THE LAW OF THE ITERATED LOGARITHM FOR BROWNIAN MOTION IN A BANACH SPACE

BY

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ABSTRACT. Strassen's version of the law of the iterated logarithm is proved for Brownian motion in a real separable Banach space. We apply this result to obtain the law of the iterated logarithm for a sequence of independent Gaussian random variables with values in a Banach space and to obtain Strassen's result.

Introduction. Let $H$ denote a real separable Hilbert space with norm $\|\cdot\|_H$ and assume $\|\cdot\|_B$ is a measurable norm on $H$ in the sense of [2]. Then there exists a constant $M > 0$ such that $\|x\|_B \leq M\|x\|_H$ for all $x \in H$, and if $B$ is the completion of $H$ in $\|\cdot\|_B$ it follows that $B$ is a real separable Banach space. We will view $H$ as a subspace of $B$ and since $\|\cdot\|_B$ is weaker than $\|\cdot\|_H$ on $H$ it follows that $B^*$, the topological dual of $B$, can be continuously injected into $H^*$, the topological dual of $H$. We call $(H, B)$ an abstract Wiener space.

For $t > 0$, let $m_t$ denote the canonical Gaussian cylinder set measure on $H$ with variance parameter $t$ and let $\mu_t$ ($t > 0$) denote the Borel probability measure on $B$ induced by $m_t$ ($t > 0$). We call $\mu_t$ the Wiener measure on $B$ generated by $H$ with variance parameter $t$.

Let $\Omega_B$ denote the space of continuous functions $w$ from $[0, \infty)$ into $B$ such that $w(0) = 0$, and let $\mathcal{F}$ be the $\sigma$-field of $\Omega_B$ generated by the functions $w \rightarrow w(t)$. Then there is a unique probability measure $P$ on $\mathcal{F}$ such that if $0 = t_0 < t_1 < \cdots < t_n$ then $w(t_j) - w(t_{j-1})$ are independent and $w(t_j) - w(t_{j-1})$ has distribution $\mu_{t_j-t_{j-1}}$ on $B$. In particular, the stochastic process $W_t$ defined on $(\Omega_B, \mathcal{F}, P)$ by $W_t(w) = w(t)$ has stationary independent Gaussian increments with transition probabilities $P(x, A) = \mu((A - x)/\sqrt{t})$ for $t > 0$. We call it Brownian motion in $B$. For a more detailed discussion see [2].

It is known from [2] that if $B$ is an arbitrary real separable Banach space, then there exists a dense subset $H$ of $B$ which is a real separable Hilbert space and the given norm on $B$ is a measurable norm on $H$. Hence any real separable Banach space can be used in the setup we described above. We also know from [5] or from [6] and [1] that if $\mu$ is any mean zero Gaussian probability measure on the Borel subsets of a real separable Banach space $B$, then there exists a real separable Hilbert space $H$ which is a subset of $B$, the given norm on $B$ is a

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measurable norm on $H$, $\mu(M) = 1$ ($M$ is the closure of $H$ in $B$), and $\mu$ is the Wiener measure on $M$ generated by $H$ with variance parameter 1. Furthermore, $H$ is unique as a subset of $B$ since it is precisely the set of vectors such that $\mu$ translated by such a vector yields a measure equivalent to $\mu$. However, the main point to be realized is that given any real separable Banach space $B$, or $B$ and a mean zero Gaussian measure $\mu$ on $B$, we can construct a Brownian motion on $B$ as indicated above. Further, in the case $\mu$ is given on $B$ we see that $\mu = \mu_1$ and that if $M$ is a proper subspace of $B$ then our Brownian motion is, with probability one, in the closed subspace $M$ satisfying $\mu(M) = 1$.

Let $C_B$ denote the continuous functions on $[0, 1]$ into $B$ which vanish at zero. Then $C_B$ is a Banach space in the norm $\|f\|_{C_B} = \sup_{0 \leq t \leq 1} \|f(t)\|_B$.

**Lemma 1.** (a) If $B$ is a real separable Banach space, then $C_B$ is a real separable Banach space in the norm $\|\cdot\|_{C_B}$.

(b) The minimal sigma-algebra $\mathcal{B}$ making the mappings $f \mapsto f(t)$ measurable consists of the Borel subsets of $C_B$.

(c) Brownian motion on $B$ induces a probability measure $P$ on $(C_B, \mathcal{B})$ which is a mean zero Gaussian measure, i.e. every linear functional in $C_B^*$ has a Gaussian distribution with mean zero.

**Proof.** (a) Let $t_j = j/2^N, j = 0, 1, \ldots, 2^N$. Let $\{x_n\}$ be a dense subset of $B$. Let $S_N$ denote the subset of $C_B$ consisting of functions which are linear on each of the subintervals $[t_{j-1}, t_j]$ with values at $t_j$ in $\{x_n\}$. Then $\bigcup_{N=1}^\infty S_N$ is a countable dense subset of $C_B$.

(b) Since $C_B$ is separable it suffices to prove that if $f_0 \in C_B$ and $\epsilon > 0$, then $U = \{f : \sup_{0 \leq t \leq 2^N} \|f(t) - f_0(t)\|_{C_B} \leq \epsilon\}$ is a set in $\mathcal{B}$. Let $I_N = \{f : \sup_{0 \leq t \leq 2^N} \|f(t) - f_0(t)\|_{C_B} \leq \epsilon\}$ for $N = 1, 2, \ldots$ and $\{t_j\}$ as in (a). Then $U = \cap_{N=1}^\infty I_N$.

