^{-j/2}\lambda\}, \tag{65}$$

where h is defined on ω^* as in (59). Applying (17) and (18), each with $q = 6$, and (60), we obtain

$$\begin{aligned}
\mu\{V(n(\omega^*), \omega^*, j)\} &\le (p^{-j/2}\lambda)^{-6}\|E^*(S^*_{n(\omega^*)}h)\|_6^6\\
&\le (p^{-j/2}\lambda)^{-6}A_6\|h\|_6^6\\
&\le A_6(p^{-j/2}\lambda)^{-6}(p^{-j+1}\lambda)^6\mu(\omega^*)\\
&= A_6 p^6(p^{-3j})\mu(\omega^*)\\
&= Ap^{-3j}\mu(\omega^*).
\end{aligned} \tag{66}$$

Let $E_2^j = \bigcup_{\Lambda_j^*}\{V(n(\omega^*), \omega^*, j)\}$ where the union is taken over all pairs $(n(\omega^*), \omega^*)$ in Λ_j^*. Then set $E_2 = \bigcup_{j=1}^{\infty} E_2^j$. Using (49) and (66) we obtain

(67)
$$\begin{aligned}\mu(E_2) &\le \sum_{j=1}^{\infty} \mu(E_2^j)\\ &\le \sum_{j=1}^{\infty} \sum_{\Lambda_j^*} \mu\{V(n(\omega^*,), \omega^*, j)\}\\ &\le A \sum_{j=1}^{\infty} p^{-3j} \sum_{\Lambda_j^*} \mu(\omega^*)\\ &\le A \sum_{j=1}^{\infty} p^{-3j}(p^{2j+2}\lambda^{-2}\|f\|_2^2)\\ &= A\lambda^{-2}\Big(\sum_{j=1}^{\infty} p^{-j}\Big)\|f\|_2^2\\ &\le A\lambda^{-2}\|f\|_2^2.\end{aligned}$$

We now set $E = E(\lambda, N, f) = E_1 \cup E_2$. Inequalities (51) and (67) imply

(68) $$\mu(E) \le A\lambda^{-2}\|f\|_2^2.$$

Changing of pairs. Let $\omega^* \not\subset E$ satisfy $p^{-j}\lambda \le \mathcal{C}_{n(\omega^*)}(\omega^*)$. We show that there exist $\tilde{n}$, $\tilde{\omega}^*$ and $\tilde{j}$ such that

(69)(i) $\tilde{n}(\varpi) = n(\varpi)$ where $\varpi^* = \omega^*$;

(ii) $\tilde{\omega}^* \supset \omega^*$;

(iii) $1 \le \tilde{j} \le j$;

(iv) $(\tilde{n}(\tilde{\omega}^*), \tilde{\omega}^*) \in \Lambda_{\tilde{j}}^*$.

If $(n(\omega^*), \omega^*) \in \Lambda_j^*$, the result is obvious by setting $\tilde{n} = n, \tilde{j} = j$, and $\tilde{\omega}^* = \omega^*$. We may therefore assume $(n(\omega^*), \omega^*) \notin \Lambda_j^*$. We first consider the case when $\omega^* = G$. Since $\omega^* \not\subset E$, (53) implies $\mathcal{C}_{n(\omega^*)}(\omega^*) < \lambda$. Hence there exists $\tilde{j}$ with $1 \le \tilde{j} \le j$ such that

(70) $$p^{-\tilde{j}}\lambda \le \mathcal{C}_{n(\omega^*)}(\omega^*) < p^{-\tilde{j}+1}\lambda.$$

Then there exists ϖ with $\varpi^* = \omega^*$ such that

$$p^{-\tilde{j}}\lambda \le |c_{n(\varpi)}(\varpi)| < p^{-\tilde{j}+1}\lambda.$$

By (35), $(n(\varpi), \varpi) \in \Lambda_{\tilde{j}}$. From (70) it follows that $\tilde{j} = \min\{j : (n(\varpi), \varpi) \in \Lambda_j, \varpi^* = \omega^*\}$. Thus $(n(\omega^*), \omega^*) \in \Lambda_{\tilde{j}}^*$. (Recall the deletion.) We now consider the case when $\omega^* \neq G$. Since $p^{-j}\lambda \le \mathcal{C}_{n(\omega^*)}(\omega^*)$ and $(n(\omega^*), \omega^*) \notin \Lambda_j^*$, there must exist ϖ with $\varpi^* = \omega^*$ and

