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UNCOMPLEMENTED $C(X)$-SUBALGEBRAS OF $C(X)$

BY

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ABSTRACT. In this paper, the uncomplemented subalgebras of the Banach algebra $C(X)$ which are isometrically and algebraically isomorphic to $C(X)$ are investigated. In particular, it is shown that if X is a 0-dimensional compact metric space with its ωth topological derivative $X^{(\omega)}$ nonempty, then there is an uncomplemented subalgebra of $C(X)$ isometrically and algebraically isomorphic to $C(X)$.

For each ordinal $\alpha \geq 1$, a class $\mathcal{C}_\alpha$ of homeomorphic 0-dimensional uncountable compact metric spaces is introduced. It is shown that each uncountable 0-dimensional compact metric space contains an open-and-closed subset which belongs to some $\mathcal{C}_\alpha$.

1. Introduction. Let X and Y be topological spaces and ϕ be a (continuous) map from X onto Y. The induced linear operator ϕ^0 is the multiplicative isometric isomorphism from $C(Y)$ into $C(X)$ that takes $f \in C(Y)$ into $f\phi$. A major result is

Theorem 4.6. *If X contains an open, 0-dimensional compact metric subspace K with its ωth topological derivative $K^{(\omega)}$ nonempty, then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is uncomplemented in $C(X)$.*

Observe that the hypothesis for X is satisfied by all uncountable, 0-dimensional compact metric spaces (e.g., the Cantor set $\mathcal{C}$) and by the space $\Gamma(\alpha)$ of ordinals not exceeding α provided $\alpha \geq \omega^\omega$.

If ϕ is a map from X onto Y then an *averaging operator* for ϕ is a continuous linear operator μ from $C(X)$ into $C(Y)$ satisfying $\mu\phi^0(f) = f$ for $f \in C(Y)$. It is easy to see that ϕ admits an averaging operator for ϕ if and only if there is a projection P of $C(X)$ onto its subalgebra $\phi^0[C(Y)]$ where μ and P are related by $P = \phi^0\mu$ [21, Corollary 2.3]. As with most of the results of this paper, the conclusion to Theorem 4.6 can be stated in terms of averaging operators (i.e., there is a map of X onto itself which does not admit an averaging operator).

The $\mathcal{C}_\alpha$-spaces introduced in §4 are formed by adding rays $\Gamma_0(\alpha) = \{\beta \mid \beta$ is an ordinal, $\beta < \alpha\}$ to the Cantor set $\mathcal{C}$ so that each point in $\mathcal{C}_\alpha$ is the limit of a

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ray $\Gamma_0(\alpha)$. A homeomorphic classification of $\mathcal{C}_\alpha$-spaces similar to the classical homeomorphic classification of the Cantor set is established (Lemma 4.3). The fact that each nondispersed 0-dimensional compact metric space contains an open-and-closed $\mathcal{C}_\alpha$-space for some α provides a new technique for working with these spaces.

In the final section of this paper, two applications of the "uncomplemented $C(X)$-subalgebras of $C(X)$" results are included. Theorem 5.1 and Theorem 5.2 are "uncomplemented" analogoues of A. Pełczyński's "complemented $C(S)$-subspaces" theorems in his paper, *On $C(S)$-subspaces of separable Banach spaces* [22].

2. **Preliminaries.** The notation and terminology is that of Dunford and Schwartz's *Linear operators*. I [14] and Kelley's *General topology* [18] with the following exception: A *decomposition* D of a topological space X is a disjoint collection of *closed* subsets of X such that $X = \bigcup\{A: A \in D\}$ and the quotient space is denoted by X/D. An isomorphism μ between two Banach algebras which is multiplicative (i.e., $\mu(fg) = \mu(f)\mu(g)$) is called an *algebra* isomorphism.

For each subspace S of X let $S^{(1)}$ denote the set of all accumulation points of S which are contained in S. Then $S^{(1)}$ is the complement in S of the set of points which are isolated in the relative topology of S. If λ is an ordinal, the *topological derivative of order* λ of S, denoted $S^{(\lambda)}$, is defined by transfinite induction as follows: $S^{(0)} = S$, $S^{(\lambda)} = (S^{(\alpha)})^{(1)}$ if $\lambda = \alpha + 1$, and $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$ if λ is a limit ordinal.