(c) $P$ is the probability measure on $(C_B, \mathcal{B})$ such that if $0 = t_0 < t_1 < \cdots < t_n \leq 1$ then $f(t_j) - f(t_{j-1})$ are independent and $f(t_j) - f(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on $B$. We now must show $P$ is a mean zero Gaussian measure on $C_B$. Let $f^* \in C_B^*$ and let $X_1, \ldots, X_n$ be independent random variables with values in $C_B$ and the same distribution as $P$. Then $X_1 + \cdots + X_n/\sqrt{n}$ has distribution $P$ since for each $t \in [0, 1]$ the law of the map $f \mapsto f(t)$ is the convolution of $\mu_{t/n}$ $n$ times yielding $\mu_t$. Hence the distribution of $f^*$ has the same distribution as

$$f^*(X_1 + \cdots + X_n/\sqrt{n}) = f^*(X_1) + \cdots + f^*(X_n)/\sqrt{n}$$

and we see by [4, p. 166] that $f^*$ is strictly stable with characteristic exponent 2 and this implies $f^*$ has a Gaussian distribution with mean zero.

Our main result is a law of the iterated logarithm for Brownian motion in a Banach space as described prior to Lemma 1. This may be regarded as a general synthesis of the two log log law which follows.

I. (Strassen [10]). Let $\Omega_k$ denote the set of continuous maps from $[0, \infty)$ into
real $k$-dimensional space ($\mathbb{R}^k$) which vanish at zero, and let $C_k$ denote the space of continuous maps vanishing at zero and mapping $[0,1]$ into $\mathbb{R}^k$ endowed with the supremum of the Euclidean norm for $\mathbb{R}^k$. If $W(t) = (W_1(t), \ldots, W_k(t))$, $0 \leq t < \infty$, is a version of the $k$-dimensional Brownian motion with sample paths in $\Omega_k$, then the sequence of random functions

$$\xi_n(t) = W(nt)/(2n \log \log n)^{1/2} \quad (0 \leq t \leq 1, n \geq 3)$$

satisfies the following log log law:

$$\{\xi_n, n \geq 3\} \subseteq C_k$$

and with probability one converges in $C_k$ to a compact set $K_k$ of $C_k$ and clusters at every point of $K_k$.

Here $K_k$ denotes those $f = (f_1, \ldots, f_k) \in C_k$ such that $f$ is coordinatewise absolutely continuous with respect to Lebesgue measure on $[0,1]$, and satisfies

$$\sum_{j=1}^k \int_0^1 [df_j(s)/ds]^2 \, ds \leq 1.$$ 

By saying $\{\xi_n, n \geq 3\}$ converges to $K_k$ we mean that for every $\epsilon > 0$ the sequence is eventually within an $\epsilon$-neighborhood of $K_k$ and since $K_k$ is compact this implies that with probability one $\{\xi_n, n \geq 3\}$ is relatively norm compact in $C_k$.

II. (LePage [8]). Suppose $B$ is a real separable Banach space and $\mu$ is a mean zero Gaussian measure on the Borel subsets of $B$. If $X_1, X_2, \ldots$ are independent identically distributed $\mathcal{B}$-valued random vectors with distribution $\mu$, then the sequence

$$\xi_n = X_1 + \cdots + X_n/(2n \log \log n)^{1/2} \quad (n \geq 3)$$

almost surely converges in $B$-norm to a closed set $K \subseteq B$ and clusters at every point of $K$, where $K$ is the unit ball of the reproducing kernel Hilbert space defined on $B^* \times B^*$ by $\mu$.

The set $K_k$ of Strassen's result may be identified as the unit ball of the reproducing kernel Hilbert space of the kernel defined for $0 < s, t < 1, 0 < i, j < k$ by

$$E(\varphi_i(s)\varphi_j(t)) = \min(s,t)\delta_{ij}.$$ 

This suggests that I may be extended to $B$-valued Brownian motion using the methods of II. As it turns out, the resulting Theorem 1 of §4 contains both I and II as special cases, and is obtained in a self-contained manner independent of I and II.

2. Some properties of Brownian motion on $B$. Here we provide some basic lemmas. The content of Lemma 2 is found in [7] and can also be expressed in slightly different terms using [6].

**Lemma 2.** For $(B, \mu)$ as in II, let $\mathcal{L}$ be the closure of $B^*$ in $L_2(B, \mu)$. For each $L \in \mathcal{L}$ the convergent Bochner integral $x_L = \int_B L(x) x \mu(dx) \in B$ exists. $H$
\[ \{x_L : L \in \mathcal{L}\} \subseteq B \] is a real separable Hilbert space isometrically isomorphic to \( \mathcal{L} \) under the inner product \( \langle x_L, x_L' \rangle = \int_B L_1(x) L_2(x) \mu(dx) \). On \( H \), \( \|\cdot\|_B \leq \|\mu\| \|\cdot\|_H \) where \( \|\mu\|^2 = \int_B \|x\|^2 \mu(dx) < \infty \). If \( y^* \in B^* \) and \( y = \int_B y^*(x)x \mu(dx) \), then \( (y, x)_H = y^*(x) \) for every \( x \in H \). If \( \{x^*_j : j \geq 1\} \subseteq B^* \) is a complete orthonormal sequence for \( \mathcal{L} \) and \( \{x_j : j \geq 1\} \subseteq H \) is the set of images \( x_j = \int_B x_j^*(x)x \mu(dx) \) (\( j \geq 1 \)), then \( \sum_{j=1}^\infty x_j^*(x)x_j \to x \) as \( k \to \infty \), everywhere on \( H \) in the sense of the \( H \)-norm and almost everywhere on \( B \) in the \( B \)-norm. The closure \( \overline{H} \) of \( H \) in \( B \) is the topological support of \( \mu \) on \( B \) and if elements of \( B \) are interpreted as (evaluation) functions on \( B^* \), \( H \) may be interpreted as the reproducing kernel Hilbert space of \( \mu \).