(71) $$\Delta(n(\varpi), \varpi) \ge p^{-j}\lambda.$$

(33) and (71) imply there exist ω' with $\omega' \supset \omega^*$ and n' with $n'(\varpi) = n(\varpi)$ such that

(72) $$|c_{n'(\omega')}(\omega')| \ge p^{-j}\lambda.$$

Consequently,

$$\mathcal{C}_{n'(\omega'^*)}(\omega'^*) \ge p^{-j}\lambda.$$

If $(n'(\omega'^*), \omega'^*) \in \Lambda_j^*$, we stop and set $\tilde{j} = j$, $\tilde{\omega}^* = \omega'^*$, and $\tilde{n} = n'$. If $(n'(\omega'^*), \omega'^*) \notin \Lambda_j^*$, we repeat the above argument and find n'', ω'' and j'' such that $\omega'' \supset \omega'^*$, $n''(\omega') = n'(\omega')$ and $\mathcal{C}_{n''(\omega''^*)}(\omega''^*) \geq p^{-j}\lambda$. Note that $n''(\omega') = n'(\omega')$ implies $n''(\overline{\omega}) = n'(\overline{\omega}) = n(\overline{\omega})$. If $(n''(\omega''^*), \omega''^*) \in \Lambda_j^*$, we stop as before. Otherwise we continue until we reach a pair $(n_0(\omega_0^*), \omega_0^*)$ in Λ_j^* or reach $\omega_0^* = G$. If $\omega_0^* = G$, $p^{-j}\lambda \leq \mathcal{C}_{n_0(\omega_0^*)}(\omega_0^*) < \lambda$ since $\omega_0^* \not\subset E_2$. Hence there exists j_0 such that $1 \leq j_0 \leq j$ and

$$p^{-j_0}\lambda \leq \mathcal{C}_{n_0(\omega_0^*)}(\omega_0^*) < p^{-j_0+1}\lambda.$$

The argument of the preceding paragraph now implies $(n_0(\omega_0^*), \omega_0^*) \in \Lambda_{j_0}^*$. Setting $\tilde{n} = n_0$, $\tilde{\omega}^* = \omega_0^*$, and $\tilde{j} = j_0$, we obtain $\tilde{n}$, $\tilde{\omega}^*$, and $\tilde{j}$ as in (69).

Thus given any $\omega^* \not\subset E$ and $\mathcal{C}_{n(\omega^*)}(\omega^*) \geq p^{-j}\lambda$, there exist $\tilde{n}$, $\tilde{\omega}^*$, and $\tilde{j}$ which satisfy (69) such that $\tilde{j}$ is minimal. It now follows that, for any $\overline{\omega}$ such that $\omega^* \subset \overline{\omega} \subset \tilde{\omega}^*$,

$$\mathcal{C}_{\tilde{n}(\overline{\omega})}(\overline{\omega}) < p^{-\tilde{j}+1}\lambda. \tag{73}$$

If (73) were false, the above argument applied to $(\tilde{n}(\overline{\omega}), \overline{\omega})$ would contradict the minimality of $\tilde{j}$.

An additional estimate. An additional estimate is required because of the above change of pairs. Let $(n(\omega^*), \omega^*) \in \Lambda_j^*$, $\omega^* \not\subset E$, $\omega^* \in G/G_s$, so that $\Pi(n(\omega^*), \omega^*, j)$ is defined. Let ω_1^* be a partition element. We wish to estimate $S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)$ where $x \in \omega_1$. We have

$$\begin{aligned} |S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| &= \left| \int_G f(t)\{D_{n(\omega_1^*)}(x-t) - D_{n(\omega_1)}(x-t)\}\, d\mu(t) \right| \\ &= \left| \int_G f(t) \left\{ \chi_{n(\omega_1^*)}(x-t) \sum_{r=s}^{\infty} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t) \right.\right. \\ &\qquad \left.\left. - \chi_{n(\omega_1)}(x-t) \sum_{r=s+1}^{\infty} D_{m_r}(x-t)\Phi_{m_r,\alpha_r}(x-t) \right\} d\mu(t) \right|. \end{aligned} \tag{74}$$