The first limit ordinal is denoted by ω and the first uncountable ordinal by Ω. We assume that all maps are continuous and that all topological spaces are Hausdorff.

3. **Construction of uncomplemented subalgebras.** This section is closely related to the author's work in [4]. Most of the terminology and notations used in that paper are needed in this section and are used without being redefined.

If D is a decomposition of a topological space X and q is the quotient map D, then q^0 is an isometric isomorphism from $C(X/D)$ onto the subalgebra of $C(X)$ consisting of the functions which are constant on each set in D. Frequently, this subalgebra of $C(X)$ is identified with $C(X/D)$ without specific reference to the isomorphism q^0. We follow Arens [3] and write $D = 0$ when D has no plural sets. The abbreviation u.s.c. is used for upper semicontinuous.

If Z is a D-saturated subset of X, then the restriction of the decomposition D to Z is denoted D_Z. If D is u.s.c., then D_Z is u.s.c. and the identity map of Z/D_Z onto $q(Z)$ is a homeomorphism (see [10, I, §5.2, Proposition 4, p. 54]). If Z is normal, then Z/D_Z is normal [15, p. 85] and $q(Z)$ is a normal subset of X/D.

Lemma 3.1 and Proposition 3.2 are basically the same as Lemma 1.2 and Theorem 1.3, respectively, in [4]. The main difference is that the assumption that X is normal is replaced by the assumption that a subspace Z of X is normal.

The proofs of these two results are omitted, since their proofs are similar to the corresponding proofs in [4] and the only additional information needed is contained in the preceding paragraph. The purpose of Lemma 3.1 is to replace Lemma 1.2 of [4] in the proof of Proposition 3.2.

Lemma 3.1. *Let X be a topological space and let D be an u.s.c. decomposition of X. Suppose there is a D-saturated, normal subspace Z of X and a plural set Y of D in* Int (Z) *such that D is contracting at Y and the boundary ∂Y of Y contains at least n points. If P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geq 3 - 2/n$.*

Moreover, if $\epsilon > 0$, if U is a neighborhood of Y and if $y_1, y_2, \cdots, y_n$ are distinct points in ∂Y, then there exist an i and a neighborhood V of y_i such that for each t in $V \sim Y$ there exists f in $C(X)$ with $f|(X \sim U) = 0$, $\|f\| = f(t) = 1$, and $Pf(t) > 3 - 2/n - \epsilon$.

The following proposition establishes a lower bound for the norms of projections of $C(X)$ onto a $C(Y)$-subalgebra of $C(X)$. This proposition demonstrates that the existence of repeated limits of plural sets in the decomposition that Y induces on X can increase the norm of projections from $C(X)$ onto this $C(Y)$-subalgebra. This result substantially generalizes R. Arens's "$3 - 2/n$ lower bound theorem" [3, Theorem 3.1] and extends a similar result obtained independently by S. Ditor to noncompact spaces [12, Corollary 5.4]. However, Ditor's result is more general in the compact case. The definition of $L_n(m_1, m_2, \cdots, m_n)$ is given in [4].

Proposition 3.2. *Let D be an u.s.c. decomposition of a topological space X and Z a normal subspace of X such that* Int (Z) *is D-saturated. If $D_{\mathrm{Int}(Z)}$ has property $L_n(m_1, m_2, \cdots, m_n)$ and P is a projection of $C(X)$ onto $C(X/D)$, then*

$$\|P\| \geq 2n + 1 - \sum_{i=1}^{n} 2/m_i.$$

Remarks 3.3. The "upper semicontinuous" requirement in Theorem 1.3 in [4] was inadvertently omitted.

Let ϕ be a map of a compact space X onto a compact (Hausdorff) space Y and let Δ_ϕ be the decomposition $\{\phi^{-1}(y) | y \in Y\}$ of X. Then Δ_ϕ is u.s.c., since ϕ is closed. It is interesting to observe that if Δ_ϕ has property $L_n(m_1, m_2, \cdots, m_n)$, then using Ditor's definition in [12], $\Delta_\phi^{(n)}(m_1, m_2, \cdots, m_n) \neq \emptyset$. Therefore, either by Proposition 3.2 or by Corollary 5.4 in [12], an averaging operator U for ϕ has

$$\|U\| \geq 2n + 1 - \sum_{i=1}^{n} 2/m_i = 1 + 2 \sum_{i=1}^{n} (1 - 1/m_i).$$

The next lemma is a more general form of the author's Lemma 2.4 in [4]. Both this lemma and the construction given in its proof are essential in the proofs of the theorems of this paper.

