DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS(1)

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ABSTRACT. Let $L_0$ be a degenerate second order elliptic operator with no zeroth order term in an $m$-dimensional domain $G$, and let $L = L_0 + c$. One divides the boundary of $G$ into disjoint sets $\Sigma_1, \Sigma_2, \Sigma_3$; $\Sigma_3$ is the noncharacteristic part, and on $\Sigma_2$ the “drift” is outward. When $c$ is negative, the following Dirichlet problem has been considered in the literature: $Lu = 0$ in $G$, $u$ is prescribed on $\Sigma_2 \cup \Sigma_3$. In the present work it is assume that $c < 0$. Assuming additional boundary conditions on a certain finite number of points of $\Sigma_1$, a unique solution of the Dirichlet problem is established.

Introduction. Consider the second order degenerate elliptic operator with smooth coefficients

$$Lu = \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i}$$

in a smoothly bounded domain $G$ in $\mathbb{R}^m$. The Dirichlet problem for the equation $Lu + c(x)u = 0$ in $G$ has been treated by many authors (see [5] and the references contained therein). In all of these approaches, the boundary $\partial G$ is decomposed as follows:

$$\Sigma_3 = \left\{ x \in \partial G; \sum_{i,j=1}^{m} a_{ij}(x) \nu_i \nu_j > 0 \right\} \quad (\nu_i = \text{outward normal}),$$

$$\Sigma_2 = \left\{ x \in \partial G \setminus \Sigma_3; \sum_{i=1}^{m} \left( b_i(x) - \frac{1}{2} \sum_{j=1}^{m} \frac{\partial a_{ij}}{\partial x_j}(x) \nu_i \right) \nu_i > 0 \right\},$$

$$\Sigma_1 = \left\{ x \in \partial G \setminus \Sigma_3; \sum_{i=1}^{m} \left( b_i(x) - \frac{1}{2} \sum_{j=1}^{m} \frac{\partial a_{ij}}{\partial x_j}(x) \nu_i \right) \nu_i \leq 0 \right\}.$$

A typical result of these theories asserts that the equation $Lu + c(x)u = 0$ has a unique solution in some function space when data are prescribed on...
The points of \( \Sigma_1 \) at which we must assign data are precisely the points to which the associated Markov process converges when \( t \to \infty \). These "distinguished boundary points" are defined in terms of the normal and tangential behavior of the diffusion process, in contrast to \( \Sigma_2 \cup \Sigma_3 \) which only depends on the normal components of diffusion and drift. For technical reasons we will only consider cases where there exist a finite number of distinguished boundary points, together with an arbitrary configuration of \( \Sigma_2 \cup \Sigma_3 \). We denote by \( \Sigma^-_1 \) the component of \( \partial G \) containing all of the distinguished boundary points.

In \( \S \S 2-5 \) we consider the case \( m = 2 \), and in \( \S 6 \) we consider the case \( m \geq 2 \). \( \S \S 1 \) and 2 contain preliminary results on the boundary behavior of solutions \( x(t) \) of the stochastic equations

\[
dx_i = \sum_{r=1}^{n} \sigma_{ir}(x) dw^r + b_i(x) dt \quad (1 \leq i \leq 2)
\]

in a special domain in the plane. For technical reasons we assume that when \( \sigma \) and \( b \) vanish simultaneously on \( \Sigma^-_1 \), they do not vanish faster than a linear function. In \( \S 3 \) we consider a general domain in the plane and show that either \( x(t) \) attains \( \Sigma_2 \cup \Sigma_3 \) in finite time or else converges to some distinguished boundary point while remaining inside \( G \) for all \( t < \infty \). In \( \S 4 \) we prove the differentiability (as a function of the starting point) of the probability that the process will converge to a given distinguished boundary point. Finally in \( \S 5 \) we consider the Dirichlet problem in a general domain in the plane, combining the results of the previous sections with known results [6] on the behavior of the diffusion process near \( \Sigma_2 \cup \Sigma_3 \).

The results of \( \S \S 2-5 \) can be extended to \( m \geq 3 \); this is briefly discussed in \( \S 6 \). However, the main result of \( \S 6 \) is a theorem which even for \( m = 2 \) is not included in \( \S \S 2-5 \).

1. Boundary behavior of stochastic solutions in annular domains. Consider a system of two stochastic differential equations

\[
dx_i = \sum_{s=1}^{n} \sigma_{is}(x) dw^s + b_i(x) dt \quad (i = 1, 2)
\]
where \(w^1(t), \ldots, w^m(t)\) are independent Brownian motions. Let \(a = (a_{ij}) = \sigma \sigma^*\) where \(\sigma = (\sigma_{is}), \sigma^* = \text{transpose of } \sigma\). We assume

(A) The functions \(\sigma_{is}(x), b_i(x)\) and their first two derivatives are continuous and bounded in \(\mathbb{R}^2\).

Let \(G\) be a bounded domain in \(\mathbb{R}^2\). For simplicity we first take

\[G = \{x; 1 < |x| < 2\}.
\]

Denote by \(\partial G\) the boundary of \(G\). We shall assume

(B) On \(\partial G\),

\[
(1.2) \quad 2 \sum_{i,j=1}^2 a_{ij} \nu_i \nu_j = 0,
\]

\[
(1.3) \quad 2 \sum_{i=1}^2 \left[ b_i \frac{1}{2} \sum_{j=1}^2 \frac{\partial a_{ij}}{\partial x_i} \nu_i \leq 0
\]

where \(\nu\) is the outward normal to \(\partial G\) (with respect to \(G\)).

Let \(R(x)\) be a positive \(C^2\) function in \(G\), which coincides with \(\text{dist}(x, \partial G)\) when the latter is sufficiently small. Let

\[
Q = \sum_{i,j=1}^2 a_{ij} \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j},
\]

\[
B = \sum_{i=1}^2 b_i \frac{\partial R}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^2 a_{ij} \frac{\partial^2 R}{\partial x_i \partial x_j},
\]

\[
Q = \frac{1}{R} \left( B - \frac{Q}{2R} \right).
\]

(C) For some \(\mu > 0\) sufficiently small,

\[
Q(x) \leq -\theta_0 < 0 \quad \text{if} \quad 1 < |x| \leq 1 + \mu \quad (\theta_0 \text{ constant}),
\]

\[
Q(x) \geq \theta_0 > 0 \quad \text{if} \quad 2 - \mu < |x| < 2,
\]

\[
Q(x) > 0 \quad \text{if} \quad 1 + \mu \leq |x| \leq 2 - \mu, \nabla R(x) \neq 0,
\]

\[
\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \quad \text{if} \quad 1 + \mu \leq |x| \leq 2 - \mu, \nabla R(x) = 0.
\]

By Theorem 1.1 of [4] and by slightly modifying the proof of Theorem 2.1 of [4] we get

Lemma 1.1. If (A)–(C) hold then, for any solution \(x(t)\) of (1.1) with \(x(0) \in G\),

\[
P|x(t)\in G \text{ for all } t > 0| = 1, \quad P|x(t)| \to 1 \text{ as } t \to \infty = 1.
\]

It is actually sufficient to assume that \(\sigma_{ij}\) are continuously differentiable in \(\mathbb{R}^2\) and twice continuously differentiable in a neighborhood of \(\partial G\).

We shall now analyze the limit set of \(x(t)\) on \(|x| = 1\) (as \(t \to \infty\)). For this,
we introduce polar coordinates \((r, \phi)\) as in [4]. We find that

\[
\begin{align*}
\frac{dr}{ds} &= \sum_{s=1}^{n} \mathcal{G}_s(r, \phi) dw^s + \mathcal{B}(r, \phi) dt, \\
\frac{d\phi}{ds} &= \sum_{s=1}^{n} \mathcal{G}_s(r, \phi) dw^s + \mathcal{B}(r, \phi) dt
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{G}_s(r, \phi) &= \sigma_{1s} \cos \phi + \sigma_{2s} \sin \phi, \\
\mathcal{B}(r, \phi) &= b_1 \cos \phi + b_2 \sin \phi + \frac{1}{2r} (a(x) \lambda_1, \lambda_1), \\
\mathcal{G}_s'(r, \phi) &= -\frac{\sin \phi}{r} \sigma_{1s} + \frac{\cos \phi}{r} \sigma_{2s}, \\
b'(r, \phi) &= -\frac{\sin \phi}{r} b_1 + \frac{\cos \phi}{r} b_2 - \frac{1}{2r} (a(x) \lambda_1, \lambda_1);
\end{align*}
\]

here

\[
\lambda = (\cos \phi, \sin \phi), \quad \lambda_1 = (-\sin \phi, \cos \phi)
\]

and

\[
(a(x)_{m, n}) = \sum_{s=1}^{n} a_{ij}(x)_{m, n}, \quad (m = (m_1, m_2), \quad n = (n_1, n_2)).
\]

Thus, if \((r(t), \phi(t))\) is a solution of (1.4) and if we define \(x_1(t) = r(t) \cos \phi(t), \quad x_2(t) = r(t) \sin \phi(t)\), then \(x(t) = (x_1(t), x_2(t))\) is a solution of (1.1).

The system (1.4) can also be written in the form (see [4])

\[
\begin{align*}
\frac{dr}{ds} &= r \left[ \sum_{s=1}^{n} \mathcal{G}_s'(r, \phi) dw^s + \mathcal{B}(r, \phi) dt \right] + \sum_{s=1}^{n} R_s dw^s + R_0 dt, \\
\frac{d\phi}{ds} &= \sum_{s=1}^{n} \mathcal{G}_s'(r, \phi) dw^s + \mathcal{B}(r, \phi) dt + \sum_{s=1}^{n} \Theta_s dw^s + \Theta_0 dt
\end{align*}
\]

where \(R_s = \sigma(r), \quad \Theta_s = \sigma(1) (0 \leq s \leq n)\) as \(r \to 1\), uniformly with respect to \(\phi\). It is useful to compare \(\phi(t)\) with the solution of the single equation

\[
\frac{d\phi}{ds} = \sigma(\phi) dw^s + b(\phi) dt
\]

where \(\sigma(\phi) = \frac{1}{\sum_{s=1}^{n} \left[ \mathcal{G}_s(\phi) \right]^2} \), \(b(\phi) = b(\phi)\).