As before, both sums vanish if $t \notin \omega_1^*$, and $\chi_{n(\omega_1^*)} = \chi_{n(\omega_1)}$ when the second sum fails to vanish. This allows us to write (74) as

$$\begin{aligned} |S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| &= \left| \int_{\omega_1^*} f(t)\overline{\chi_{n(\omega_1^*)}(t)} D_{m_s}(x-t)\Phi_{m_s,\alpha_s}(x-t)\, d\mu(t) \right| \\ &= \left| \int_{\omega_1^*} f(t)\overline{\chi_{n(\omega_1^*)}(t)} m_s \left(\sum_{k=p_{s+1}-\alpha_s}^{p_{s+1}-1} \chi_{km_s}(x-t) \right) d\mu(t) \right| \\ &\leq \sum_{k=p_{s+1}-\alpha_s}^{p_{s+1}-1} \mu(\omega_1^*)^{-1} \left| \int_{\omega_1^*} f(t)\overline{\chi_{n(\omega_1^*)}(t)}\chi_{km_s}(x-t)\, d\mu(t) \right| \\ &\leq \sum_{k=p_{s+1}-\alpha_s}^{p_{s+1}-1} \sum_{\overline{\omega}_1^* = \omega_1^*} \mu(\overline{\omega}_1)^{-1} \left| \int_{\overline{\omega}_1} f(t)\overline{\chi_{n(\omega_1^*)}(t)}\, d\mu(t) \right|. \end{aligned} \tag{75}$$

In the last line we made use of the fact that χ_{km_s} is constant on cosets of G_{s+1}. For each $\bar{\omega}_1$ with $\bar{\omega}_1^* = \omega_1^*$ we have

$$\mu(\bar{\omega}_1)^{-1}\left|\int_{\bar{\omega}_1} f(t)\chi_{n(\omega_1^*)}(t)\,d\mu(t)\right| \leq \mathcal{C}_{n(\omega_1^*)}(\omega_1^*).$$

Thus we have

$$|S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| \leq p^2\mathcal{C}_{n(\omega_1^*)}(\omega_1^*). \tag{76}$$

Proof of L^2 result. We now prove that if $x \notin E = E(\lambda, N, f)$, $|S_nf(x)| \leq A\lambda$, $1 \leq n \leq m_N$. Let $\omega_0^* = G$. We may assume $\mathcal{C}_n(\omega_0^*) > 0$. Then there exists $\tilde{j}_0$ such that $p^{-\tilde{j}_0}\lambda \leq \mathcal{C}_n(\omega_0^*) < p^{-\tilde{j}_0+1}\lambda$ since $x \notin E$ (see (53)). Then $(n, \omega_0^*) \in \Lambda_{\tilde{j}_0}^*$ and the partition $\Pi(n, \omega_0^*, \tilde{j}_0)$ is defined. Let ω_1^* denote the partition element such that $x \in \omega_1$. Then by (64) and (65), we have

$$|S_nf(x) - S_{n(\omega_1^*)}f(x)| \leq p^{-\tilde{j}_0/2}\lambda. \tag{77}$$

If $n(\omega_1^*) = 0$, we stop. Otherwise we continue with a typical step: $n(\omega_1^*) \neq 0$ implies $\omega_1^* \notin G/G_N$. Since $\omega_1^* \in \Pi(n(\omega_0^*), \omega_0^*, \tilde{j}_0)$ and $x \notin E$ (see (53)) we have $p^{-\tilde{j}_0+1}\lambda \leq \mathcal{C}_{n(\omega_1^*)}(\omega_1^*) < \lambda$. Hence there exists j_1 such that, $1 \leq j_1 < \tilde{j}_0$,

$$p^{-j_1}\lambda \leq \mathcal{C}_{n(\omega_1^*)}(\omega_1^*) < p^{-j_1+1}\lambda.$$