**Proof.** That \( H \) is separable follows from [6] and the remainder is given in [7].

By [1, Theorems 2 and 3] and the fact that \( \|\cdot\|_B \leq M\|\mu\|_H \) we have that \( \|\cdot\|_B \) is a measurable norm in the sense of Gross [2] and hence \( (H,H) \) is an abstract Wiener space.

**Lemma 3.** Let \( B \) be a real separable Banach space with norm \( \|\cdot\|_B \). Let \( H \) be a subspace of \( B \) which is a real Hilbert space in the norm \( \|\cdot\|_H \) and assume \( \|\cdot\|_B \) is a measurable norm on \( H \). Let \( K \) denote the unit ball of \( H \), i.e. \( K = \{x \in H : \|x\|_H \leq 1\} \). Then \( K \) is a compact subset of \( B \).

**Proof.** First we show \( K \) is a closed subset of \( B \). Let \( \{y_n\} \subseteq K \) and assume \( \lim_n y_n = y \in B \) in the norm \( \|\cdot\|_B \). Now \( \{y_n\} \subseteq K \) implies there is a subsequence \( \{y_{n_k}\} \) such that \( \{y_{n_k}\} \) converges weakly in \( H \) to \( z \in H \). Now \( \|z\|_H \leq 1 \) by the uniform boundedness principle, and since \( \{y_n\} \) also converges to \( y \) in \( \|\cdot\|_B \) we have \( \{y_{n_k}\} \) converging weakly to \( y \) and to \( z \) in \( B \) because \( B^* \) can be viewed as a subset of \( H^* \). That is, since \( \|\cdot\|_B \) is a measurable norm on \( H \) we have a constant \( M \) such that \( \|\cdot\|_B \leq M\|\cdot\|_H \), and hence \( B^* \) can be continuously injected into \( H^* \). Now \( B^* \) separating points of \( B \) implies \( y = z \), and hence \( z \in K \) implies \( y \in K \). Hence \( K \) is closed in \( B \).

Now we show \( K \) is compact in \( B \). To do this we note that since \( \|\cdot\|_B \) is a measurable norm on \( H \) we can construct a second measurable norm on \( H \) as in [2], call it \( \|\cdot\|_\| \), such that for \( r > 0 \), \( V_r = \{x \in H : \|x\|_\| \leq r\} \) has a compact closure in \( B \). Now \( \|\cdot\|_\| \) measurable on \( H \) implies there exists an \( M > 0 \) such that \( \|x\|_\| \leq M\|x\|_H \) for all \( x \in H \), and hence \( K \subseteq \{x \in H : \|x\|_\| \leq M\} \). Thus \( K \) has compact closure in \( B \) and since \( K \) is closed we have \( K \) compact.

There are three separable Banach spaces, each with a mean zero Gaussian measure situated on its Borel subsets, which figure in our analysis. Of these, \( (C_B, \mu) \) and \( (B, \mu) \) have already been introduced. The third is \( (C, \nu) \) where \( C \) is the space of real-valued continuous maps on \([0,1]\) which vanish at zero (with the supremum norm) and \( \nu \) is Wiener measure. Each of \( (C_B, \mu), (C, \nu), (B, \mu) \) satisfies the hypotheses of Lemma 2. Let \( \mathcal{H} \subseteq C_B, H_0 \subseteq C, H \subseteq B \) denote the respective Hilbert spaces given for each of these spaces by Lemma 2, and let \( \mathcal{K}, K_0, K \) be the respective unit balls of these spaces. Then Lemma 3 applies to \( \mathcal{K}, K_0, K \).
Using Lemma 2 one may prove the following familiar characterization of 
\( H_0 : \phi \in H_0 \) iff \( \phi(0) = 0, \phi \) is absolutely continuous with respect to Lebesgue
measure on \([0,1]\) and \( \int_0^1 [(d/dt)\phi(t)]^2 \, dt < \infty. \) The inner product on \( H_0 \) is

\[
(\phi_1, \phi_2)_{H_0} = \int_0^1 \frac{d}{dt} \phi_1(t) \frac{d}{dt} \phi_2(t) \, dt.
\]

Our next result enables us to interpret \( \mathcal{H} \) as a denumerable direct sum of copies of \( H_0. \)

**Lemma 4.** \( \mathcal{H} \) has the following characterization in terms of any set \( \{x_j^* : j \geq 1\} \subseteq B^* \) such that \( \{x_j : j \geq 1\} \) is a complete orthonormal set for \( H; f \in \mathcal{H} \) iff \( f(0) = 0, f(t) \in H \) for each \( t \in [0,1], \) each \( x_j^*(f) \in H_0, \) and

\[
\sum_j \int_0^1 [(d/dt)x_j^*(f)(t)]^2 \, dt < \infty.
\]

The inner product on \( \mathcal{H} \) is given by

\[
(f_1, f_2)_{\mathcal{H}} = \sum_j \int_0^1 \frac{d}{dt} x_j^*(f_1(t)) \frac{d}{dt} x_j^*(f_2(t)) \, dt \quad \text{for } f_1, f_2 \in \mathcal{H}.
\]

In somewhat greater detail we have the following:

(a) \( \mathcal{H} = H_0 \otimes H \) (the tensor product).