In case \(\sigma(\phi) \neq 0\) for all \(\phi\) and \(\int_0^{2\pi} b(x)/\sigma^2(x) dx \neq 0\), it was proved in [4] that the algebraic angle \(\phi(t)\) (i.e., the component \(\phi(t)\) of the solution \((r(t), \phi(t))\) of (1.4)) satisfies

\[
P\left\{ \lim_{t \to \infty} \frac{\phi(t)}{t} = c \right\} = 1 \quad (c \text{ constant} \neq 0).
\]

Suppose \(\sigma(x)\) is degenerate, but it has only a finite number of zeros \(a_1, \ldots, a_k\) \((k \geq 1)\) in the interval \([0, 2\pi]\). Then the conclusion (1.7) is still valid [4] provided \(b(a_j) > 0\) for all \(i\), or \(b(a_j) < 0\) for all \(i\), and provided the following condition holds:

(i) For some \(r > 0\),

\[
\sum_{s=1}^{n} \left[ \mathcal{G}_s(r, \phi) \right]^2 = \sum_{s=1}^{n} \left[ \mathcal{G}_s(\phi) \right]^2 [1 + \eta(r, \phi)] \quad (1 \leq r \leq 1 + \varepsilon)
\]

where \(\eta(r, \phi) \to 0\) if \(r \to 1\), uniformly with respect to \(\phi\).

We shall now consider the degenerate case in situations where the \(b(a_j)\) may
vanish. The condition (i) will not be assumed in the sequel.

Our basic assumptions are:

(D) If \( b(\alpha_j) = 0 \) for some \( j \) (\( 1 < j < k \)) then there is a simple \( C^3 \) curve \( \Delta_{\alpha_j} \) given by \( r = r^*(t), \phi = \phi^*(t) \) \( t_1 \leq t \leq t_2 \) such that \( (r^*(t_1), \phi^*(t_1)) = (1, \alpha_j), \)
\( (r^*(t_2), \phi^*(t_2)) \) lies outside \( \bar{G} \), and \( (r^*(\tau), \phi^*(\tau)) \), for some \( t_1 < \tau < t_2 \), is a point on \( \partial G \) different from \( (1, \alpha_j) \), and such that
(i) a part \( \{(r^*(t), \phi^*(t)); t_1 \leq t \leq t_1 + \epsilon_1 \} \) coincides with the segment \( 1 \leq r \leq 1 + \tau, \phi = \alpha_j \), and
(ii) the following relations hold at each point of \( \Delta_{\alpha_j} \):

\[
(1.8) \quad \sum_{i,i=1}^{2} a_{ij} \nu_i \nu_j = 0, \quad \sum_{i=1}^{2} b_i - \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial a_{ij}}{\partial x_j} \nu_i = 0
\]

where \( (\nu_1, \nu_2) \) is the normal to \( \Delta_{\alpha_j} \).

Finally, if \( b(\alpha_j) = b(\alpha_{j'}) = 0 \) and \( b(\alpha_j) \neq 0 \) for all \( \alpha_j \) between \( \alpha_j \) and \( \alpha_{j'} \),
then the points \( (r, \phi) \) with \( r = 1 + \tau, \phi \) in the interval \( (\alpha_j, \alpha_{j'}) \) cannot be connected (in \( G \)) to points \( (r, \phi) \) with \( r = 1 + \tau, \phi \) outside the interval \( (\alpha_j, \alpha_{j'}) \),
without crossing either \( \Delta_{\alpha_j} \) or \( \Delta_{\alpha_{j'}} \).

Note that the conditions in (1.8) along the ray \( 1 \leq r \leq 1 + \epsilon, \phi = \alpha_j \) hold if
and only if

\[
\mathbf{G}_s(r, \alpha_j) = 0 \quad (1 \leq s \leq n), \quad \mathbf{b}(r, \alpha_j) = 0 \quad \text{for} \quad 1 \leq r \leq 1 + \tau.
\]

(E) If \( b(\alpha_j) = 0 \) for some \( j \) (\( 1 < j < k \)), then \( b(z), \sigma(z) \) vanish at \( z = \alpha_j \), to the first order only, and

\[
Q_{\alpha_j} = \lim_{z \to \alpha_j} \frac{2(z - \alpha_j) b(z)}{\sigma^2(z)} \neq 1.
\]

Note that the limit exists since \( b(z), \sigma(z) \) are continuously differentiable.

We first consider the case where \( b(z) \) vanishes at two consecutive points, say \( \alpha = \alpha_{j'}, \beta = \alpha_{j'+1} \). Introduce straight segments

\[
l_{\alpha} = \{(r, \alpha); 1 \leq r \leq 1 + \tau\}, \quad l_{\beta} = \{(r, \beta); 1 \leq r \leq 1 + \tau\}.
\]

Denote by \( m_{\eta} \) \( (\eta \geq 0) \) the curve \( \{(r, \phi); r = 1 + \eta, \alpha \leq \phi \leq \beta\} \), and by \( \Omega_{\eta} \) \( (\eta > 0) \) the domain bounded by \( m_0, m_\eta, l_{\alpha}, l_{\beta} \). Denote by \( A_{\alpha\beta} \) the set of points in the probability space for which \( \alpha \leq \phi(t) \leq \beta \) for a sequence of \( t \)'s converging to \( \infty \). By (D), and the proof of Theorem 1.1 (or rather Theorem 1.1) of [4], if \( x(0) \not\in (\Delta_\alpha \cup \Delta_\beta) \) then the solution \( (r(t), \phi(t)) \) never intersects \( \Delta_\alpha \cup \Delta_\beta \). It follows (by the last part of (D)) that on the set \( A_{\alpha\beta} \)

\[
(1.9) \quad \alpha < \phi(t) < \beta
\]
for all \( t \) sufficiently large.
Lemma 1.2. Let (A)-(E) hold and let \( x(0) \not\in (\Delta_\alpha \cup \Delta_\beta) \). If \( Q_\alpha > 1, Q_\beta < 1 \) then, a.s. on the set \( A_{\alpha, \beta}, \phi(t) \to \beta \) if \( t \to \infty \).

Proof. Consider the function

\[
g(x) = \int_{a^*}^{x} \frac{1}{\beta(y)} \left[ \int_{y}^{a^*} \frac{2\beta(z)}{\sigma^2(z)} \, dz \right] \, dy \quad \text{in} \quad (\alpha + \epsilon_0, \beta - \epsilon_0)
\]

for some small \( \epsilon_0 > 0 \), where

\[
\beta(y) = \exp \left\{ \int_{a^*}^{y} \frac{2\beta(z)}{\sigma^2(z)} \, dz \right\}
\]

It is easily seen that

\[
\hat{L}g = \frac{1}{2} \sigma^2(x)g''(x) + b(x)g'(x) = -1 \quad \text{in} \quad (\alpha + \epsilon_0, \beta - \epsilon_0).
\]

Further,

\[
g'(x) > 0 \quad \text{at} \quad x = \alpha + \epsilon_0,
\]

\[
g'(x) < 0 \quad \text{at} \quad x = \beta - \epsilon_0
\]

if \( \alpha + \epsilon_0 < a^* < \beta - \epsilon_0 \);

\[
g'(x) > 0 \quad \text{at} \quad x = \alpha + \epsilon_0, \quad x = \beta - \epsilon_0
\]

if \( \alpha < a^* < \alpha + \epsilon_0 \);

\[
g'(x) < 0 \quad \text{at} \quad x = \alpha + \epsilon_0, \quad x = \beta - \epsilon_0
\]

if \( \beta - \epsilon_0 < a^* < \beta \).

Set

\[
f(x) = \begin{cases} 
  -A_1 \log(x - \alpha) + B_1 & \text{in} \quad (\alpha, \alpha + \epsilon_0), \\
  A_2 \log(\beta - x) + B_2 & \text{in} \quad (\beta - \epsilon_0, \beta), \\
  g(x) & \text{in} \quad [\alpha + \epsilon_0, \beta - \epsilon_0].
\end{cases}
\]

If \( A_1 > 0, A_2 > 0 \) then, by using the assumptions \( Q_\alpha > 1, Q_\beta < 1 \) we find that \( \hat{L}f(x) \leq -\nu < 0 \) in \( (\alpha, \alpha + \epsilon_0), (\beta - \epsilon_0, \beta) \) where \( \nu \) is a positive constant, provided \( \epsilon_0 \) is sufficiently small. Choose \( a^* \) so that (1.13) holds, and determine the constants \( A_1, B_1 \) in such a way that \( f(x) \) is continuously differentiable at \( x = \alpha + \epsilon_0, x = \beta - \epsilon_0 \). We then find that \( A_1 > 0, A_2 > 0 \).

Let \( \{F_m\} \) be a sequence of continuous functions which approximate \( f'' \) in the following manner:

\[
F_m(x) = f''(x) \quad \text{if} \quad |x - (\alpha + \epsilon_0)| > 1/m, \quad |x - (\beta - \epsilon_0)| > 1/m,
\]

and \( F_m(x) \) connects linearly \( f''(\alpha + \epsilon_0 - 1/m) \) to \( f''(\alpha + \epsilon_0 + 1/m) \) and \( f''(\beta - \epsilon_0 - 1/m) \) to \( f''(\beta - \epsilon_0 + 1/m) \). Let
$f_m(x) = f(x^*) + f'(x^*)(x - x^*) + \int_{x^*}^{x} \int_{x^*}^{y} F_m(z) \, dz \, dy.$

Then $f_m(x) - f(x) \to 0$, $f'_m(x) - f'(x) \to 0$ uniformly in the interval $(\alpha, \beta)$, and $f''(x) - f''(x) = 0$ outside the intervals with centers $\alpha + \epsilon_0$, $\beta - \epsilon_0$ and length $2/m$.