By a change of pairs, we obtain $\tilde{n}_1$, $\tilde{\omega}_1^*, \tilde{j}_1$ such that $\tilde{n}_1(\omega_1) = n(\omega_1)$, $\tilde{\omega}_1^* \supset \omega_1^*$, $(\tilde{n}(\tilde{\omega}_1^*)\tilde{\omega}_1^*) \in \Lambda_{\tilde{j}_1}^*$, and $\tilde{j}_1$ is minimal. Then the partition $\Pi(\tilde{n}_1(\tilde{\omega}_1^*), \tilde{\omega}_1^*, \tilde{j}_1)$ is defined. Let ω_2^* be the partition element such that $x \in \omega_2$. Since

$$\mathcal{C}_{\tilde{n}_1(\omega_1^*)}(\omega_1^*) = \mathcal{C}_{n(\omega_1^*)}(\omega_1^*) < p^{-j_1+1}\lambda \leq p^{-\tilde{j}_1+1}\lambda, \tag{78}$$

it follows that $\omega_2^* \subsetneq \omega_1^*$. Hence $\tilde{n}_1(\omega_1) = n(\omega_1)$ implies $\tilde{n}_1(\omega_2^*) = n_1(\omega_2^*)$. We have

$$\begin{aligned}
&|S_{n(\omega_1^*)}f(x) - S_{n(\omega_2^*)}f(x)| \\
&\quad \leq |S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| + |S_{\tilde{n}_1(\omega_1)}f(x) - S_{\tilde{n}_1(\tilde{\omega}_1^*)}f(x)| \\
&\qquad + |S_{\tilde{n}_1(\omega_1^*)}f(x) - S_{\tilde{n}_1(\omega_2^*)}f(x)| \\
&\quad \leq 2p^{-\tilde{j}_1/2}\lambda + |S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)|
\end{aligned} \tag{79}$$

by (64) and (65). Now (76) and (78) imply

$$|S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| \leq p^2\mathcal{C}_{n(\omega_1^*)}(\omega_1^*) \leq p^{-\tilde{j}_1+3}\lambda. \tag{80}$$

Combining (79) and (80), we have

$$|S_{n(\omega_1^*)}f(x) - S_{n(\omega_2^*)}f(x)| \leq \{2p^{-\tilde{j}_1/2} + p^{-\tilde{j}_1+3}\}\lambda. \tag{81}$$

Combining (77) and (81), we obtain

$$|S_nf(x) - S_{n(\omega_2^*)}f(x)| \leq p^{-\tilde{j}_0/2}\lambda + \{2p^{-\tilde{j}_1/2} + p^{-\tilde{j}_1+3}\}\lambda.$$

If $n(\omega_2^*) = 0$, we stop. If $n(\omega_2^*) \neq 0$, we repeat the above step until we reach

$$G = \omega_0^* \supsetneqq \omega_1^* \supsetneqq \omega_2^* \supsetneqq \ldots \supsetneqq \omega_r^*,$$

with $n(\omega_i^*) \neq 0$, $i = 1, 2, \ldots, r-1$, $n(\omega_r^*) = 0$, and $j_0 > j_1 \geq \tilde{j}_1 > j_2 \geq \tilde{j}_2 > \ldots > j_r \geq \tilde{j}_r \geq 1$ with

$$|S_{n(\omega_i^*)}f(x) - S_{n(\omega_{i+1}^*)}f(x)| < \{2p^{-\tilde{j}_i/2} + p^{-\tilde{j}_i+3}\}\lambda.$$

Then

$$\begin{aligned} |S_nf(x)| &\leq \sum_{i=0}^{r-1} |S_{n(\omega_i^*)}f(x) - S_{n(\omega_{i+1}^*)}f(x)| \\ &\leq 2\left(\sum_{j=1}^{\infty} p^{-j/2}\right)\lambda + p^3\left(\sum_{j=1}^{\infty} p^{-j}\right)\lambda = A\lambda. \end{aligned}$$

This completes the proof of the L^2 result.