(b) If \( x^* \in B^* \) and \( f \in \mathcal{H} \) then \( x^*f \in H_0 \) and, for every \( \phi \in H_0, \) \( (x^*f, \phi)_{H_0} = (f, \phi x^*)_{H_0} \) where \( (x^*f)(t) = x^*(f(t)), t \in [0,1], \frac{\|x^*f\|_{H_0}}{\|\phi\|_{H_0}} \leq \frac{\|x^*\|_H}{\|\phi\|_H}. \)

(c) If \( f \in \mathcal{H} \) and \( t \in [0,1] \) then \( f(t) \in H \) and, for every \( x^* \in B^*, \) \( (f(t), x)_{H} = (f, x(t))_{\mathcal{H}} \frac{\|x\|_{\mathcal{H}}}{\|f(t)\|_{H}} \leq \|f(t)\|_{H}. \)

(d) For \( \{x_j : j \geq 1\} \subseteq B^* \) and \( \{x_j : j \geq 1\} \subseteq H \) as above, \( \sum_{j=1}^\infty x_j^*(f)x_j \rightarrow f \) as \( k \rightarrow \infty \) everywhere on \( \mathcal{H} \) in the sense of \( \mathcal{H} \) norm and almost everywhere on \( C_B \) in the sense of \( C_B \) norm. That is, if \( P \) is the Gaussian measure induced on the Borel subsets of \( C_B \) by Brownian motion on \( B, \) then with \( P \)-probability one for \( f \in C_B \)

\[
\sup_{0 \leq t \leq 1} \left\| f(t) - \sum_{j=1}^k x_j^*(f(t))x_j \right\|_B \rightarrow 0 \quad \text{as } k \rightarrow \infty,
\]

and the law of \( x^*f(t) (j \geq 1) \) is that of mutually independent one dimensional Brownian motions normalized as usual.

**Proof.** For each \( t \in [0,1], \) \( x^* \in B^*, \) let \( \Lambda_{t,x^*}(f) = x^*(f(t)) \) for \( f \in C_B. \) Then \( \Lambda_{t,x^*} \in C_B \) and these functionals separate points of \( C_B \). To prove \( \mathcal{H} = H_0 \otimes H, \) suppose \( t \in [0,1], \) \( x^* \in B^*. \) We first show that the element of \( C_B \) defined by \( \min(t, \cdot)x \) is the Bochner integral \( \int_{C_B} \Lambda_{t,x^*}(f) fP(df) \) and therefore by Lemma 2 applied to \((C_B, P)\) we have \( \min(t, \cdot)x \in \mathcal{H}. \) We proceed by evaluation of the two expressions. If \( s \in [0,1], \) \( x^* \in B^*, \) then

(2) \( \Lambda_{t,x^*}(\min(t, \cdot)x) = \min(t,s)y^*(x) = \min(t,s)(y, x)_H \)
by Lemma 2. Now
\[
\Lambda_{t,x^*} \left( \int_{c_t} \Lambda_{t,x^*} (f) f P(df) \right) = \int_{c_t} x^* (f(t)) y^* (f(s)) P(df)
\]
(3)
\[
= \int_{c_t} x^* (f(\min(t,s))) y^* (f(\min(t,s))) P(df)
\]
by independence of increments, and using the stationarity of the increments of Brownian motion on B we have (3) equal to
\[
\min(t,s) \int_{c_t} x^* (f(1)) y^* (f(1)) P(df) = \min(t,s)(y,x)_H
\]
(4) since \( f \to f(1) \) induces the measure \( \mu = \mu_1 \) on B. Combining (2) and (4) we have
\[
\min(t, \cdot) x \in \mathcal{H}. \text{ From (3) and (4) we have the factorization}
\]
\[
\min(s, \cdot) y, \min(t, \cdot) x = \min(s,t)(x,y)_H
\]
(5)
\[
= (\min(s, \cdot), \min(t, \cdot))_{H_k}(x,y)_H.
\]
This proves \( \mathcal{H} = H_0 \otimes H \) provided the elements \{\min(t, \cdot)x: 0 \leq t \leq 1, x^* \in B^* \} can be shown to span \( \mathcal{H} \) (for a discussion of tensor products of reproducing kernel Hilbert spaces see [9]). To see this, suppose \( f_0 \in \mathcal{H} \) and \( (f_0, \min(t, \cdot) x)_H = 0 \) for all \( t \in [0,1] \), \( x^* \in B^* \). By Lemma 2 there is an element \( L_0 \) belonging to the closure of the subspace of \( L^2(C_B, P) \) spanned by \( C_B^* \) for which \( f_0 = \int_{c_B} L_0(f) f P(df) \). Then
\[
\Lambda_{t,x^*} (f_0) = \int_{c_t} L(f) \Lambda_{t,x^*} (f) P(df) = (f_0, \min(t, \cdot) x)_H = 0
\]
for all \( t \) and \( x^* \). Hence \( f_0 = 0 \) in \( C_B \) and in \( \mathcal{H} \). This completes the proof of \( \mathcal{H} = H_0 \otimes H \).