Denote by $f^\delta_m(x)$, $f^\delta(x)$ any $C^2$ $(2\pi)$-periodic functions of $x \in R^1$ which coincide, respectively, with $f_m(x)$ and $f(x)$ in $(\alpha + \delta, \beta - \delta)$; $\delta$ is any positive number smaller than $\epsilon_0$. Denote by $R(r)$ any $C^2$ function satisfying

$$R(r) = 1 \quad \text{if} \quad 1 < r < 1 + \eta_0, \quad R(r) = 0 \quad \text{if} \quad 1 + \eta_1 < r < \infty,$$

where $0 < \eta_0 < \eta_1$ and $\eta_1 < \tau$. Let

$$\Phi^\delta_s(r, \phi) = \left[ \Phi^\delta_s(r, \phi) \right]^2 + \epsilon^{1/2} \quad (\epsilon > 0).$$

Denote by $(r^\delta, \phi^\delta)$ the solution of (1.4) when $\Phi^\delta_s$ is replaced by $\Phi^\delta_s\phi$. Denote by $L_\epsilon$ the elliptic generator corresponding to the process $(r^\delta, \phi^\delta)$. Set

$$\Phi^\delta_m(r, \phi) = R(r)/\Phi^\delta_m(\phi), \quad \Phi^\delta(r, \phi) = R(r)/\Phi^\delta(\phi).$$

By Itô's formula,

$$(1.14) = \int_0^t \nabla \Phi^\delta_m(r^\delta, \phi^\delta(r)) \cdot \sigma^\delta(r^\delta, \phi(r)) \, dw(r) + \int_0^t L_\epsilon \Phi^\delta_m(r^\delta, \phi^\delta(r)) \, dr$$

where $\nabla \Phi$ is the gradient of $\Phi$ and $\sigma^\delta(r, \phi)$ is the matrix corresponding to $\Phi^\delta_s(r, \phi), \Phi^\delta_s(r, \phi)$. Since $L_\epsilon$ is uniformly elliptic, with bounded and uniformly Hölder continuous coefficients, the corresponding parabolic operator has a fundamental solution (see [2]). We can therefore go to the limit with $m \to \infty$ in (1.14) (cf. [4]) and conclude that

$$R(\gamma(t))/\Phi^\delta_m(\phi^\delta) - R(r(0))/\Phi^\delta(\phi(0)).$$

$$\Phi^\delta_m(r^\delta, \phi^\delta(r)) \cdot \sigma^\delta(r^\delta, \phi(r)) \, dw(r) + \int_0^t L_\epsilon \Phi^\delta_m(r^\delta, \phi^\delta(r)) \, dr.$$  

$$(1.15) = \int_0^t \nabla \Phi^\delta_m(r^\delta(r), \phi^\delta(r)) \cdot \sigma^\delta(r^\delta(r), \phi^\delta(r)) \, dw(r) + \int_0^t L_\epsilon \Phi^\delta_m(r^\delta(r), \phi^\delta(r)) \, dr.$$  

We shall now consider the behavior of $(r^\delta(t), \phi^\delta(t))$ on the set $A_{a, \beta}$. Given $0 < \eta < \eta_0$, let $T_\eta$ be the last time $(r(t), \phi(t))$ is outside $\Omega_{\eta}/2$. With $t$ fixed, and a.s. in $A_{a, \beta}$,

$$(r^\delta(t), \phi^\delta(r)) \to (r(t), \phi(t)) \quad \text{uniformly in} \, \tau, \, 0 \leq \tau \leq t,$$

for a subsequence $\epsilon = \epsilon' \downarrow 0$. Hence, if $T_\eta \leq \tau \leq t$, $(r^\delta(t), \phi^\delta(t)) \in \Omega_{\eta}$ for all $\epsilon = \epsilon' \leq \epsilon^*(\omega)$. Given $0 \leq \delta^*(\omega)$ sufficiently small, we have for any $\epsilon = \epsilon'$ sufficiently small and for any $\eta > 0$,
\[
\int_{T_\eta}^{t} L_\epsilon \Phi^{\delta}(r, \phi(r)) \, dr = \int_{T_\eta}^{t} L_\epsilon \Phi(r, \phi(r)) \, dr \\
= \int_{T_\eta}^{t} (L_\epsilon f)(r, \phi(r)) \, dr \\
= \int_{T_\eta}^{t} \left\{ \left[ \frac{1}{2} \sum_{s=1}^{n} [\beta_s'(r, \phi)]^2 + \epsilon \right] f''(\phi) + \beta(r, \phi) f'(\phi) \right\} \, dr \\
= \int_{T_\eta}^{t} \left\{ \frac{1}{2} \sigma^2(\phi) + \epsilon \right\} f''(\phi) + b(\phi) f'(\phi) + \theta r \right\} \, dr \\
\leq -(t - T_\eta) \nu + \int_{T_\eta}^{t} \epsilon''(\phi) \, dr + \theta y(t - T_\eta)
\]

where \(|\theta| \leq 1, |\theta| \leq 1\), provided \(\eta \leq \eta^\epsilon(y)\). Here \(\nu\) is any positive number such that \(L f(x) \leq -\nu\) for all \(x \neq \alpha + \epsilon_0, x \neq \beta - \epsilon_0\). It follows that

\[\text{(1.16)} \quad \lim_{\epsilon \to 0} \int_{T_\eta}^{t} L_\epsilon \Phi^{\delta}(r, \phi(r)) \, dr \leq -(t - T_\eta) \frac{\nu}{2}\]

if \(\gamma < \nu/2\). Since \((r(r), \phi(r))\) does not intersect the set \(I_{a+} \cup I_{b} \) for \(r = 0\), \(f''(\phi(r)) \) \((0 \leq r \leq T_\eta)\) remains bounded as \(\epsilon = \epsilon' \downarrow 0\). We conclude that

\[\text{(1.17)} \quad \int_{T_\eta}^{t} L_\epsilon \Phi^{\delta}(r, \phi(r)) \, dr \leq C - \frac{\nu}{2} t\]

where \(C\) is a.s. finite valued random variable.

Consider next the stochastic integral in (1.15). If \(T_\eta \leq r \leq t\), then the vector

\[\text{(1.18)} \quad b_\epsilon^{\delta}(r) = \nabla \Phi^{\delta}(r, \phi(r)) \cdot \alpha f'(r, \phi(r)) \]

has components \(|d/d\phi|b_\epsilon(\phi(r))|d/\sigma_s f'(r, \phi(r))|^2 + \epsilon|b_\epsilon(\phi(r))|^2\). If we let \(\epsilon = \epsilon' \downarrow 0\) through an appropriate subsequence \(\epsilon''\), then we obtain a.s. (cf. \([4, \S 2]\))

\[\lim_{\epsilon \to \epsilon'' \downarrow 0} \int_{T_\eta}^{t} b_\epsilon^{\delta}(r) \cdot dw(r) = \int_{T_\eta}^{t} b_\epsilon^{\delta}(r) \cdot dw(r) = \int_{T_\eta}^{t} b_0(r) \cdot dw(r)\]

\[= \sum_{s=1}^{n} \int_{T_\eta}^{t} f'(\phi) \frac{\partial}{\partial r_s} (r, \phi(r)) \, dw(r)\]

where \(b_0(r)\) is defined by (1.18) with \(\Phi^{\delta}\) replaced by \(\Phi\) and with \(\epsilon = 0\).

If \(0 \leq r \leq T_\eta\), then as \(\epsilon \downarrow 0\) through an appropriate subsequence of \(\epsilon''\),

\[\int_{T_\eta}^{t} b_\epsilon^{\delta}(r) \cdot dw(r) \rightarrow \int_{T_\eta}^{t} b_0(r) \cdot dw(r) = \int_{T_\eta}^{t} b_0(r) \cdot dw(r) = \tilde{C}\]

where \(b_0(r)\) has a more complicated expression than in (1.19) (involving \(R(r)\))
and its first derivative; \( \hat{C} \) is a.s. finite. We conclude from this and from (1.15),
(1.17), (1.19) that, a.s. on \( \Omega_{a,\beta} \),
\[
f(\phi(t)) - f(\phi(0)) = C - \frac{\nu}{2} t + \hat{C} + \sum_{s=1}^{n} \int_{T^n} f''(\phi(r)) \tilde{\mathcal{G}}(r, \phi(r)) \, dw^s(r).
\]
By Lemma 1.3 of [3], the last integral is \( o(t) \). Hence
\[
\lim_{t \to \infty} \frac{f(\phi(t))}{t} \leq -\frac{\nu}{2} < 0 \quad \text{a.s. in } \Omega_{a,\beta}.
\]
This implies that \( \phi(t) \to \beta \) as \( t \to \infty \), a.s. in \( \Omega_{a,\beta} \).

2. Boundary behavior of stochastic solutions (continued). Divide the zeros
\( \alpha_1, \ldots, \alpha_k \) of \( \sigma(x) \) in \( [0, 2\pi) \) into blocks
\[
B_j = \{ \alpha_{j,1}, \ldots, \alpha_{j,k_j} \} \quad (k_j > 1)
\]
where \( \alpha_{j,1} < \alpha_{j,i+1}, \alpha_{j,k_j} = \alpha_{j+1,1}, \) (Here we agree that \( \alpha_k < \alpha_1 \)). For each block
\( B_j \), \( b(\alpha_{j,1}) = 0, b(\alpha_{j,k_j}) = 0, \) and \( b(\alpha_{j,i}) \neq 0 \) if \( 2 \leq i \leq k_j - 1 \). Let
\[
A_j = \{ \omega; \alpha_{j,1} < \phi(t) < \alpha_{j,k_j} \text{ for all } t \text{ sufficiently large} \}.
\]
In view of the Lemma 1.1 and the fact that \( (r(t), \phi(t)) \) never crosses the segments
\( \{r, \alpha_{j,1} \}; \ 1 \leq r \leq 1 + \varepsilon \} \) we conclude that \( \Sigma P(A_j) = 1 \).

Consider now a block \( B_j \), and set \( \alpha = \alpha_{j,1}, \beta = \alpha_{j,k_j} \). Suppose
\[
b(\alpha) = 0, \quad b(\beta) = 0, \quad b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_j - 1),
\]
\[
(2.1) \quad Q_\alpha > 1, \quad Q_\beta < 1.
\]

Lemma 2.1. Let (A)-(E) and (2.1), (2.2) hold. If \( x(0) \notin (\Delta_\alpha \cup \Delta_\beta) \), then a.s.
in \( A_j \), \( \phi(t) \to \beta \) as \( t \to \infty \).

Proof. Let
\[
f(x) = \begin{cases} 
-A_1 \log(x - \alpha) + B_1 & \text{in } (\alpha, \alpha + \epsilon_0), \\
A_2 \log(\beta - x) + B_2 & \text{in } (\beta - \epsilon_0, \beta), \\
g(x) & \text{in } [\alpha + \epsilon_0, \beta - \epsilon_0];
\end{cases}
\]
the function \( g(x) \) consists of three parts:
\[
A_3 g_1(x) + B_3 \quad \text{in } [\alpha + \epsilon_0, \alpha_{j,2} - \epsilon'], \quad \\
A_4 g_2(x) + B_4 \quad \text{in } [\alpha_{j,2} - \epsilon', \alpha_{j,k_j-1} + \epsilon'], \\
g_3(x) \quad \text{in } (\alpha_{j,k_j-1} + \epsilon', \beta - \epsilon_0].
\]
where \( \epsilon' > 0 \) is sufficiently small. The function \( g_2 \) is constructed as the function \( f \) in the proof of Theorem 3.2 in [3]; thus \( Lg_2 \leq -\nu < 0 \) in \([a, a - \epsilon', \alpha_{j, k_j - 1} + \epsilon']\) and \( g'(x) < 0 \) at the endpoints. The function \( g_1 \) is defined as the function \( g \) in (1.10), (1.13) with \( \beta \) replaced by \( \alpha_{j, k_j - 1} + \epsilon' \). Finally, the function \( g_3 \) is defined as the function \( g \) in (1.10), (1.13) with \( \alpha \) replaced by \( \alpha_{j, k_j - 1} + \epsilon' \). We can choose the constants \( A_i, B_i \) so that \( f(x) \) is continuously differentiable; the \( A_i \) are all positive.