Lemma 3.3. (Construction of subspaces of $C(X)$ with high projection norm). *Let X be a topological space and n a positive integer. Suppose Z is a normal subspace of X and S is a subset of* Int (Z) *with $S^{(n)} \neq \emptyset$ such that each point in $S^{(1)}$ has a countable neighborhood base. Then for each positive integer $k > 1$ there exists an u.s.c. decomposition D of X such that*

(1) *Each plural set in D consists of k elements of S.*

(2) *X/D is Hausdorff. Moreover, X/D is, respectively, normal, compact, first-countable, or compact and metrizable, provided X has the corresponding property.*

(3) *If q is the quotient map from X onto X/D, then q^0 is an algebraic isometric isomorphism from $C(X/D)$ into $C(X)$.*

(4) *The decomposition $D^{(j)}$ of X contains a plural set if and only if $j < n$.*

(5) *If P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geq 2n - 1 - (2n - 2)/k$.*

Proof. Let $x \in S^{(n)}$ and let G be a closed neighborhood of x included in Int (Z). By induction, we select nonempty families $C_1, C_2, \cdots, C_{n+1}$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_{n+1}$ of subsets of Int (G) such that if $1 \leq m \leq n+1$ (and $C_0 = \emptyset$) then

(a) C_1 consists of a singleton subset of S and each set in C_m for $m > 1$ consists of k elements from $S^{(n-m+1)}$.

(b) If $a \in A$ for some $A \in C_m$ and U is a neighborhood of a, then U includes a set in C_{m-1} [i.e., a is an accumulation point of sets in C_{m-1}].

(c) $\mathfrak{U}_m$ is a family of disjoint, closed subsets such that for each A in C_m, there is a neighborhood U_A of A in $\mathfrak{U}_m$ which does not include any other set in C_m.

(d) If $U \in \mathfrak{U}_m$ then U does not intersect any set in C_j for $1 \leq j < m$.

(e) $\mathfrak{U}_m$ implies $\mathfrak{U}_{m-1}$ for $m > 1$.

(f). The decomposition D_m of X consisting of the plural sets in $(\bigcup_{i=1}^{n-1} C_i) \cup \mathfrak{U}_m$ is contracting and each set in $\mathfrak{U}_m$ is a nonlimit set of D_m.

Let C_1 be the family consisting of the singleton set $\{x\}$ and let $\mathfrak{U}_1 = \{G\}$. It is easy to see that conditions (a)–(f) are satisfied for $m = 1$.

Next, suppose $C_1, C_2, \cdots, C_m$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_m$ have been selected and $m \leq n$. Let A be a set in C_m. We may suppose $\overline{A} = \{a_1, a_2, \cdots, a_z\}$ where each a_i is in $S^{(n-m+1)}$ where $z = 1$ if $m = 1$ and $z = k$ if $m > 1$. There exists a neighborhood U_A of A in $\mathfrak{U}_m$ which does not intersect any other set in C_m. There is a family $\{U_i\}_{i=1}^{z}$ of closed disjoint sets such that for each i, U_i is a neighborhood of a_i and is a subset of U_A. Let $\{V_{ij}\}_{j=1}^{\infty}$ be a closed monotone neighborhood base for a_i with $V_{ij} \subset U_i$ for each j. In fact, we can suppose $\{V_{ij}\}_{j=1}^{\infty}$ is selected so that, for each i and j, $V_{ij} \sim V_{i(j+1)}$ is a neighborhood of a point a_{ij} in $S^{(n-m)}$. Then there is a family $\{W_{ij}\}_{j=1}^{\infty}$ of disjoint closed sets such that W_{ij} is a neighborhood of a_{ij} included in $V_{ij} \sim V_{i(j+1)}$. Then, we define

$$A_{ij} = \{a_{i(jk+r)} | 1 \leq r \leq k\} \text{ for } 1 \leq i \leq z,$$

$$U(A_{ij}) = \bigcup_{r=1}^{k} W_{i(jk+r)} \text{ for } 1 \leq i \leq z \text{ and } j = 0, 1, 2, \cdots,$$

$$C_{m+1} = \{A_{ij} | A \in C_m, 1 \leq i \leq z, \text{ and } j = 0, 1, 2, \cdots\},$$

$$\mathfrak{U}_{m+1} = \{U(A_{ij}) | A \in C_m, 1 \leq i \leq z, \text{ and } j = 0, 1, 2, \cdots\}.$$