To prove (b) suppose \( x^* \in B^* \), \( f \in \mathcal{H}, \phi \in H_0 \). If \( f \) is of the special form
\[
f = \phi^*_x x \text{ with } \phi^*_x \in C^*, x^*_t \in B^*, \text{ then } x^* f = \phi^*_x x^*_t \in H_0 \text{ and } (f, \phi x)_H = (\phi^*_x x^*_t, \phi)_{H_k} = (x^*_t, (\phi^*_x \phi)_H)_{H_k} = (x^* (\phi^*_x x^*_t), \phi)_{H_k} = (x^* f, \phi)_{H_k}.
\]
Thus for every \( f \) expressible as a finite sum of elements of the form \( \phi^*_x x \) we have \( x^* f \in H_0 \) and \( (x^* f, \phi)_{H_k} = (f, \phi x) \). To extend this to all \( f \) in \( \mathcal{H} \) we need for \( f \) of the type of sum just considered the inequality \( \|x^* f\|_{H_k} \leq \|f\|_{\mathcal{H}} \cdot \|x\|_H \). To prove this note that if \( \{\phi^*_x: j \geq 1\} \subseteq C^* \) and \( \{\phi_j: j \geq 1\} \subseteq H_0 \) is complete and orthonormal for \( H_0 \) then for \( f \) of the above type,
\[
\|x^* f\|^2_{H_k} = \sum_j (x^* f, \phi_j)_{H_k}^2 = \sum_j (f, \phi_j x)^2 \leq \|f\|^2_{\mathcal{H}} \cdot \|x\|^2_H
\]
since \( \{\phi_j: j \geq 1\} \subseteq \mathcal{H} \) are orthogonal in the tensor product and each have norm squared equal to \( \|x\|^2_H \). If \( f \in \mathcal{H} \) then there exists \( f_n \in \mathcal{H} \) of the above type tending to \( f \) in \( \mathcal{H} \). Now \( \mathcal{H} \subseteq C_B \) and \( x^* \in B^* \) implies \( x^* (f_n(t)) \to x^* (f(t)) \) as \( n \to \infty \) since \( f_n \to f \) in \( \mathcal{H} \) implies \( f_n \to f \) in \( C_B \) by Lemma 2. Further, by the above inequality \( x^* f_n \) converges in \( H_0 \) as \( n \to \infty \). Combining the last two statements we have \( x^* f \in H_0 \) and \( x^* f_n \to x^* f \) in \( H_0 \). Finally,
The iterated logarithm for Brownian motion

\((x^*f, \phi)_{\mathcal{H}_0} = \lim_n (x^*f_n, \phi)_{\mathcal{H}_n} = \lim_n (f_n, x)_{\mathcal{H}} = (f, \phi x)_{\mathcal{H}}\)

and

\[\|x^*f\|_{\mathcal{H}_0}^2 = \lim_n \|x^*f_n\|_{\mathcal{H}_n}^2 \leq \lim_n \|f_n\|_{\mathcal{H}_n}^2 \|x\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 \|x\|_{\mathcal{H}}^2.\]

The proof of (c) is analogous to (b). Suppose \(f \in \mathcal{H}\) and \(t \in [0, 1]\). If \(f\) is of the form \(\phi_t x_t\) for some \(\phi_t^* \in C^*\), \(x_t^* \in B^*\) then \(f(t) = \phi_t(x_t) \in H\) and if \(x \in H\) then from Lemma 2 \((f(t), x)_H = x^*f(t) = (f, \min(t, \cdot)x)_{\mathcal{H}}\) from part (a).

If \(f\) is a finite sum of elements of the above type then \(f(t) \in H\), \((f(t), x)_H = (f, \min(t, \cdot)x)_{\mathcal{H}}\) and

\[\|f(t)\|_H^2 = \sum_j (f(t), x_j)_H^2 = \sum_j (f, \min(t, \cdot)x_j)_H^2 \leq \|f\|_{\mathcal{H}}^2 \|\min(t, \cdot)\|_H^2 = \|f\|_{\mathcal{H}}^2.\]

Suppose \(f_n\) are such finite sums and \(f_n \to f\) in \(\mathcal{H}\). Then \(f_n(t)\) converges to \(f(t)\) in \(B\) and \(x_n^*f_n(t) \to x^*f(t)\) for each \(x^* \in B^*\). By the previous inequality \(f_n(t)\) converges in \(H\), and hence \(f(t) \in H\) and \(f_n(t) \to f(t)\) in \(H\). By passage to the limit we get \((f(t), x)_H = (f, \min(t, \cdot)x)_{\mathcal{H}}\) and \(\|f(t)\|_H \leq \|f\|_{\mathcal{H}}\sqrt{t}\).

To prove (d) assume \(\{x^*_j: j \geq 1\} \subseteq B^*\) and \(\{x_j: j \geq 1\} \subseteq H\) is a complete orthonormal set for \(H\). Likewise suppose \(\{\phi^*_n: n \geq 1\} \subseteq C^*\) and \(\{\phi_n: n \geq 1\}\) \(\subseteq \mathcal{H}_0\) is complete and orthonormal for \(\mathcal{H}_0\). Then from (a) \(\{\phi_n, x_j: n \geq 1, j \geq 1\}\) \(\subseteq \mathcal{H}\) is complete and orthonormal for \(\mathcal{H}\). For arbitrary \(n \geq 1, j \geq 1\) we see that for every \(\phi \in \mathcal{H}_0, x \in H\), \(\phi^*_n x^*_j(\phi x) = (\phi_n, \phi)_{\mathcal{H}_0}(x_j, x)_H\) and hence \(\phi_n^* x^*_j\) yields \(\phi_n x_j\) by Bochner integration on \(C_B\). Then everywhere on \(\mathcal{H}\) and in \(\mathcal{H}\) norm we have \(f = \sum_n (f, \phi_n x_j)_{\mathcal{H}}\phi_n x_j = \sum_n \phi_n^* (x^*_j(f)) \phi_n x_j\). For each \(j \geq 1\) the series may be summed on \(n\) in \(\mathcal{H}\) obtaining

\[f = \sum_j \left(\sum_n (f, \phi_n x_j)_{\mathcal{H}}\phi_n x_j = \sum_j \left(\sum_n (x^*_j f, \phi_n)_{\mathcal{H}_0} \phi_n x_j = \sum_j x^*_j(f) x_j.\right)\right)\]