We can now proceed similarly to the proof of Lemma 1.2.

Suppose now that (2.1), (2.2) are replaced by

\[(2.3) \quad b(\alpha) = 0, \quad b(\beta) = 0, \quad b(a_{ji}) > 0 \quad (2 \leq i \leq k_i - 1), \]
\[(2.4) \quad Q_\alpha < 1, \quad Q_\beta > 1. \]

Lemma 2.2. Let (A)-(E) and (2.3), (2.4) hold. If \( x(0) \notin (\Delta_\alpha \cup \Delta_\beta) \), then a.s. in \( A_j \), \( \phi(t) \to \alpha \) as \( t \to \infty \).

The proof is similar to the proof of Lemma 2.1. Here one takes \( f(x) = A_1 \log(x - \alpha) + B_1 \) in \((\alpha, \alpha + \epsilon_0)\), \( f(x) = -A_2 \log(\beta - x) + B_2 \) in \((\beta - \epsilon_0, \beta)\).

Consider next the cases where

\[(2.5) \quad b(\alpha) = 0, \quad b(\beta) = 0, \quad Q_\alpha < 1, \quad Q_\beta < 1. \]

We further assume that one of the following three conditions holds:

\[(2.6) \quad b(a_{ji}) > 0 \quad (2 \leq i \leq k_j - 1), \]
\[(2.7) \quad b(a_{ji}) < 0 \quad (2 \leq i \leq k_j - 1), \]
\[(2.8) \quad b(a_{ji}) < 0 \quad (2 \leq i \leq i_0), \]
\[b(a_{ji}) > 0 \quad (i_0 + 1 \leq i \leq k_j - 1). \]

Lemma 2.3. Let (A)-(E) and (2.5) hold, and let one of the conditions (2.6), (2.7), (2.8) hold. If \( x(0) \notin (\Delta_\alpha \cup \Delta_\beta) \), then a.s. in \( A_j \), either \( \lim_{t \to \infty} \phi(t) = \alpha \) or \( \lim_{t \to \infty} \phi(t) = \beta \).

The proof is similar to the proof of Lemma 2.1. One takes \( f(x) = A_1 \log(x - \alpha) + B_1 \) in \((\alpha, \alpha + \epsilon_0)\), \( f(x) = A_2 \log(\beta - x) + B_2 \) in \((\beta - \epsilon_0, \beta)\). In case (2.8) holds one takes \( g_2 \) to be the function occurring in the proof of Theorem 4.2 in [3].

The case \( b(\alpha) = 0, b(\beta) = 0, Q_\alpha > 1, Q_\beta < 1 \) will not be considered in this paper. In this case \( \phi(t) \) may oscillate between \( \alpha \) and \( \beta \) without having a limit, as suggested by the case of linear equations [3].

3. Behavior of solutions in general domains. We shall now extend the results of §2 to a general bounded domain \( G \). A point \( x_0 \) on the boundary \( \partial G \) of \( G \) is said to belong to \( \Sigma_3 \) if \( \Sigma_{i,j} a_{ij}(x_0)^{i-j} > 0 \). It belongs to \( \Sigma_2 \) if (1.2) and
\[ \sum_i \left[ b_i - \frac{1}{2} \sum_j \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i > 0 \text{ at } x_0 \]

hold. Finally, \( x_0 \) belongs to \( \Sigma_1 \) if (1.2), (1.3) hold at \( x_0 \).

Denote by \( R(x) \) a continuous function in \( G, C^2 \) and positive in \( G \cup \Sigma_2 \cup \Sigma_3 \), that coincide with \( \text{dist}(x, \Sigma_1) \) when the latter is sufficiently small. With \( R(x) \) fixed from now on, we define \( G, \mathcal{B}, Q \) as in §1.

We shall need the following assumption:

\( (P) \) \( \partial G \) consists of a finite number of curves \( \Gamma_1, \ldots, \Gamma_g \). Each curve belongs entirely to either \( \Sigma_2 \cup \Sigma_3 \) or to \( \Sigma_1 \). A curve \( \Gamma_j \) of \( \Sigma_2 \cup \Sigma_3 \) is in \( C^2 \), and a curve \( \Gamma_j \) of \( \Sigma_1 \) is in \( C^2 \). There is a positive constant \( \mu \) such that if a curve \( \Gamma_j \) belongs to \( \Sigma_1 \) then either (i) \( Q(x) \leq -\theta_0 < 0 \) (\( \theta_0 \) constant) for all \( x \in G \) whose distance to \( \Gamma_j \) is \( \leq \mu \) [we then say that \( \Gamma_j \) belongs to \( \Sigma_1^- \)], or (ii) \( Q(x) > \theta_0 > 0 \) (\( \theta_0 \) constant) for all \( x \in G \) whose distance to \( \Gamma_j \) is \( \leq \mu \) [we then say that \( \Gamma_j \) belongs to \( \Sigma_1^+ \)]. Finally, \( \Sigma_1^- \) is nonempty.

We shall maintain the assumptions (A), drop the assumption (B), and replace (C) by

\( (C^*) \) \( \mathcal{Q}(x) > 0 \) for all \( x \in G \) with \( \text{dist}(x, \Sigma_1) \geq \mu, \) \( \nabla_x R(x) \neq 0; \)

\[ \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \text{ for all } x \in G \text{ with } \text{dist}(x, \Sigma_1) \geq \mu, \]

\[ \nabla_x R(x) = 0. \]

By slightly modifying the construction of \( R(x) \) in the proof of Lemma 2.1 of [4], one can show that if the exterior boundary of \( G \) is not in \( \Sigma_1 \), then there actually exists a function \( R(x) \) with \( \nabla_x R(x) \neq 0 \) everywhere in \( G \).

The proof of Theorem 2.2 of [4] can be modified to yield the following extension of Lemma 1.1.

**Theorem 3.1.** Let (A), (P), (C*) hold. Then, with probability 1, either (i) \( x(t) \) exits \( G \) in finite time by crossing \( \Sigma_2 \cup \Sigma_3 \), or (ii) \( x(t) \in G \) for all \( t > 0 \) and \( \text{dist}(x(t), \Sigma_1^-) \to 0 \) as \( t \to \infty \).

Suppose for definiteness that \( \Gamma_1 \subset \Sigma_1^- \), and \( \Gamma_1 \) is not the outer boundary of \( G \). If \( x_1 = f(r), x_2 = g(r) \) are parametric equations for \( \Gamma_1 \) (\( r \) = length parameter), then we can introduce new variables

\[ y_1 = (1 + \rho) \cos(2\pi r/L), \quad y_2 = (1 + \rho) \sin(2\pi r/L) \quad (L = \text{length of } \Gamma_1) \]

where the "polar coordinates" \( \rho, r \) are defined by

\[ x_1 = f(r) + \rho g'(r), \quad x_2 = g(r) - \rho f'(r). \]
As in [4] we can extend this mapping into a diffeomorphism from the exterior of \( \Gamma_1 \) onto the set \( \{ y : |y| \geq 1 \} \). In the new coordinates

\[
dp = \sum_{s=1}^{n} \sigma_s \, dw_s + b^t \, dt,
\]

\[
d\phi = \sum_{s=1}^{n} \sigma_s \, dw_s + b^t \, dt \quad \left( \phi = \frac{2\pi \tau}{L} \right),
\]

and

\[
\frac{L}{2\pi} \sigma_s (0, \phi) = \int \sigma_{1s} b^t + \sigma_{2s},
\]

\[
\frac{L}{2\pi} \sigma_s (0, \phi) = (\tilde{b}_1 + \hat{b} \tilde{b}_2) - (\hat{y}, -\hat{f}) \begin{pmatrix} \Sigma_{1s} \sigma_{2s} \\ \Sigma_{1s} \sigma_{2s} \end{pmatrix} \begin{pmatrix} \hat{y} \\ \hat{f} \end{pmatrix}.
\]

Set \( \sigma (\phi) = \left| \sum_{s=1}^{n} [\sigma_s (0, \phi)]^2 \right|^{1/2} \), \( b (\phi) = b (0, \phi) \).

We now assume:

(D\textsuperscript{'} ) The condition (D) holds with \( r = 1 + \rho \). More precisely: \( \sigma (x) \) vanishes at a finite number of points \( a_1, \ldots, a_k \) (\( k \geq 1 \)). If \( b (a_j) = 0 \) for some \( j \), then there is a simple \( C^3 \) curve \( \Delta_{a_j} \), given by \( x = \chi(t) \) (\( t_1 \leq t \leq t_2 \)) such that \( x(t_1) = (\chi(a_j), \chi(a_j)) \), \( x(t_2) \) lies outside \( \Sigma \), and \( x(t) \), for some \( t \in (t_1, t_2) \), lies on \( \partial G \) and is different from \( (\chi(a_j), \chi(a_j)) \), and such that

(i) a part \( \{ x(t), t_1 < t < t_1 + \epsilon \} \) of \( \Delta_{a_j} \) lies in \( G \) and is nontangential to \( \partial G \) at \( t = t_1 \);

(ii) the relations (1.8) hold along \( \Delta_{a_j} \).

Finally, if \( b (a_j) = b (a_j) = 0 \) and \( b (a_j) \neq 0 \) for all the \( a_i \) between \( a_j \) and \( a_k \), then points of \( G \) corresponding to \( (\rho, \phi) \) with \( \rho = \epsilon, \phi \) in the interval \( (a_j, a_k) \) \( [\rho > 0 \) small], cannot be connected (in \( G \)) to points of \( G \) corresponding to \( (\rho, \phi) \) with \( \rho = \epsilon, \phi \) outside the interval \( (a_j, a_k) \), without crossing either \( \Delta_{a_j} \) or \( \Delta_{a_k} \).

(E\textsuperscript{'} ) The condition (E) holds. Furthermore, \( b (a_j) = 0 \) for at least one value of \( j \).

Suppose (A), (P), (C\textsuperscript{*}) and (D\textsuperscript{' }), (E\textsuperscript{' }) hold. Denote by \( A^1 \) the set where \( x(t) \in G \) for all \( t > 0 \) and \( \text{dist} (x(t), \Gamma_1) \to 0 \). Let \( A^1_{a_j} \) be the subset of \( A^1 \) for which \( \phi(t) < \phi(t) < \phi(t) \) holds for all \( t \) sufficiently large. Suppose a portion of each \( \Delta_{a_j} \) initiating at \( (\chi(a_j), \chi(a_j)) \) coincides with the normal to \( \partial G \) at that point. Then the proof of Lemmas 2.1–2.3 remain valid (in the \( y \)-coordinates). Here we use the fact that the diffeomorphism \( x \to y \) given above does not affect the condition (D\textsuperscript{' }), i.e., the conditions in (1.8) are invariant under a diffeomorphism. If \( \Delta_{a_j} \) does not contain the normal, then we perform a different local diffeomorphism from the \( x \)-space onto the \( y \)-space, such that \( \Gamma_1 \) is mapped onto the unit circle.
and such that the image of a portion of $\Delta^1_{a_j}$ does coincide with the normal to this circle. The new diffeomorphism does not affect the tangential stochastic equation, i.e., the functions $\sigma(\phi), b(\phi)$ remain the same.