Using these definitions, it is easy to see that hypotheses (a)–(e) are satisfied by C_{m+1} and $\mathfrak{U}_{m+1}$. The proof that (f) is satisfied is given in [4, paragraphs 2 and 3, p. 96]. This completes the inductive selection of $C_1, C_2, \cdots, C_{n+1}$ and $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_{n+1}$.

Let D be the decomposition of X consisting of the plural sets in $\bigcup_{j=1}^{n+1} C_{.j}$. It follows from Lemma 2.2 in [4] that since D_{n+1} is contracting, D is also contracting. (Let $M = D_{n+1}$ in Lemma 2.2.) This selection of D is easily seen to satisfy conclusions (1) and (3). Let q denote the quotient map of D. Since D is u.s.c. and G is a neighborhood of each plural set in D, D_G (the restriction of D to G) is u.s.c. and the identity map of G/D_G onto $q(Z)$ is a homeomorphism [10, I, §5.2, Proposition 4, p. 53]. Since G is normal, G/D_G is normal and $q(G)$ is a normal subspace of X/D. A similar argument with G replaced by Z shows $q(Z)$ is a normal subspace of X/D. But $q(G) \subset q(\text{Int } Z) = \text{Int } q(Z)$, and q is a homeomorphism on $X \sim G$. Since $X \sim G$ is Hausdorff, it follows that X/D is Hausdorff. If X is normal or compact, then X/D has the corresponding property [15, pp. 85 and 104]. If X is both compact and metrizable, it follows by a theorem of K. Morita and S. Hanai [19, Theorem 1] and also by A. H. Stone [26, Theorem 1] that X/D is also compact and metrizable. If X is first countable, X/D is also first countable since each decomposition set contains at most k elements. Thus D satisfies conclusion (2) also.

The following generalization of (5) is established next:

(5′) If M is an u.s.c. decomposition of X such that M contains each plural set of D and M is contracting at each plural set in D, then each projection P of $C(X)$ onto $C(X/M)$ has $\|P\| \geq 2n - 1 - (2n - 2)/k$.

To establish (5′), we let $S_i = D^{(i)} \sim D^{(i+1)}$ for $i = 1, 2, \cdots, n - 1$. Since each set in S_i is a plural set in D and M is contracting at each of these sets, M satisfies property $L_{n-1}(k, k, \cdots, k)$. By Proposition 3.2, each projection

P of $C(X)$ onto $C(X/M)$ has $\|P\| \geq 2n - 1 - (2n-2)/k$. This proves (5′). Letting $D = M$, we obtain (5).

To establish (4), observe that it follows by induction that the decomposition $D^{(m)}$ satisfies the following properties provided $0 \leq m \leq n$:

(i) $\bigcup_{j=2}^{n-m+1} C_j$ is the family of plural sets in $D^{(m)}$.

(ii) The family of plural sets in C_{n-m+1} is the set of nonlimit plural sets in $D^{(m)}$.

By (i), $D^{(m)}$ contains a plural set if and only if $n - m + 1 \geq 2$ or $m \leq n - 1$. This proves (4).

In case the collection $S^{(n)}$ in Lemma 3.3 contains an isolated point with respect to its subset topology, the decomposition D of Lemma 3.3 can be selected so that two additional properties are satisfied.

Lemma 3.4. *If $S^{(n)} \sim S^{(n+1)}$ is nonempty in Lemma 3.3, then the decomposition D can be also selected so that*

(1′) *Each plural set of $D^{(j)} \sim D^{(j+1)}$ consists of k elements of $S^{(j)} \sim S^{(j+1)}$.*

(6) *For each ordinal number α, $t \in S^{(\alpha)}$ if and only if $q(t) \in q(S)^{(\alpha)}$.*

Proof. If $S^{(n)} \sim S^{(n+1)}$ is nonempty, then "$S^{(t)}$" can be replaced with "$S^{(t)} \sim S^{(t+1)}$" for each t in the proof of Lemma 1. [In this case, let $x \in (S^{(n)} \sim S^{(n+1)})$.] By the revised form of inductive hypothesis (a), it follows that each plural set in C_{n-m+1} consists of k-points from $S^{(m)} \sim S^{(m+1)}$. This establishes (1′).