The argument may be repeated almost everywhere on \(C_B\) in \(C_B\) norm and hence \(f = \sum_j x^*_j(f) x_j\) almost everywhere on \(C_B\).

Using the explicit description of \(H_0\) given previous to the present lemma the characterization of \(\mathcal{H}\) with which we began the statement of the lemma follows easily from the above series representation.

Since \(P\) is a mean zero Gaussian measure on \(C_B\) it follows easily from (3) and (4) (since the joint distributions are all Gaussian) that \(x^*_j f(t) (j \geq 1)\) are independent one dimensional Brownian motions.

For every \(\varepsilon > 0\) let \(\mathcal{K}_\varepsilon\) denote the open \(\varepsilon\)-neighborhood of \(\mathcal{K}\) in \(C_B\).

**Lemma 5.** For each \(\varepsilon > 0\), there exists \(r > 1\) such that

\[P\{f \in C_B: f/\sqrt{2 \log \log s} \notin \mathcal{K}_\varepsilon\} \leq \exp(-r^2 \log \log s)\]

for all sufficiently large \(s\).
Proof. This result can be proved just as Proposition 1 of [8] is obtained.

**Lemma 6.** If \( \epsilon > 0 \) one may choose \( c > 1 \) sufficiently close to one so that for every \( f \in \Omega_n \) the statements \( c^n \leq s \leq c^{n+1} \) and \( f((c^{n+1}) \cdot)/(2[c^{n+1}]\log \log [c^{n+1}])^{1/2} \in K_n \) together imply \( f(s \cdot)/\sqrt{2} s \log s \in K_n \) for all sufficiently large \( n \).

**Proof.** Suppose \( \epsilon > 0 \) and choose \( c > 1 \) so that for all sufficiently large \( n \),

\[
\gamma_n \epsilon + (\gamma_n - 1)\|P\| < 2\epsilon
\]

where

\[
\gamma_n = \left( \frac{[c^{n+1}]\log \log [c^{n+1}]}{[c^n]\log \log [c^n]} \right)^{1/2}.
\]

This is possible because \([c^{n+1}] < c^2[c^n]\) for all large \( n \). If \( h \in K, f \in C_B \),

\[
\|f((c^{n+1}) \cdot)/(2[c^{n+1}]\log \log [c^{n+1}])^{1/2} - h(\cdot)\|_{C_B} < \epsilon,
\]

then

\[
\left\| f(s \cdot)/(2[s \log \log s])^{1/2} - h\left( \frac{s \cdot}{c^{n+1}} \right) \right\|_{C_B} < \epsilon
\]

and

\[
\left\| h\left( \frac{s \cdot}{c^{n+1}} \right) \right\|_{C_B} \leq \frac{s \cdot}{c^{n+1}} \|h\|_{C_B} \leq 1.
\]

Hence \( h((s/[c^{n+1}]) \cdot) \in K \) and

\[
\left\| f(s \cdot)/(2[s \log \log s])^{1/2} - h\left( \frac{s \cdot}{c^{n+1}} \right) \right\|_{C_B} \leq \left\| f(s \cdot)/(2[s \log \log s])^{1/2} - \left( \frac{[c^{n+1}]\log \log [c^{n+1}]}{s \log s} \right)^{1/2} - h\left( \frac{s \cdot}{c^{n+1}} \right) \right\|_{C_B} + \|P\| \left\| \left( \frac{[c^{n+1}]\log \log [c^{n+1}]}{s \log s} \right)^{1/2} - h\left( \frac{s \cdot}{c^{n+1}} \right) \right\|_{C_B}
\]

\[
\leq \gamma_n \epsilon + (\gamma_n - 1)\|P\| < 2\epsilon
\]

if \( n \) is sufficiently large.

For what follows we assume \( \{x_j^* : j \geq 1\} \subseteq B^* \) and \( \{x_j : j \geq 1\} \subseteq H \) is complete and orthonormal for \( H \). For each \( k \geq 1 \) and \( f \in C_B \) let \( f^{(k)} = \sum_{j=1}^k x_j^*(f) x_j \).

**Lemma 7.** For each \( \epsilon > 0 \) and \( r > 1 \) there exists \( k \) sufficiently large so that

\[
P\left( f \in C_B : \|f - f^{(k)}\|_{C_B} \geq \epsilon \sqrt{2 \log \log s} \right) \leq \exp(-r^2 \log \log s)
\]

for all sufficiently large \( s \).
Proof. By (d) of Lemma 4 this result follows just as in Lemma 4 of [8].

The main theorem and some corollaries. Our basic theorem is the following and implies the result of Strassen mentioned in I and that of LePage in II.