IV. THE L^q RESULT

Basic result. To obtain the L^q result for $1 < q < 2$, some properties of Lorentz spaces [6, p. 236] and an interpolation theorem of R. Hunt [5] reduce the problem to the following

Basic result. *Let $1 < q < \infty$, $q \neq 2$, $\lambda > 0$, and F be a measurable set in G. Then there exists a constant $A_q > 0$, independent of λ and F, such that*

$$\mu\{x \in G : MI_F(x) > \lambda\} \leq A_q\lambda^{-q}\mu(F) \tag{82}$$

where I_F is the characteristic function of F.

Since the proof of the basic result follows the L^2 proof closely, we shall only indicate the necessary modifications. We shall borrow all the notation of the L^2 proof. We may also assume $\mu(F) < \lambda^q$.

Proof of basic result. We begin by defining

$$\begin{aligned} \bar{E}_1 &= \left\{\omega : \mu(\omega)^{-1} \int_\omega I_F(t)\,d\mu(t) \geq \lambda^q\right\} \quad \text{and} \\ E_1 &= \{\omega^* : \omega \in \bar{E}_1\}. \end{aligned} \tag{83}$$

Then (6) and (83) imply

$$\begin{aligned} \sum_{\omega^* \in E_1} \mu(\omega^*) &\leq p \sum_{\omega \in \bar{E}_1} \mu(\omega) \\ &\leq p\lambda^{-q} \sum_{\omega \in \bar{E}_1} \int_\omega I_F(t)\,d\mu(t) \\ &\leq p\lambda^{-q}\mu(F). \end{aligned} \tag{84}$$

Let $L = L_q = [2q^2/(q-1)] + 1$ where $[x]$ denotes the greatest integer not greater than x. Then if $(n(\omega), \omega) \in \Lambda_j$, $\omega \not\subset E_1$, we have

$$\lambda^{-2} \leq p^{jL}\lambda^{-q}. \tag{85}$$

To see this we consider the cases $1 < q < 2$ and $q > 2$ separately. If $1 < q < 2$, we have

$$p^{-j}\lambda \leq |c_{n(\omega)}(\omega)| \leq \mu(\omega)^{-1} \int_\omega I_F(t)\, d\mu(t) \leq \lambda^q.$$

This yields

$$\lambda^{1-q} \leq p^j. \tag{86}$$

Now for $1 < q < 2$,

$$(q-2)(1-q)^{-1} \leq 2q^2(q-1)^{-1} \leq L. \tag{87}$$

From (86) and (87) we have

$$\lambda^{q-2} = (\lambda^{1-q})^{(q-2)(1-q)^{-1}} \leq (p^j)^L = p^{jL}$$

which is (85). In the case $q > 2$, we have

$$\begin{aligned} p^{-j}\lambda &\leq |c_{n(\omega)}(\omega)| \\ &\leq \mu(\omega)^{-1} \int_\omega I_F(t)\, d\mu(t) \\ &= \mu(\omega \cap F)/\mu(\omega) \leq 1. \end{aligned}$$

Thus $\lambda \leq p^j$ and so $\lambda^{q-2} \leq p^{j(q-2)} \leq p^{jL}$, since $q - 2 \leq L$, which is (85). Hence (85) is established. Applying (85) to (49), we obtain

$$\begin{aligned} \sum_{\Lambda_j^*} \mu(\omega^*) &\leq p^{2j+2}\lambda^{-2}\mu(F) = p^{2j+2}\lambda^{-q}(\lambda^{q-2})\mu(F) \\ &\leq p^{2j+2}\lambda^{-q}(p^{jL})\mu(F) = p^{2j+2+jL}\lambda^{-q}\mu(F) \\ &\leq p^{4jL}\lambda^{-q}\mu(F). \end{aligned} \tag{88}$$

As before, we use (88) to estimate $\mu(E_2)$.