Next, we establish (6) by transfinite induction. It is obvious if $\alpha = 0$. Suppose (6) is valid for all $\alpha < \gamma$ where $1 < \gamma$. Let $x \in S^{(\gamma)}$. Then for each $\alpha < \gamma$, there exists a sequence $\{x_n\}$ of distinct points in $S^{(\alpha)}$ such that $x_n \to x$. By inductive hypothesis, $\{q(x_n)\} \subset q(S)^{(\alpha)}$ and since $q(x_n) \to q(x)$, $q(x) \in q(S)^{(\gamma)}$.

Conversely, let $q(x) \in q(S)^{(\gamma)}$. Then, for each $\alpha < \gamma$, there exists a sequence $\{y_n\}$ of distinct points in $q(S)^{(\alpha)}$ such that $y_n \to q(x)$. Choose $x_n \in S$ with $q(x_n) = y_n$. By inductive hypothesis, $x_n \in S^{(\alpha)}$. If $q(x)$ is a singleton set, then $x_n \to x$ and it follows that $x \in S^{(\gamma)}$. If x is a plural set, then $\gamma < \omega$ by (1′) and $x = \{z_1, z_2, \cdots, z_k\}$ for some choice of z_i in S. Let $U_1, U_2, \cdots, U_k$ be disjoint neighborhoods of $z_1, z_2, \cdots, z_k$, respectively. Since D is contracting, we may assume each U_i is $(D \sim \{q(x)\})$-saturated. As before, there is a sequence $\{x_n\}$ of distinct points in $S^{(\gamma-1)}$ such that $q(x_n) \to q(x)$. But an infinite subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is included in some U_j. Then $x_{n_i} \to z_j$, so $z_j \in S^{(\gamma)}$. But by (1′), $z_j \in S^{(\gamma)}$ implies $\{z_1, z_2, \cdots, z_k\} \subset S^{(\gamma)}$. Since $x \in q(x)$, $x \in S^{(\gamma)}$.

Lemmas 3.5 and 3.6 are, respectively, the uncomplemented analogues of Lemmas 3.3 and 3.4.

Lemma 3.5. (Construction of uncomplemented subspaces of $C(X)$). *Suppose Z is a normal subspace of a topological space X and S is a subset of* Int (Z) *with $S^{(\omega)} \neq \emptyset$ such that each point in $S^{(1)}$ has a countable neighborhood base. Then for each positive integer $k > 1$ there exists an u.s.c. decomposition such that conclusions* (1)–(3) *of Lemma* 3.3 *are satisfied. Moreover,*

(4) *The decomposition $D^{(j)}$ of X contains a plural set if and only if $j < \omega$.*

(5) *The subspace $C(X/D)$ of $C(X)$ is uncomplemented in $C(X)$.*

Proof. Suppose $x \in S^{(\omega)}$. Let $\{O_n\}_{n=1}^{\infty}$ be an open monotone neighborhood base for x with $O_1 \subset \text{Int}(Z)$. We may assume this neighborhood base is selected so that $O_n \sim O_{n+1}$ is a neighborhood of a point $x_n \in S^{(n)}$. Let E_n be a closed neighborhood of x_n included in $O_n \sim O_{n+1}$. If R is the decomposition of X consisting of the plural sets $\{E_n\}_{n=1}^{\infty}$, then R is clearly contracting. Let $S_n = S \cap \text{Int}(E_n)$. Since $x_n \in S_n$, it follows by (5$'$) of Lemma 3.3 that there is a contracting decomposition M_n of X with each plural set in M_n a subset of S_n such that if M is a contracting decomposition containing each plural set in M_n and P is a projection of $C(X)$ onto $C(X/M)$, then $\|P\| \geq n$. Moreover, each M_n can be selected so that each plural set contains exactly k points of S_n. Let D be the decomposition consisting of the plural sets in M_n. By Lemma 2.2 in [4], D is contracting. Thus, there does not exist a projection P of $C(X)$ onto $C(X/D)$, since $\|P\| \geq n$ for each positive integer n is impossible. This proves (5). Parts (1) and (3) are trivial, and the proof of (2) is the same as in Lemma 3.3. Part (4) follows, since the decomposition M_n has the property that $M_n^{(j)}$ contains a plural set if and only if $j < n$.