We conclude: If $x \not\in \bigcup_j \Delta^1_{a_j}$, then almost surely on $A^1$, either

(i) $\phi(u) \to \beta$ if (2.1), (2.2) hold; or

(ii) $\phi(u) \to \alpha$ if (2.3), (2.4) hold; or

(iii) $\phi(u) \to \alpha$ or $\phi(u) \to \beta$ if (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

In what follows we assume:

(Q) For each block $B_j$, either (2.1), (2.2) or (2.3), (2.4) or (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

The segment $\{x, y\}; \phi = \alpha_{j,1}, 0 \leq \rho \leq 7\}$ in the $y$-space is mapped onto an arc $l_j$ in the $x$-space. $l_j$ initiates at a point $y_{j1}$ on $\Gamma_j$, is nontangential to $\Gamma_j$ at $y_{j1}$, it is contained in $\Delta^1_{a_{j,1}^1}$, and it lies in the interior of $G$ (with the exception of its endpoint $y_{j1}$). It divides a small $G$-neighborhood $N_j$ of $y_{j1}$ into domains: $N^+_{j1}$ and $N^-_{j1}$.

Definition. The point $y_{j1}$ is called a distinguished boundary point of $G$ if at the corresponding point $a_{j,1}^1, Qa_{j,1} < 1$.

If $\Gamma_1 \subset \Sigma^-$ and the interior of $\Gamma_1$ contains $G$, then the above considerations remain valid with trivial changes; the assertions (i)–(iii) are unchanged.

Consider now the general case. We index the $\Gamma_i$ so that

\[ \Sigma^-_1 = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p, \]
\[ \Sigma^+_1 = \Gamma_{p+1} \cup \cdots \cup \Gamma_{p+h}, \]
\[ \Sigma_2 \cup \Sigma_3 = \Gamma_{p+h+1} \cup \cdots \cup \Gamma_q. \]

We assume

(D*) The condition (D') holds for each $\Gamma_j$, $1 \leq j \leq p$.

(E*) The condition (E') holds for each $\Gamma_j$, $1 \leq j \leq p$.

(Q*) The condition (Q) holds for each $\Gamma_j, 1 \leq j \leq p$.

We define distinguished boundary points on $\Gamma, \Gamma_2, \ldots, \Gamma_p$ in the same way as for $\Gamma_1$. Denote by $\zeta_j$ ($1 \leq j \leq k$) the set of all distinguished boundary points on $\Sigma^-_1$. With each $\zeta_j$ we associate two "half" $G$-neighborhoods $N^+_{\zeta_j}, N^-_{\zeta_j}$ of $\zeta_j$, in the same way that we have associated $N^+_{j1}, N^-_{j1}$ with $y_{j1}$.

In the condition (D*) there appear curves $\Delta^l_{\zeta_j} (1 \leq l \leq p)$ defined analogously to the curves $\Delta^l_{a_j}$. Denote these curves by $\Delta_j (1 \leq j \leq l)$ and set $\Lambda_j = \Delta_j \cap \overline{G}$.

Each $\zeta_j$ is an endpoint of some $\Delta_j$.

We sum up the previous considerations in the following theorem.
Theorem 3.2. Let the conditions (A), (P), (C*), and (D*), (E*), (Q*) hold. If \( x \not\in \bigcup_{j=1}^{k} \Lambda_j \), then the probability space is a finite disjoint union

\[
\Omega_0 \cup \left( \bigcup_{j=1}^{k} \Omega^+_j \right) \cup \left( \bigcup_{j=1}^{k} \Omega^-_j \right),
\]

such that the following holds almost surely: if \( \omega \in \Omega_0 \), \( x(t) \) exits from \( G \) in finite time by crossing \( \Sigma_2 \cup \Sigma_3 \); if \( \omega \in \Omega^+_j \) then \( x(t) \in G \) for all \( t > 0 \), and \( x(t) \in N_j^+, x(t) \to \zeta_i \) as \( t \to \infty \); if \( \omega \in \Omega^-_j \) then \( x(t) \in G \) for all \( t > 0 \), and \( x(t) \in N_j^-, x(t) \to \zeta_i \) as \( t \to \infty \). The decomposition (3.1) depends on \( x(0) \).

Definition. If \( x(t) \in G \) for all \( t > 0 \), and \( x(t) \in N_j^+, x(t) \to \zeta_i \) as \( t \to \infty \), then we shall write: \( x(t) \to \zeta_i^+ \) as \( t \to \infty \). Similarly we define the concept: \( x(t) \to \zeta_i^- \) as \( t \to \infty \). We denote by \( p_i^+(x) \) (\( p_i^-(x) \)) the probability that \( x(t) \to \zeta_i^+ \) (\( x(t) \to \zeta_i^- \)) as \( t \to \infty \), given \( x(0) = x \in G \).

Clearly \( p_i^+(x) \geq 0, p_i^-(x) \geq 0, \sum_{i=1}^{k} p_i^+(x) + \sum_{i=1}^{k} p_i^-(x) \leq 1 \). If \( \Sigma_2 \cup \Sigma_3 \) is empty, then the last sum is equal to 1 (by Theorem 3.2) if \( x \not\in \bigcup_{j=1}^{k} \Lambda_j \).

Definition. Denote by \( q_i(x) \) (\( 1 \leq i \leq p \)) the probability that \( x(t) \in G \) for all \( t > 0 \) and \( \text{dist}(x(t), \Gamma_i) \to 0 \) as \( t \to \infty \), given \( x(0) = x \in G \).

Theorem 3.3. Let the conditions (A), (P), (C*) hold. Then \( q_i(x) \to 1 \) if \( \text{dist}(x, \Gamma_i) \to 0 \) as \( t \to \infty \).

Proof. For any \( \lambda > 0 \) sufficiently small, let \( \Gamma_{i \lambda} \) be the curve in \( G \) parallel to \( \Gamma_i \) at a distant \( \lambda \). Denote by \( G_\lambda \) the domain bounded by \( \Gamma_i, \Gamma_{i \lambda} \). Denote by \( L \) the elliptic operator corresponding to the diffusion process (1.1). Then

\[
L[R(x)]\xi = \left[ e^{2\xi^2/2R^2 + eQ} \right] R(x)|R(x)|^\xi \quad (\xi > 0).
\]

Since \( \hat{G} = O(R^2) \) in \( G_\lambda \), \( LR \leq 0 \) in \( G_\lambda \) provided \( \lambda \) and \( \varepsilon \) are sufficiently small. Denote by \( r_\lambda \) the exit time from \( G_\lambda \). Then, by Itô's formula,

\[
E[R(x(r_\lambda))]\xi - [R(x)]\xi = E \int_0^{r_\lambda} L[R(x(\tau))]\xi \, d\tau \leq 0.
\]

Since \( x(r_\lambda) \in \Gamma_{i \lambda}, R(x(r_\lambda)) = \lambda \). Hence

\[
[1 - q_i(x)]\xi \leq [R(x)]\xi = [\text{dist}(x, \Gamma_i)]\xi,
\]

and the assertion follows.

The above proof is valid also in any number of dimensions.

4. Regularity of the functions \( p_i^+(x) \). Let

\[
Lu = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i}.
\]
Let $\Lambda_{1}', \ldots, \Lambda_{p}',$ be disjoint $C^3$ curves (the endpoints are included) lying in $G$, and set

$$\Lambda = \bigcup_{i=1}^{l} \Lambda_{i}, \quad \Lambda' = \bigcup_{i=1}^{l'} \Lambda_{i}'.$$

The elliptic operator $L$ will be allowed to degenerate in $G$ only at the points of $\Lambda \cup \Lambda'$. We shall call $\Lambda_{i}$ a “boundary spoke” and $\Lambda_{i}'$ an “interior spoke.”

Consider a parametric representation for $\Lambda_{i}$:

$$x_1 = f(s), \quad x_2 = g(s) \quad (0 < s < L_{i}).$$

where $s$ is the length parameter. Let

$$b_{s}(s) = \sqrt{(b_2 - gb_1)^2 - \frac{1}{2} (g_2 - g_1)^2} \sum_{r=1}^{n} (\sigma_{1r} + \sigma_{2r})^2$$

where the argument of $b_{s}, \sigma_{ir}$ is $(x_1, x_2)$ given by (4.2).

We shall need the following assumption regarding the degeneracy of $L$ in $G$:

(G) On each $\Lambda_{i}', b_{s}(s) \neq 0$ for $0 < s < L_{i}$. The elliptic operator $L$ may degenerate on each arc $\Lambda_{i}'$, and in a sufficiently small $\delta_0$-neighborhood of each $\Lambda_{i}'$, at all the remaining points of $G$, $L$ is nondegenerate.

The number $\delta_0$ is positive and depends only on upper bounds on the first derivatives of $\sigma_{ik}, b_{s}$, and on a positive lower bound on the $|b_{s}(s)|$. Its precise nature will emerge from the proof of Theorem 4.1 below. Denote by $\Lambda_{i}^{\infty}$ the $\delta_0$-neighborhood of $\Lambda_{i}$.

**Theorem 4.1.** Let the conditions (A), (P), (C*) and (G) hold. Then $p_{i}(x)$ ($1 \leq i \leq p$) are Lipschitz continuous functions in $G - \Lambda$, and $C^2$ solutions of $Lu = 0$ in $G - (\Lambda \cup \Lambda_{i}^{\infty})$.

**Proof.** We shall combine classical regularity theorems with the method of Freidlin [1]. Denote by $p_{i}(x)$ any one of the functions $p_{i}^{\pm}(x)$. Consider first a point $x_0 \in G$ where $L$ is nondegenerate. Let $N$ be a small disc with center $x_0$ such that $L$ is nondegenerate in $N$. By the strong Markov property, for any $x \in N$,

$$p_{i}(x) = E_x p_{i}(r_{N}) = \int_{\partial N} p_{i}(y) P_{x}(r_{N} = y) \, dS_{y}$$

where $r_{N}$ is the exit time from $N$, and $dS_{y}$ is the length element on $\partial N$. Note, by a standard argument, that $p_{i}(y)$ is a Borel function on $\partial N$.