The partitions $\Pi(n(\omega^*), \omega^*, j)$, the basic estimate, and the changing of pairs are the same as in the L^2 proof. We modify the set E_2 somewhat to compensate for the above estimate. Consider the operator $E^*(S^*_{n(\omega^*)})$ which is sublinear and has strong type (q, q) for $1 < q < \infty$. Recall that $\|E^*\|_q = O(1)$ as $q \to \infty$ and $\|S^*_{n(\omega^*)}\|_q = O(q)$ as $q \to \infty$ independent of $n(\omega^*)$. Hence $\|E^* S^*_{n(\omega^*)}\|_q = O(q)$ as $q \to \infty$ independent of $n(\omega^*)$. By extrapolation [16, p. 119, vol. 2], there exist positive constants A_1 and A_2 such that

$$\mu\{x \in \omega^* : |E^* S^*_{n(\omega^*)} h(x)| > A_1 \lambda\} \leq \exp\{-A_2 \lambda \|h\|_\infty^{-1}\}\mu(\omega^*). \tag{89}$$

For the moment, let $\mathcal{C}$ denote an absolute constant to be determined later. We define

$$V(n(\omega^*), \omega^*, j) = \{x \in \omega^* : |E^*(S^*_{n(\omega^*)} h)(x)| > A_1 \mathcal{C} j L p^{-j+1} \lambda\}. \tag{90}$$

Then by (89) and (60), we have

$$\begin{aligned} \mu\{V(n(\omega^*), \omega^*, j)\} &\leq \exp\{-A_2 \mathcal{C} j L p^{-j+1} \lambda \|h\|_\infty^{-1}\}\mu(\omega^*) \\ &\leq \exp\{-A_2 \mathcal{C} j L\}\mu(\omega^*). \end{aligned} \tag{91}$$

We now choose $\mathcal{C}$ such that $A_2 \mathcal{C} \geq 5 \log p$. Then from (91) we obtain

$$\begin{aligned} \mu\{V(n(\omega^*), \omega^*, j)\} &\leq \exp\{-A_2 \mathcal{C} j L\}\mu(\omega^*) \\ &\leq \exp\{-5j \log Lp\}\mu(\omega^*) \\ &= p^{-5jL}\mu(\omega^*). \end{aligned} \tag{92}$$

Summing over Λ_j^* and using (88) and (92), we obtain

$$\begin{aligned} \mu(E_2^j) &\leq \sum_{\Lambda_j^*} \mu\{V(n(\omega^*), \omega^*, j)\} \\ &\leq p^{-5jL} \sum_{\Lambda_j^*} \mu(\omega^*) \\ &\leq (p^{-5jL})(p^{4jL}\lambda^{-q})\mu(F) \\ &= p^{-jL}\lambda^{-q}\mu(F). \end{aligned} \tag{93}$$

Summing (93) over all j, we have

$$\begin{aligned} \mu(E_2) &\leq \sum_{j=1}^{\infty} \mu(E_2^j) \\ &\leq \left(\sum_{j=1}^{\infty} p^{-jL}\right)\lambda^{-q}\mu(F) \leq \left(\sum_{j=1}^{\infty} p^{-j}\right)\lambda^{-q}\mu(F). \end{aligned}$$

We finally consider $x \notin E = E_1 \cup E_2$. As before, we assume $\mathcal{C}_n(\omega_0^*) > 0$ where $\omega_0^* = G$. Then there exists $\tilde{j}_0 \geq 1$ with $p^{-\tilde{j}_0}\lambda \leq \mathcal{C}_{n(\omega_0^*)}(\omega_0^*) < p^{-\tilde{j}_0+1}\lambda$ since $\omega_0^* \not\subset E$. Let ω_1^* denote the partition element such that $x \in \omega_1$. Then (64) and (90) imply

$$|S_n f(x) - S_{n(\omega_1^*)} f(x)| \leq A_2 \mathcal{C} j p^{-\tilde{j}_0+1}\lambda, \tag{94}$$

where $f = I_F$. If $n(\omega_1^*) = 0$, we stop. Otherwise we continue with a typical step. Since $\omega_1^* \notin G/G_N$, there exists j_1 with $1 \leq j_1 < \tilde{j}_0$ such that

$$p^{-j_1}\lambda \leq \mathcal{C}_{n(\omega_1^*)}(\omega_1^*) < p^{-j_1+1}\lambda. \tag{95}$$