Lemma 3.6. *If $S^{(\omega)} \sim S^{(\omega+1)}$ is nonempty in Lemma* 3.5, *then the decomposition D can also be selected so that*

(1$'$) *Each plural set of $D^{(j)} \sim D^{(j+1)}$ consists of k elements of $S^{(j)} \sim S^{(j+1)}$.*

(6) *For each ordinal number α, $t \in S^{(\alpha)}$ if and only if $q(t) \in q(S)^{(\alpha)}$.*

Proof. If $S^{(\omega)} \sim S^{(\omega+1)}$ is nonempty, then the point x in the proof of Lemma 3.5 can be selected from $S^{(\omega)} \sim S^{(\omega+1)}$. Also, the points x_n in that proof can be selected from $S^{(n)} \sim S^{(n+1)}$. Then by Lemma 3.4, each of the decompositions M_n in the proof of Lemma 3.5 can be selected so that M_n satisfies (1$'$). Since each plural set of D is contained in some M_n, (1$'$) is established.

The proof of (6) is identical to the proof of (6) in Lemma 3.4.

4. **Existence of uncomplemented $C(X)$-subalgebras of $C(X)$.** In this section, we apply the lemmas of the last section to certain topological spaces X to construct a subalgebra of $C(X)$ isometrically and algebraically isomorphic to $C(X)$ that either has a large lower bound for the norms of projections from $C(X)$ or is uncomplemented in $C(X)$.

A topological space X is *dispersed* (*scattered*) if it does not contain a perfect subset. If X is dispersed, there is a least ordinal α such that $X^{(\alpha)}$ is either finite or empty. The *characteristic* (characteristic system) of X is the ordered pair (α, n) where n is the cardinality of X. Note that $n \geq 1$ if X is compact. If ξ is an ordinal, $\Gamma(\xi)$ denotes the space of ordinals not exceeding ξ with the interval topology and $\Gamma_0(\xi)$ denotes the subspace of $\Gamma(\xi)$ consisting of the ordinals strictly less that ξ.

Theorem 4.1. *Let n be a positive integer. Suppose a topological space X includes a compact first-countable set K such that* Int $(K)^{(n)}$ *contains infinitely many isolated points. Then for each $\epsilon > 0$, there is a map ϕ of X onto itself such that if P is a projection of $C(X)$ onto $\phi^0[C(X)]$, then $\|P\| \geq 2n + 1 - \epsilon$.*

Proof. Let x be an accumulation point of the set of isolated points in Int $(K)^{(n)}$. Suppose $\{U_j\}_{j=1}^{\infty}$ is a neighborhood base for x. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of distinct isolated points of Int $(K)^{(n)}$ with x_j in U_j. By induction, there is a sequence $\{V_j\}_{j=1}^{\infty}$ of disjoint closed sets such that V_j is a neighborhood of x_j included in Int $(K) \cap U_j$. Since $X^{(n+1)}$ is closed and $X^{(n)} \sim X^{(n+1)}$ is discrete in its subset topology, we may assume $V_j \cap X^{(n)} = \{x_j\}$. Each V_i is dispersed and compact; hence, it is 0-dimensional [20]. Thus, there exists an open-and-closed (in V_j) neighborhood W_j of x_j included in Int (V_j). Since $W_j \subset$ Int (V_j), W_j is open-and-closed in X. Let $W = \bigcup_{i=1}^{\infty} W_i$ and $S = W \cup \{x\}$. The set S is first-countable, compact and has characteristic $(n + 1, 1)$, so it follows by a theorem due to Z. Semadeni [24] (see [5, Corollary 2]) that S is homeomorphic to $\Gamma(\omega^{n+1})$.