Let $A$ be an interval on $\partial N$. Denote by $I_{A}$ the characteristic function of $A$, and by $\eta, \zeta$ the endpoints of $A$. By classical theorems [2], there exists a unique solution $u$ of
A. FIELDMAN AND M. A. PINSKY

\[ Lu = 0 \text{ in } N, \quad u \in C^2(N), \]

\[ u(x) \to I_A(y) \quad \text{if } x \to y \in \partial N, \quad y \neq \eta, \quad y \neq \zeta, \]

\[ u(x) \text{ remains bounded as } x \to \eta, \text{ or } x \to \zeta. \]

We can write \( u \) in terms of Green’s function \[2\]

\[ u(x) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} ds_y \]

where \( \nu_y \) is the inward normal. Denote by \( N_\epsilon \) the disc with center \( x_0 \) and radius \( r_0 - \epsilon \), where \( r_0 \) is the radius of \( N \). Let \( \tau_{N_\epsilon} \) be the exit time from \( N_\epsilon \).

By Itô’s formula,

\[ u(x) = E_x \{ u(x_{\tau_{N_\epsilon}}) \} \quad (\epsilon > 0). \]

Since \( L \) is nondegenerate in \( \overline{N} \), \( x(\epsilon) \) exists \( N \) at \( \zeta \) or at \( \eta \) with probability 0.

Hence, taking \( \epsilon \to 0 \) in (4.6), we arrive at the formula

\[ u(x) = E_x \{ u(x_{\tau_N}) \} = E_x \{ I_A(x_{\tau_N}) \} = P_x (x_{\tau_N} \in A). \]

Comparing this with (4.5), we conclude that

\[ P_x (x_{\tau_N} \in A) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} ds_y. \]

This implies that

\[ P_x (x_{\tau_N} \in dS_y) = \frac{\partial G(x, y)}{\partial \nu_y} ds_y. \]

Hence (4.4) gives

\[ p_i(x) = \int_{\partial N} p_i(y) \frac{\partial G(x, y)}{\partial \nu_y} ds_y. \]

This shows that \( p_i(x) \) is continuous in \( N \). By decreasing \( N \) we may assume that \( p_i(x) \) is continuous in \( \overline{N} \).

The solution \( \nu \) of \( L \nu = 0 \) in \( N \), \( \nu = p_i \) on \( \partial N \) is also given by the right-hand side of (4.7). Hence \( \nu = p_i \) in \( N \). Since \( \nu \) belongs to \( C^2(N) \), the same is true of \( p_i \). This completes the proof of the second assertion of Theorem 4.1.

To prove the first assertion, consider first the case of an interior spoke \( A_j \) having the form \( \phi = \phi_0, \quad r_0 \leq r \leq r_1 \). Let \( B_\delta \) be the domain \( |\phi - \phi_0| < \delta, \quad r_0 < r < r_1 \). The condition \( b_i^+ \neq 0 \) reduces to \( b_i^+ r, \phi \neq 0 \) where \( b_i^+(r, \phi) \) is defined as in \( \S 1 \). Suppose, for definiteness, that \( b_i^+(r, \phi) \geq \beta > 0 \) inside \( B_\delta \). By Itô’s formula we have
\[ \phi(t) = \phi_0 + \sum_{s=1}^{n} \int_0^t \sum_{r} \phi_s(r, \phi) dw_s + \int_0^t \beta(r, \phi) dr, \]

and hence
\[ \phi(t \wedge r_{B_\delta}) \geq \phi_0 + \sum_{s=1}^{n} \int_0^t \sum_{r} \phi_s(r, \phi) dw_s + \beta(t \wedge r_{B_\delta}). \]

Thus
\[ \beta E_x(t \wedge r_{B_\delta}) \leq \sup_{\phi \in B_\delta} |\phi - \phi_0| = \delta. \]

It follows that
\[ E_x(r_{B_\delta}) \leq \delta/\beta = C \quad (x \in B_\delta). \]

By a standard iteration argument it follows that
\[ P_x(r_{B_\delta} \geq nt_0) \leq (C/t_0)^n \quad (n = 1, 2, \ldots). \]

Consequently,
\[ P_x(r_{B_\delta} \geq t) \leq e^{-at}, \quad a = -(1/t_0) \log (C/t_0). \]

Taking \( t_0 = e \) we get
\[ P_x(r_{B_\delta} \geq t) \leq e^{-at}, \quad a = (1/e) \log (\beta e/\delta). \]

We may choose \( \delta > 0 \) sufficiently small to apply the following result of Freidlin [1, p. 1349] (which we state only in \( R^2 \)):

Theorem. Suppose \( \sigma_{ij}, b_i \) are continuously differentiable in \( R^2 \) with
\[ \max_{i,j,k} \left\{ \left| \frac{\partial \sigma_{ij}}{\partial x_k} \right|, \left| \frac{\partial b_i}{\partial x_k} \right| \right\} = K. \]

Let \( a_1 = 8K^2 + 4K. \) Suppose the boundary is uniformly normally regular, and the boundary function \( \psi \) is the restriction to \( \partial B_\delta \) of a \( C^2 \) function in a neighborhood of \( \partial B_\delta. \) Then the function \( E_x|\psi(x)_{r_{B_\delta}}| \) is Lipschitz continuous in \( B_\delta, \)

provided \( a > a_1. \)

By choosing \( \delta_0 \) (in the condition (G)) sufficiently small we can ensure that, for some \( \delta > \delta_0, \) \( L \) is nondegenerate on the boundary of \( B_\delta, \) and \( a > a_1. \)

The uniform normal regularity of \( \partial B_\delta \) means that \( E_x(r_{B_\delta}) \leq C_0|x - x_0| \) for all \( x_0 \in \partial B_\delta, x \in B_\delta \) where \( C_0 \) is a constant. This property is guaranteed by the nonvanishing of the normal diffusion on \( \partial B_\delta \) (see [1]). Further, since \( L \) is nondegenerate on \( \partial B_\delta, p_i(x) \) is \( C^2 \) in a neighborhood of \( \partial B_\delta. \) Hence we can apply Freidlin's theorem to deduce (upon recalling the first equation of (4.4), which holds for \( N = B_\delta \)) that \( p_i(x) \) is Lipschitz continuous in \( B_\delta, \)

To handle the case of a general spoke \( A_j, \) we introduce new coordinates \((\rho, s)\) by the equations
where $-\rho_0 < \rho < \rho_0$ ($\rho_0$ is positive and sufficiently small) and $f, g$ are as in (4.2). The stochastic differentials $ds, dp$ are given by the formulas

$$
dp = \sum_{r=1}^{n} \bar{d}_r dw_r + \bar{b}_r dt, \quad d\phi = \sum_{r=1}^{n} \bar{d}_r dw_r + \bar{b}_r dt \quad (\phi = 2\pi s/L_j),$$

Explicit calculation gives (cf. [4]) $b_j(s, \phi) = 2nb_j(s)/L_j$. Since $b_j(s) \neq 0$, we can repeat the argument given in the previous special case.

**Remark.** Suppose $\sigma_{ij}, b_i$ belong to $C^m(R^2)$. Using Theorem 3 of Freidlin [1] (instead of the above quoted theorem of [1]) we conclude that the $p_j(x)$ have $m - 1$ Lipschitz continuous derivatives in $G - A$. Here the constant $\delta_0$ occurring in the condition (G) depends also on $m$.

**Definition.** If $x \to \zeta_i, x \in N_i^+$ then we write $x \to \zeta_i^+$. Similarly we write $x \to \zeta_i^-$ if $x \to \zeta_i^-$, $x \in N_i^-$. 

**Theorem 4.2.** Let the conditions (A), (P), (C*) and (D*), (E*), (Q*) hold. Then $p_j^+(x) \to 1$ if $x \to \zeta_i^+$, and $p_j^-(x) \to 1$ if $x \to \zeta_i^-$. 

This theorem is of the same type as Theorem 3.3. The method of proof is also the same as for Theorem 3.3.

**Proof.** It suffices to prove the assertion for $p_j^+(x)$. Consider first the special case where the distinguished boundary point $\zeta_i$ lies in some $\Gamma_j$, say $\Gamma_1$, which is the circle $r = 1$, and $G$ lies in the exterior of $\Gamma_1$. Let $N$ be "half $G$-neighborhood" of $\zeta_i$ given by $\zeta_i < \phi < \phi_1$, $1 < r < 1 + \delta$. Consider the function

$$
f(R, \phi) = R^\epsilon + (\phi - \zeta_i)^\epsilon \quad \text{in} \quad N$$

where $R = r - 1$, and $\epsilon > 0$ is sufficiently small. It is easily seen that $L/\leq 0$ if $\delta$ and $\epsilon$ are sufficiently small. Let $r$ be the exit time from $N$. By Itô's formula

$$
E[f(R(t \wedge r), \phi(t \wedge r))] \leq f(R(0), \phi(0)) \quad (t > 0).
$$

Now, $x(t)$ cannot leave $N$ in finite time by crossing either $R = 0$ or $\phi = \zeta_i$. On the other hand, on the remaining boundary of $N$, $f(R, \phi)$ is bounded below by some constant $\gamma > 0$ ($\gamma$ depends on $\delta, \epsilon$). Hence, taking $t \to \infty$ in (4.10), we obtain the inequality

$$
yP^x(r < \infty) \leq f(R(0), \phi(0)) \quad (x = x(0)).$$
Since \( f(R(0), \phi(0)) \rightarrow 0 \) if \( x \rightarrow \zeta_i^+ \), we conclude that
\[
P_x(r < \infty) \rightarrow 0 \quad \text{if} \quad x \rightarrow \zeta_i^+.
\]
Since, by Theorem 3.2, \( p_i^+(x) = 1 - P_x(r < \infty) \), the proof is complete.

Remark. By Theorem 4.2, the \( p_i^+(x) \) are discontinuous at the points of the boundary spoke initiating at \( \zeta_i \), which are in some small neighborhood of \( \zeta_i^+ \).