**Theorem 1.** Let \( \{W(t): 0 \leq t < \infty\} \) be Brownian motion on \( B \) and for each \( t \in [0,1] \), \( s \geq 3 \), let

\[
\xi_s(t) = W(st) / \sqrt{2s \log \log s}.
\]

Then the net \( \{\xi_s: s \geq 3\} \) is a subset of \( C_B \) and with probability one converges in \( C_B \) to the compact set \( K \) and clusters at every point of \( K \), where \( K \) is the unit ball of the reproducing kernel Hilbert space (equivalently, \( K \) is the unit ball of the Hilbert subspace of \( C_B \) which generates \( P \)).

Proof. That \( K \) is compact in \( C_B \) follows from Lemma 3 by applying the lemma to \( C_B, A, \) and \( \|\cdot\|_{C_B} \). For every \( \epsilon > 0 \), there exists \( r > 1 \) such that

\[
\Pr(\xi_s \not\in K) = P(f \in C_B: \|f - K\|_{C_B} \geq \epsilon \sqrt{2 \log \log s}) \leq \exp(-r^2 \log \log s)
\]

for all sufficiently large \( s \) by Lemma 5. Hence by the Borel-Cantelli lemma for \( c > 1 \) there is a set \( A \) of probability one such that the sequence \( \xi_{s_n} \in K_s \) for all but finitely many \( n \). Therefore by Lemma 6 \( \xi_s \in K_{s_n} \) for all \( s \) sufficiently large on the set \( A \). Letting \( \epsilon \) converge to zero through a countable set we have

\[
\Pr(\xi_s \rightarrow K \text{ as } s \rightarrow \infty \text{ in } C_B) = 1.
\]

To prove \( K \) is almost surely the set of cluster points of \( \{\xi_s: s \geq 3\} \) it suffices by the separability of \( K \) to prove that if \( h \in K, \|h\|_{C_B} < 1 \) and \( \epsilon > 0 \) there is a \( c > 1 \) so that with probability one \( \|\xi_{s_n} - h\|_{C_B} < \epsilon \) for infinitely many \( n \). By Lemma 7 choose \( r > 1 \) and \( k \) sufficiently large so that (5) holds with \( \epsilon \) replaced by \( \epsilon / 3 \) for all sufficiently large \( s \). By Lemma 4(d) choose \( k \) large enough so that

\[
\|h - h^{(k)}\|_{C_B} < \epsilon / 3.
\]

Then for every \( c > 1 \), applying these estimates and the Borel-Cantelli lemma, we have with probability one that

\[
\|\xi_{s_n} - h\|_{C_B} \leq 2\epsilon / 3 + \|\xi_{s_n}^{(k)} - h^{(k)}\|_{C_B} \leq 2\epsilon / 3 + \|P\| \sup_{0 \leq t \leq 1} \|\xi_{s_n}^{(k)}(t) - h^{(k)}(t)\|_H
\]

for all sufficiently large \( n \).

It now suffices to show that with probability one

\[
\sup_{0 \leq t \leq 1} \|\xi_{s_n}^{(k)}(t) - h^{(k)}(t)\|_H < \epsilon / 3 \|P\|
\]

for infinitely many \( n \). Our argument follows an idea due to Strassen.
Let \( m \geq 2 \) be an integer, \( 0 < \delta < 1 \), and assume \( Z_{jk}\) and \( h_j \) \((j = 1, \ldots, k)\) are the \( j \)-th-coordinates of \( \zeta^{(k)} \) and \( h^{(k)} \). We define the event

\[
A_n = \{w: |(Z_{jk}(w, i/m) - Z_{jk}(w, i - 1/m)) - (h_j(i/m) - h_j(i - 1/m))| < \delta/k \ \text{for} \ i = 2, \ldots, m \ \text{and} \ j = 1, \ldots, k\}.
\]

Then

\[
\Pr(A_n) \geq \prod_{i=2}^{m} \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{b_y}^{a_y} e^{-s^2/2} ds
\]

where (letting \( LL \) denote \( \log \log \))

\[
a_y = |h_j(i/m) - h_j(i - 1/m)|\sqrt{2mLL[c^\theta]},
\]

\[
b_y = \sqrt{(|h_j(i/m) - h_j(i - 1/m)| + \delta/k)|\sqrt{2mLL[c^\theta]}},
\]

for \( i = 2, \ldots, m; j = 1, \ldots, k \). Using the estimate

\[
\int_a^b \exp(-s^2/2) ds \geq \frac{\exp(-a^2/2)}{b} (1 - \exp(-(b^2 - a^2))/2) \quad \text{for} \ 0 \leq a < b
\]

we have a constant \( \gamma > 0 \) such that

\[
\Pr(A_n) \geq \gamma \prod_{i=2}^{m} \prod_{j=1}^{k} \frac{\exp(-a_y^2/2)}{b_y}
\]

for all \( n \) sufficiently large (because \( 0 \leq a_y < b_y \) implies \( b_y - a_y \geq (b_y - a_y)^2 \geq (\delta^2/k^2)2mLL[c^\theta] \)). Hence there is a constant \( \gamma_1 > 0 \),

\[
\Pr(A_n) \geq \gamma_1 \exp\left\{\sum_{i=2}^{m} \sum_{j=1}^{k} (h_j(i/m) - h_j(i - 1/m))^2 mLL[c^\theta]\right\}
\]

\[
\geq \gamma_1 \exp\left\{-\|h^k\|_{\theta}^2 \cdot LL[c^\theta]\right\},
\]

and since \( \theta = \|h^k\|_{\theta} < 1 \) we have

\[
\Pr(A_n) \geq \frac{\gamma_1}{(\log[c^\theta]\sqrt{(2mLL[c^\theta])}^{mk/2}}.
\]

Hence for \( c = m \) we have \( A_1, A_2, \ldots \) independent and

\[
\Pr(A_n) \geq \frac{\gamma_1}{(n \log m)^{\theta}(2m(\log n + LLm))^{mk/2}} \geq \frac{\gamma_2}{n^{\theta}(\log n)^{mk/2}}
\]

for all \( n \) sufficiently large.