Let $\epsilon > 0$ and choose a positive integer k sufficiently large so that $(2n)/k < \epsilon$. Since $S^{(n+1)} \neq \emptyset$, it follows by Lemma 3.3 that there is an u.s.c. decomposition D of X such that each projection P of $C(X)$ onto $C(X/D)$ has $\|P\| \geq 2n + 1 - (2n)/k > 2n + 1 - \epsilon$. Since $S^{(n+1)} \sim S^{(n+2)} = \{x\}$, it follows by Lemma 3.4 that D can also be selected so that if q denotes the quotient map, then $x \in S^{(\alpha)}$ if and only if $q(x) \in q(S)^{(\alpha)}$ for each ordinal α. Thus, $q(S)$ also has characteristic $(n + 1, 1)$. The subset $q(S)$ is compact because S is compact. But $q(S)$ is first-countable as each plural set of D is finite, so, by [5, Corollary 2], $q(S)$ is also homeomorphic to $\Gamma(\omega^{n+1})$.

Let μ be a homeomorphism of S onto $q(S)$. As topological derivatives are preserved by homeomorphisms [23, Lemma 3.1 (e)], $\{\mu(x)\} = \mu(S^{(n+1)}) = \mu(S)^{(n+1)} = q(S)^{(n+1)} = \{q(x)\}$ and $\mu(x) = q(x)$. Since x is the only possible accumulation point of $X \sim S$ contained in S, we can extend μ to a map of X into $q(X)$ by letting $\mu(x) = q(x)$ for x in $X \sim S$. Then μ is one-to-one and onto X/D. Since $q(x)$ is a singleton set and q^{-1} is continuous on $q(X) \sim q(S)$, μ is a homeomorphism of X onto X/D.

Thus, $\phi = \mu^{-1} \circ q$ is a map of X onto X and $\phi^0[C(X)] = q^0[C(X/D)]$. Consequently, each projection P of $C(X)$ onto $\phi^0[C(X)]$ has $\|P\| \geq 2n + 1 - \epsilon$.

Our first "uncomplemented $C(X)$-algebra of $C(X)$" result is contained in the next theorem. This theorem is essentially the uncomplemented version of Theorem 4.1.

Theorem 4.2. *Suppose a topological space X includes a first-countable compact subset K such that* Int $(K)^{(n)}$ *contains an isolated point for each positive integer n. Then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is an uncomplemented subalgebra of $C(X)$.*

Proof. Let t_n be an isolated point in Int $(K)^{(n)}$ for each positive integer n. Let x be an accumulation point of $\{t_n\}_{n=1}^{\infty}$. Suppose $\{U_n\}$ is a neighborhood base for x. For each n, let x_n be an isolated point of Int $(K)^{(n)}$ contained in U_n. By induction, there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of disjoint sets such that V_n is a neighborhood of x_n included in Int $(K) \cap U_n$. Since $X^{(n+1)}$ is closed in its subset topology, we may assume $V_n \cap X^{(n)} = \{x_n\}$ for each n.

The remainder of the proof follows the proof of Theorem 4.1, except that $n + 1$ is replaced with ω and Lemmas 3.5 and 3.6 are used in place of Lemmas 3.3 and 3.4 respectively. □

Next, for each denumerable ordinal number α, we construct a compact subspace $\mathcal{C}_\alpha$ of the unit interval satisfying the two following properties: (1) $(\mathcal{C}_\alpha)^{(\alpha)} = \mathcal{C}$, and (2) each point in the subset $\mathcal{C}$ of $\mathcal{C}_\alpha$ is the limit of a well-ordered sequence in $\mathcal{C}_\alpha \sim \mathcal{C}$ homeomorphic to $\Gamma_0(\omega^\alpha)$. Let $I_{n,k}$ denote the kth open interval removed (counting from left to right) in the nth step of the construction of the Cantor set $\mathcal{C}$ [17, p. 70]. Let $a_{n,k}$ and $b_{n,k}$ denote the left and right endpoint, respectively, of $I_{n,k}$ and let $m_{n,k}$ be the midpoint of $I_{n,k}$. In each interval $[a_{n,k}, m_{n,k})$ select a well-ordered sequence $A_{n,k} = \{a_\mu\}_{\mu<\omega^\alpha}$ homeomorphic to $\Gamma_0(\omega^\alpha)$ with $\lim_{\mu<\omega^\alpha} a_\mu = a_{n,k}$. Similarly, select a well-ordered sequence $B_{n,k} = \{b_\mu\}_{\mu<\omega^\alpha}$ in $(m_{n,k}, b_{n,k}]$ homeomorphic to $\Gamma_0(\omega^\alpha)$ with $\lim_{\mu<\omega^\alpha} b_\mu = b_{n,k}$. Then let $\mathcal{C}_\alpha$ be the subspace of $[0, 1]$ defined by

$$\mathcal{C}_\alpha = \mathcal{C} \cup \left[\bigcup_{n,k=1}^{\infty} (A_{n,k} \cup B_{n,k}) \right].$$