5. The Dirichlet problem.

Lemma 5.1. Let \( x(t) = (x_1(t), \ldots, x_l(t)) \) be a solution of a system of \( l \) stochastic equations of the form (1.1), with uniformly Lipschitz continuous coefficients \( \sigma_{ij} \), \( b_i \). Let \( r \) be any finite valued random variable. Suppose the range of \( x(t) \), \( 0 < t < r \), is contained in an open set \( D \subset R^l \). Let \( f(x) \) be a \( C^2 \) function in \( D \). Then Itô's formula holds:
\[
f(x(r)) - f(x(0)) = \sum_{i=1}^l \sum_{j=1}^n \int_0^r f_{x_i}(x(s)) \sigma_{ij}(x(s)) \, dw^j + \int_0^r Lf(x(s)) \, ds
\]
where
\[
Lf(x) = \frac{1}{2} \sum_{i,j=1}^l a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^l b_i(x) \frac{\partial f}{\partial x_i} \quad [(a_{ij}) = \sigma \sigma^*].
\]

Proof. For any \( \delta > 0 \), modify \( f \) into a function \( f^\delta(x) \) in \( C^2(R^l) \), coinciding with \( f(x) \) if \( \text{dist}(x, R^l - D) > \delta \). Apply Itô's formula to \( f^\delta(x(t)) \), substitute \( t = r \), and take \( \delta \downarrow 0 \).

Now let the assumptions of Theorem 4.2 hold. Consider the Dirichlet problem
\[
(Lu = 0 \quad \text{in} \quad G - \Lambda,
\]
\[
u = g \quad \text{on} \quad \Sigma_2 \cup \Sigma_3,
\]
\[
\begin{cases}
u(x) \rightarrow f_i^+ \quad \text{if} \quad x \rightarrow \zeta_i^+ \\nu(x) \rightarrow f_i^- \quad \text{if} \quad x \rightarrow \zeta_i^- \quad (1 \leq i \leq k).
\end{cases}
\]
Here \( L \) is defined by (4.1), \( g \) is a given continuous function on \( \Sigma_2 \cup \Sigma_3 \), and \( f_i^\pm \) are given numbers.

If \( u \) is continuous in \( (G \cup \Sigma_2 \cup \Sigma_3) - \Lambda \) and is in \( C^2(G - \Lambda) \), and if it satisfies (5.1)–(5.3), then we call it a classical solution of the Dirichlet problem (5.1)–(5.3). Notice that \( u \) is not required to be continuous on \( \Sigma_1^+ \cup \Sigma_1^- \cup \Lambda \). Since \( u(x) \rightarrow f_i^\pm \) as \( x \rightarrow \zeta_i^\pm \), \( u \) cannot be continuous at the points of \( \Lambda \) near \( \zeta_i \) if \( f_i^+ \neq f_i^- \).

We shall prove in this section the existence and uniqueness of a classical solution.
Theorem 5.2. Let (A), (P), (C*), (D*), (E*), (Q*) hold. Then there exists at most one classical solution of the Dirichlet problem.

Proof. Let $G_\varepsilon = \{x \in G, \text{dist}(x, \partial G) > \varepsilon\}, \varepsilon > 0$. Denote by $\tau$ the exit time from $G$, and denote by $\tau_\varepsilon$ the exit time from $G_\varepsilon$. Let $u$ be a classical solution. Since $x(t)$ (with $x(0) \in G - \Lambda$) remains in $G - \Lambda$ for $0 \leq t \leq T \wedge \tau_\varepsilon$, where $T < \infty$, $\varepsilon > 0$, we can apply Lemma 5.1. This gives, after taking the expectation,

$$u(x) = E_x[u(x(T \wedge \tau_\varepsilon))].$$

Taking $\varepsilon \downarrow 0$, $T \to \infty$ and using the continuity of $u$ at $\Sigma_2 \cup \Sigma_3$ and Theorem 3.2, we get

$$u(x) = E_x[g(x(\tau))]I_{\{\tau < \infty\}} + \sum_{i=1}^{k} \int_{i}^+ p_i^+(x) + \sum_{i=1}^{k} \int_{i}^- p_i^-(x)$$

where $I_A$ is the indicator function of $A$. This implies that $u(x)$ is uniquely determined (in $G - \Lambda$).

We shall now prove the existence of a solution.

Theorem 5.3. Let (A), (P), (C*), (G), (D*), (E*), (Q*) hold, and let $g$ be continuous on $\Sigma_2 \cup \Sigma_3$. Then the function $u(x)$ given by (5.4) is continuous in $(G \cup \Sigma_2 \cup \Sigma_3) - \Lambda$, Lipschitz continuous in $G - \Lambda$, and $C^2$ in $G - (\Lambda \cup \Lambda_0^\ast)$, and it satisfies (5.1) in $G - (\Lambda \cup \Lambda_0^\ast)$ and (5.2), (5.3).

Proof. In Theorem 4.1 we proved that $p_i^\pm(x)$ is Lipschitz continuous in $G - \Lambda$, and is a $C^2$ solution of (5.1) in $G - (\Lambda \cup \Lambda_0^\ast)$. The same proof works also for the function $E_x[g(x(\tau))]I_{\{\tau < \infty\}}$. Hence, the function $u$, given by (5.4), is Lipschitz continuous in $G - \Lambda$ and is a $C^2$ solution of (5.1) in $G - (\Lambda \cup \Lambda_0^\ast)$. The assertion $u(x) \to f_i^\pm$ as $x \to \zeta_i^\pm$ follows from Theorem 4.2 and the fact that

$$\sum_{j=1}^{k} p_j^+(x) + \sum_{j=1}^{k} p_j^-(x) + E_x[I_{\{\tau < \infty\}}] = 1.$$  

(This is the assertion of Theorem 3.2.) Finally, the assertion that $u(x)$ is continuous at the points of $\Sigma_2 \cup \Sigma_3$ and it satisfies (5.2) follows from Theorem 2 of Pinsky [6].

Remark. The function $u(x)$ is a weak solution of (5.1) in $G$, in the sense that

$$u(x) = \int_{\partial N} u(y)p_x(y) \, dS_y$$

where $N$ is a disc in $G$, $x \in N$, and $\tau_N$ is the exit time from $N$. The proof is the same as for (4.4).

We shall now strengthen the assumptions of Theorem 5.3 in order to achieve a classical solution.
(G*) The condition (G) holds and \( a_{ij}, b_i \) are in \( C^2(A_{\delta_0}) \). The positive constant \( \delta_0 \) occurring in the condition (G) will now be smaller; it will be as in the remark following the proof of Theorem 4.1, with \( m = 2 \).

**Theorem 5.4.** Let (A), (P), (C*), (G*), (D*), (E*), (Q*) hold, and let \( g \) be a continuous function on \( \Sigma_2 \cup \Sigma_3 \). Then (5.4) is the unique classical solution of the Dirichlet problem (5.1)-(5.3).

Indeed, we only have to verify that \( u \) is in \( C^2(G - \Lambda) \) and \( Lu = 0 \) in \( G - \Lambda \). For \( \rho_i^\pm(x) \) this follows from the remark following the proof of Theorem 4.1. For \( \rho_i^0(x) \) the proof is the same.

**Remark.** Theorems 5.2-5.4 extend to the Dirichlet problem consisting of

\[
L u + c(x) u = 0 \quad \text{in } G - \Lambda
\]

and (5.2), (5.3), provided \( c(x) < 0 \) in \( G \). Instead of (5.4) we have

\[
u(x) = E_x \left\{ g(x(r)) \exp \left[ \int_0^r c(x(s)) \, ds \right] I_{[r < \infty]} \right\} + \sum_{i=1}^k \left[ \frac{c(x)}{2} \right] I_{[\rho_i^0(x) > 0]}
\]

If \( c(x) \leq -c_0 < 0 \) then the last two sums vanish, so that no boundary conditions on \( \Sigma_1 \) need to be given. This is in agreement with the treatment in [5], [7] (and the references given there) where \( c_0 \) is assumed to be positive.

6. The Dirichlet problem in \( m \)-dimensional domains. In subsection 6.1 we prove a theorem for \( m \geq 2 \) which even when \( m = 2 \) is not contained in §§2—5. In subsections 6.2, 6.3 we discuss the generalizations of the results of §§2—5 to \( m \geq 2 \).

6.1. Consider a system of \( m \) stochastic equations

\[
dx_i = \sum_{r=1}^n a_{ir}(x) \, dw^r + b_i(x) \, dt \quad (1 \leq i \leq m)
\]

and let \( L \), given by (0.1), be the corresponding elliptic operator, i.e., \( a_{ij} = (a_{ij}^*) \). We shall denote the analogs of the conditions (A), (P), (C*) for \( m \geq 2 \) by (A_m), (P_m), (C*_m) respectively. Assuming that these conditions hold, the assertion of Theorem 3.1 remains valid.

With \( G \) a bounded \( m \)-dimensional domain, and \( \Gamma_1, \ldots, \Gamma_p \) its boundary hypersurfaces, we index the \( \Gamma_i \) as in §3. Thus, \( \Sigma_i^- \) is made up of \( \Gamma_1, \ldots, \Gamma_p \). Denote by \( \Gamma_i^\epsilon (\epsilon > 0) \) the intersection of \( G \) with \( \epsilon \)-neighborhood of \( \Gamma_i \). We assume
(R) On each $\Gamma_l$ ($1 \leq l \leq p$) there is a finite number of points $\xi_{lj}$ such that $a_{ij}(\xi_{lj}) = 0$, $b_{ij}(\xi_{lj}) = 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Let $R_{lj}(x) = \|x - \xi_{lj}\|$ if $|x - \xi_{lj}| < \epsilon'$ (for some $\epsilon' > 0$), and define $Q_{lj}(x)$ as $Q(x)$ in §1 when $R(x)$ is replaced by $R_{lj}(x)$. Then

$$Q_{lj}(x) \leq -\theta_0 < 0 \quad \text{if} \quad |x - \xi_{lj}| < \epsilon', \; x \in G.$$

For any $1 \leq l \leq p$, let $R^{*l}(x)$ be a positive $C^2$ function for $x \in \Gamma_l^{e_0} \cup \Gamma_l$, $x \neq \xi_{lj}$ (for some $\epsilon_0 > 0$) such that $R^{*l}(x) = R_{lj}(x)$ if $|x - \xi_{lj}| < \epsilon'$. We shall assume

(S) For all $x \in \Gamma_l^{e_0} \cup \Gamma_l$, $\min_j |x - \xi_{lj}| > \epsilon'$ ($1 \leq l \leq p$),

$$\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial R^{*l}}{\partial x_i} \frac{\partial R^{*l}}{\partial x_j} > 0 \quad \text{if} \quad \nabla_x R^{*l}(x) \neq 0;$$

$$\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 R^{*l}}{\partial x_i \partial x_j} < 0 \quad \text{if} \quad \nabla_x R^{*l}(x) = 0.$$

Notice that $R^{*l}(x)$ can be constructed such that $\nabla_x R^{*l}(x)$ is nonzero if $x \in \Gamma_l^{e_0} \cup \Gamma_l$, $x \neq \xi_{lj}$, and $\nabla_x R^{*l}(x)$ is not normal to $\Gamma_l$ if $x \in \Gamma_l$, $x \neq \xi_{lj}$. Hence, if $L$ is nondegenerate in $\Gamma_l^{e_0}$ and if the stochastic equations induced by (6.1) on $\Gamma_l$ have a nondegenerate diffusion matrix [i.e., if the elliptic operator induced by $L'$ on $\Gamma_l$ is nondegenerate] at each $x \neq \xi_{lj}$, then $\nabla_x R^{*l}(x) \neq 0$ if $x \in \Gamma_l^{e_0} \cup \Gamma_l$, $x \neq \xi_{lj}$ and (6.2) holds.