Now \( \theta < 1 \) implies \( \sum_{n=1}^{\infty} \Pr(A_n) = \infty \) so by Borel-Cantelli \( \Pr(\lim sup_n A_n) = 1. \)
Using the fact that the H-norm and the B-norm are equivalent on the finite dimensional subspace of B generated by \( \{x_1, \ldots, x_k\} \) we have by the first part of the proof that with probability one \( \xi_n^{(k)}(i) \) is eventually within \( \delta \) of \( \mathcal{K}^{(k)} \). Here \( \mathcal{K}^{(k)} \) is the subset of \( \mathcal{K} \) consisting of functions of the form \( \sum_{j=1}^{k} x_j^g(w(t))x_j \).

Hence with probability one

\[
\sup_{n \geq n_0} \|\xi_n^{(k)}(i) - \xi_n^{(k)}(j)\|_H \leq \sqrt{|t-s|} + \delta
\]

for all \( 0 \leq s, t \leq 1 \) and all \( n \) sufficiently large. Now if \( y \in C_B[0,1] \) and \( y \) satisfies

(a) \( \|y^{(k)}(i) - y^{(k)}(s)\|_H \leq \sqrt{|t-s|} + \delta \) \( (0 \leq s, t \leq 1) \),

(b) \( |y_j(i/m) - y_j((i - 1)/m)) - (h_j(i/m) - h_j((i - 1)/m))| < \delta/k \)

for all \( j = 1, \ldots, k, 2 \leq i < m \) where \( y_j \) is the \( j \)th coordinate of \( y \), then

\[
\sup_{0 \leq s \leq 1} \|y^{(k)}(i) - h^*(i)\|_H < \epsilon/3\|P\|
\]

provided \( m \) is sufficiently large and \( \delta \) is sufficiently small. Using the definition of \( A_n \) and (7) we see that with probability one (6) holds for infinitely many \( n \). This concludes the proof.

The next corollary follows immediately from Theorem 1.

**Corollary 1.** Let \( \theta \) be a continuous function on \( C_B \) into a Hausdorff topological space \( Y \) and assume the notation of Theorem 1. Then with probability one \( \{\theta \circ \xi_n: s \geq 3\} \) converges to the compact set \( \theta(\mathcal{K}) \) and clusters at each point of \( \theta(\mathcal{K}) \).

**Corollary 2.** If \( \{W(t): 0 \leq t < \infty\} \) is Brownian motion on \( B \), then

\[
\Pr \left( \lim_{s \to \infty} \frac{\|W(s)\|_B}{\sqrt{2s \log \log s}} = \sup_{x \in \mathcal{K}} \|x\|_B \right) = 1.
\]

**Proof.** Since \( \|W(s)\|_B/\sqrt{2s \log \log s} = \|\xi_n(1)\|_B \) this result follows from Corollary 1 with \( \theta(f) = \|f(1)\|_B \) and by showing that \( \sup_{f \in \mathcal{K}} \|f(1)\|_B = \sup_{x \in \mathcal{K}} \|x\|_B \).

Now if \( f \in \mathcal{K} \), then by Lemma 4(c) \( \|f(1)\|_H \leq \|f\|_H \leq 1 \) and hence \( f(1) \in \mathcal{K} \subseteq H \). On the other hand, if \( x \in \mathcal{K} \) we can set \( f(i) = tx \) and then \( \|f\|_\mathcal{K} = (x,x)_H \leq 1 \). Hence \( f(i) = tx \in \mathcal{K} \) and \( \theta(f) = \|x\|_B \). By combining the above we have

\[
\sup_{f \in \mathcal{K}} \|f(1)\|_B = \sup_{x \in \mathcal{K}} \|x\|_B,
\]

and the proof is complete.

For the following recall the statements I and II of the introduction.

**Corollary 3.** I holds.

**Proof.** If \( B = \mathbb{R}^k \) then \( H = B, \mathcal{K} = K_k \) and the result in I follows immediately.
Corollary 4. II holds.

Proof. Construct Brownian motion in $B$, call it $\{W(t): 0 \leq t < \infty\}$, such that $\mu_t = \mu$. Let $\theta(f) = f(1)$ for $f \in C_B$. Using the stationary independent increments of $\{W(t)\}$ it follows that the joint distributions of $\{\xi_n: n \geq 3\}$ are identical to those of $\{\theta(\xi_n): n \geq 3\}$ where $\xi_n$ is as in Theorem 1. Hence with probability one $\{\xi_n: n \geq 3\}$ converges to the set $\theta(\mathcal{K})$ and clusters at each point of $\theta(\mathcal{K})$ by Corollary 1. Now by the argument given in Corollary 2 $\theta(\mathcal{K}) = K$ and hence the proof is complete.

Remark. In view of Lemma 3 it follows that $K$ is compact in $B$ and hence Corollary 4 actually implies the sequence $\{\xi_n: n \geq 3\}$ is relatively norm compact with probability one and that its limit points consist precisely of $K$ (with probability one). This is slightly stronger than II. Finally II generalizes the law of the iterated logarithm of [3] to Gaussian random variables in $B$.

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