By a change of pairs, we obtain $\tilde{n}_1$, $\tilde{\omega}_1^*$, and $\tilde{j}_1$ such that $\tilde{n}(\omega_1) = n(\omega_1)$, $\tilde{\omega}_1^* \supset \omega_1^*$, $(\tilde{n}(\tilde{\omega}_1^*), \tilde{\omega}_1^*) \in \Lambda_{\tilde{j}_1}^*$, and $\tilde{j}_1$ is minimal. Then $\Pi(\tilde{n}(\tilde{\omega}_1^*), \tilde{\omega}_1^*, \tilde{j}_1)$ is defined. Let ω_2^* be the partition element such that $x \in \omega_2$. Then as before $\omega_2^* \subsetneqq \omega_1^*$, and $\tilde{n}_1(\omega_1) = n(\omega_1)$ implies $\tilde{n}_1(\omega_2^*) = n(\omega_2^*)$. We have, by (64), (76), (90) and (95),

$$\begin{aligned}
&|S_{n(\omega_1^*)}f(x) - S_{n(\omega_2^*)}f(x)| \\
&\qquad \le |S_{n(\omega_1^*)}f(x) - S_{n(\omega_1)}f(x)| + |S_{\tilde{n}_1(\omega_1)}f(x) - S_{\tilde{n}_1(\tilde{\omega}_1^*)}f(x)| \\
&\qquad\quad + |S_{\tilde{n}_1(\tilde{\omega}_1^*)}f(x) - S_{n(\omega_2^*)}f(x)| \\
&\qquad \le 2A_2\mathcal{C}\tilde{j}_1 Lp^{-\tilde{j}_1+1}\lambda + |S_{n(\omega_1^*)} - S_{n(\omega_1)}| \\
&\qquad \le 2A_2\mathcal{C}\tilde{j}_1 Lp^{-\tilde{j}_1+1}\lambda + p^2\mathcal{C}_{n(\omega_1^*)}(\omega_1^*) \\
&\qquad \le 2A_2\mathcal{C}\tilde{j}_1 Lp^{-\tilde{j}_1+1}\lambda + p^{-\tilde{j}_1+3}\lambda
\end{aligned} \tag{96}$$

since $\tilde{j}_1$ is minimal. Combining estimates (94) and (96), we have

$$|S_nf(x) - S_{n(\omega_2^*)}f(x)| \le 2A_2\mathcal{C}L\{\tilde{j}_0 p^{-\tilde{j}_0+1} + \tilde{j}_1 p^{-\tilde{j}_1+1} + p^{-\tilde{j}_1+3}\}\lambda.$$

If $n(\omega_2^*) = 0$, we stop. If $n(\omega_2^*) \ne 0$, we repeat the above procedure until we reach

$$G = \omega_0^* \supsetneqq \omega_1^* \supsetneqq \omega_2^* \supsetneqq \cdots \supsetneqq \omega_r^*$$

with $n(\omega_i^*) \ne 0$, $i = 1, 2, \ldots, r-1$, $n(\omega_r^*) = 0$ and $\tilde{j}_0 > j_1 \ge \tilde{j}_1 > j_2 \ge \tilde{j}_2 > \cdots > j_r \ge \tilde{j}_r \ge 1$, with

$$|S_{n(\omega_i^*)}f(x) - S_{n(\omega_{i+1}^*)}f(x)| \le \{2A_2\mathcal{C}\tilde{j}_i L + p^3\}p^{-\tilde{j}_i}\lambda.$$

Then

$$\begin{aligned}
|S_nf(x)| &\le \sum_{r=0}^{r-1} |S_{n(\omega_i^*)}f(x) - S_{n(\omega_{i+1}^*)}f(x)| \\
&\le \left\{2A_2\mathcal{C}L\left(\sum_{j=1}^{\infty} jp^{-j}\right) + p^3\left(\sum_{j=1}^{\infty} p^{-j}\right)\right\}\lambda.
\end{aligned}$$

This establishes (82), the basic result, and completes the proof.

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