So far, assuming $(A_m)$, $(P_m)$, and $(C^*)$, we already know that $x(t)$ does not intersect $\Sigma_1$ in finite time, and $\text{dist}(x(t), \Sigma_1) \to 0$ as $t \to \infty$, provided $x(0) \in G$ for all $t > 0$. We now employ the assumptions (R), (S) to construct a function

$$f(x) = \Phi(R^{*l}(x)) \quad (x \in \Gamma_l^{e_0} \cup \Gamma_l, \; x \neq \xi_{lj})$$

such that $Lf \leq -\nu$ (as in Theorem 2.2 of [4]), and then use it to deduce that, on the set where $\text{dist}(x(t), \Gamma_l) \to 0$, $\min_j |x(t) - \xi_{lj}| \to 0$.

We shall denote the set of all the points $\xi_{lj}$ by $\xi_1, \ldots, \xi_k$, and call them distinguished boundary points. We pose the Dirichlet problem

$$Lu = 0 \quad \text{in} \; G,$$

$$u = g \quad \text{on} \; \Sigma_2 \cup \Sigma_3,$$

$$u = f_i \quad \text{at} \; \xi_i \quad (1 \leq i \leq k)$$

where $g$ is a given continuous function on $\Sigma_2 \cup \Sigma_3$ and the $f_i$ are given numbers.

Theorem 6.1. Let the conditions $(A_m)$, $(P_m)$, $(C^*)$ and (R), (S) hold, and let $L$ be nondegenerate in $G$. Then there exists a unique solution of the Dirichlet problem (6.4)–(6.6).
In fact, the solution is given by

\[ u(x) = E_x \left[ f(x(r)) \right]_{r < \infty} + \sum_{i=1}^{k} f_i p_i(x) \]

where \( r \) is the exit time from \( G \), and \( p_i(x) \) is the probability that \( x(t) \in G \) for all \( t > 0 \) and \( x(t) \to \zeta_i \) as \( t \to \infty \), given \( x(0) = x \). The regularity of the terms on the right-hand side of (6.7) can be proved as in the case \( m = 2 \) (in §4).

Remark 1. Theorem 6.1 can be extended to the case where \( L \) may degenerate in a small neighborhood of a finite number of "interior spokes," as in the case \( m = 2 \). This can be proved by the same method as for \( m = 2 \).

Remark 2. Theorem 6.1 extends to the Dirichlet problem in which (6.4) is replaced by \( L u + c(x) u = 0 \) in \( G \), and \( c(x) \leq 0 \); cf. the remark at the end of §5.

6.2. Let the conditions \( (A_m) \), \( (P_m) \), \( (C^*_m) \) hold and consider the Dirichlet problem

\[ \begin{align*}
L u &= 0 \quad \text{in } G, \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \Sigma_2 \cup \Sigma_3, \\
u &= f_i \quad \text{on } \Gamma_i \ (1 \leq i \leq p)
\end{align*} \]

where the \( f_i \) are constants; the \( \Gamma_i \ (1 \leq i \leq p) \) constitute the \( \Sigma^* \) boundary of \( G \). If \( L \) is nondegenerate in \( G \), then the function

\[ u(x) = E_x \left[ g(x(r)) \right]_{r < \infty} + \sum_{i=1}^{p} f_i q_i(x) \]

is the unique classical solution of the Dirichlet problem (6.8). The proof of uniqueness is the same as the proof of Theorem 5.2. The proof that \( u(x) \) is in \( C^2(G) \) is the same as the corresponding proof for \( p^+_i(x) \). The assertion that \( u(x) \to f_i \) is \( \text{dist} (x, \Gamma_i) \to 0 \) is a consequence of Theorem 3.3 (which holds in any number of dimensions). Finally the assertion that \( u(x) \to g(y) \) if \( x \to y, y \in \Sigma_2 \cup \Sigma_3 \) follows from [6, Theorem 2].

6.3. All of the results of §§2–5 can be generalized to the case \( m \geq 3 \). The conditions needed, however, take a more complicated form. In order to clarify the procedure, we shall describe only a special case, namely, \( m = 3 \) and \( G \) is a shell given by \( 1 < r < 2 \). We further assume that the conditions of Lemma 1.1 hold for \( m = 3 \) so that the assertion of Lemma 1.1 is valid, with respect to the system in polar coordinates

\[ \begin{align*}
\bar{a} r &= \sum_{j=1}^{n} \bar{a}_j dw^j + \bar{b}_1 dt, \\
\bar{a}\theta &= \sum_{j=1}^{n} \bar{a}_j dw^j + \bar{b}_2 dt, \\
\bar{a}\phi &= \sum_{j=1}^{n} \bar{a}_j dw^j + \bar{b}_3 dt.
\end{align*} \]
On \( t = 1 \), this system reduces to

\[
\begin{align*}
\frac{d\theta}{dt} &= \sum_{j=1}^{n} \mathcal{G}_{2j}(1, \theta, \phi) dw^j + \widetilde{\mathcal{G}}_{2}(1, \theta, \phi) dt, \\
\frac{d\phi}{dt} &= \sum_{j=1}^{n} \mathcal{G}_{3j}(1, \theta, \phi) dw^j + \widetilde{\mathcal{G}}_{3}(1, \theta, \phi) dt.
\end{align*}
\]

(6.11)

We shall assume

(T1) Along the closed curve \( \Gamma: (\theta = \theta'_0, 0 \leq \phi \leq 2\pi) \) we have

\[
\sum_{i,j=2}^{3} a_{ij} \nu_i \nu_j = 0, \quad \sum_{i=2}^{3} \left[ \frac{d}{d\theta} \sum_{j=1}^{2} \frac{\partial^2}{\partial \phi^2} \right] \nu_i = 0
\]

when \( \theta_1 = \theta, \theta_2 = \phi, (\nu_2, \nu_3) \) is normal to \( \Gamma \), and \( \widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}^{*} \).

(T2) The condition (C*) holds with \( x \) replaced by \( \langle \theta_1, \theta_2 \rangle \) and \( R(x) \) replaced by a positive function \( R^*(\theta_1, \theta_2) \) coinciding with the distance function from \( \Gamma \) when the latter is sufficiently small.

(T3) Define \( Q(\theta_1, \theta_2) \) with respect to (6.11) and \( R(\theta_1, \theta_2) \) in the same way that \( Q(x) \) was defined with respect to the system (1.2) with respect to \( R(x) \). Then \( Q(\theta_1, \theta_2) \leq -\nu < 0 \) \( (0 \leq \phi \leq 2\pi, |\theta - \theta'_0| < \epsilon') \) for some \( \epsilon' > 0 \).

(T4) No solution of (6.10) crosses the conical surface \( S: \theta = \theta'_0 \). This is the case if and only if \( \Sigma_{j=1}^{3} a_{ij} \nu_i = 0, \Sigma_{j=1}^{3} \left[ b_i - \frac{1}{2} \Sigma_{j=1}^{2} \frac{\partial^2}{\partial \phi^2} \right] \nu_i = 0 \) on \( \theta = \theta'_0 \), where \( \nu_i \) is the normal to \( S \).

Using the condition (T1) we can prove (as in [4]) that the solution \( (\theta(\tau), \phi(\tau)) \) of (6.11) never crosses \( \Gamma \). Using also the conditions (T2), (T3) we can construct a function \( V(\theta, \phi) \) for \( \theta \neq \theta'_0 \) such that \( LV \leq -\nu < 0 \) and \( V \to -\infty \) if \( \theta \to \theta'_0 \). If we apply Itô's formula to \( V(\theta(t), \phi(t)) \), where \( (r(t), \theta(t), \phi(t)) \) is a solution of the system (6.10) with \( \theta(0) \neq \theta'_0 \), then we conclude (as in [4]) that

\[
r(t) \to 1, \quad \theta(t) \to \theta'_0 \quad (t \to \infty).
\]

Next we consider the restriction of (6.11) to \( \theta = \theta'_0 \), i.e.

\[
\frac{d\phi}{dt} = \sum_{j=1}^{n} \mathcal{G}_{3j}(1, \theta'_0, \phi) dw^j + \widetilde{\mathcal{G}}_{3}(1, \theta'_0, \phi) dt
\]

and assume

(T5) Conditions analogous to (D), (E) and (Q) hold for (6.11) with respect to \( \Gamma \).

Thus, through each point where \( \widetilde{\mathcal{G}}_{3j}(1, \theta'_0, \phi) = 0 \) \( (1 \leq j \leq n) \) and \( \widetilde{\mathcal{G}}_{3}(1, \theta'_0, \phi) = 0 \) there passes a "boundary spoke" \( \Lambda_j \) lying in the sphere. \( \Lambda_j \) connects a pair of adjacent \( \alpha_j \)'s. With the aid of the condition (T5), we construct a function \( f(\phi) \) with \( Lf \leq -\nu < 0 \) as in §§6 and 7. Applying Itô's formula to \( f(\phi(t)) \) where \( (r(t), \theta(t), \phi(t)) \) is a solution of (6.10), we can then show that \( \phi(t) \to \alpha^\pm_i \)
(1 \leq i \leq k) \) with probability \( p_i^\pm(x) (\sum_{i=1}^{k} [p_i^+(x) + p_i^-(x)] = 1) \).

The points \( \zeta_i = (1, \theta_0, a_i) \) are called distinguished boundary points. We can now pose the Dirichlet problem

\[
Lu = 0 \quad \text{in } G,
\]

\[
 u(x) \rightarrow f_i^\pm, \quad x \rightarrow \zeta_i^\pm \quad (1 \leq i \leq k).
\]

Theorem 6.2. Let the assumptions \((T_1 - T_5)\) hold and let \( L \) be nondegenerate in \( G - S \). Then there exists a unique solution of the Dirichlet problem (6.13). It is given by the formula

\[
u(x) = \sum_{i=1}^{k} f_i^+ p_i^+(x) + \sum_{i=1}^{k} f_i^- p_i^-(x) \quad (x \in S).
\]

This theorem extends easily to general domains \( G \) with a \( \Sigma_2 \cup \Sigma_3 \) boundary component.

A subsequent treatment by one of us (M.P.) shows that condition \((D)\) is superfluous. The details will appear in a forthcoming publication.

REFERENCES