Clearly, the choice of each $A_{n,k}$ and $B_{n,k}$ is possible but not unique. The fact that $\mathcal{C}_\alpha$ is independent up to homeomorphism of the choices of $A_{n,k}$ and $B_{n,k}$ is established in Proposition 4.4. It follows from Lemma 1 in [5] and the construction of $\mathcal{C}_\alpha$ that $\mathcal{C}_\alpha$ satisfies properties (1) and (2) of the preceding paragraph. Compact metric spaces which satisfy (1) and (2) will be called $\mathcal{C}_\alpha$-spaces. More specifically,

Definition. Let α be an ordinal number. A topological space X is called a $\mathcal{C}_\alpha$-*space* if and only if X is an uncountable 0-dimensional compact metric space, $X^{(\alpha)}$ is perfect, and each point in $X^{(\alpha)}$ is the limit of a well-ordered sequence in $X \sim X^{(\alpha)}$ homeomorphic to $\Gamma_0(\omega^\alpha)$.

If X is a topological space, there is a least ordinal λ such that $X^{(\lambda)} = X^{(\lambda+1)}$. Let Ker $(X) = X^{(\lambda)}$ and observe that Ker (X) is the largest perfect subset of X.

An alternate characterization of $\mathcal{C}_\alpha$-spaces is given by the following lemma.

Lemma 4.3. *Let X be a compact 0-dimensional metric space. If α is an ordinal number, then X is a $\mathcal{C}_\alpha$-space if and only if*

$$\mathrm{Ker}(X) \subset \mathrm{Cl}\,[X^{(\gamma)} \sim \mathrm{Ker}(X)]$$

for each $\gamma < \alpha$, but $X^{(\alpha)} = \mathrm{Ker}\,(X) \neq \emptyset$.

Proof. Suppose X is a $\mathcal{C}_\alpha$-space. Since $X^{(\alpha)}$ is perfect, $X^{(\alpha)} = \mathrm{Ker}\,(X)$. By Lemma 1 in [5] it follows that $\mathrm{Ker}\,(X) \subset \mathrm{Cl}\,[X^{(\gamma)} \sim \mathrm{Ker}\,(X)]$ for each $\gamma < \alpha$.

Conversely, suppose $\mathrm{Ker}\,(X) \subset \mathrm{Cl}\,[X^{(\gamma)} \sim \mathrm{Ker}\,(X)]$ for each $\gamma < \alpha$, but $X^{(\alpha)} = \mathrm{Ker}\,(X) \neq \emptyset$. Thus, $X^{(\alpha)}$ is perfect. The proof that each point in $X^{(\alpha)}$ is the limit of a well-ordered sequence of points in $X \sim X^{(\alpha)}$ homeomorphic to $\Gamma_0(\omega^\alpha)$ is similar to the proof of Theorem 2 in [7] and is omitted.

Each uncountable, 0-dimensional, perfect, compact metric space is homeomorphic to $\mathcal{C}$. The following proposition establishes a similar homeomorphic characterization for each $\mathcal{C}_\alpha$.

Proposition 4.4. *All $\mathcal{C}_\alpha$-spaces are homeomorphic.*

Proof. Suppose X and Y are $\mathcal{C}_\alpha$-spaces. Since $\mathrm{Ker}\,(X) = X^{(\alpha)}$ and $\mathrm{Ker}\,(Y) = Y^{(\alpha)}$ are nonempty, 0-dimensional, perfect metric spaces, they are both homeomorphic to the Cantor set. Thus, there is a homeomorphism ϕ of $\mathrm{Ker}\,(X)$ onto $\mathrm{Ker}\,(Y)$. Let $G_1 = X \sim \mathrm{Ker}\,(X)$ and $G_2 = Y \sim \mathrm{Ker}\,(Y)$. Since $G_1^{(\gamma)}$ and $G_2^{(\gamma)}$ are infinite for each $\gamma < \alpha$ and $G_1^{(\alpha)} = G_2^{(\alpha)} = \emptyset$, it follows by Theorem 3 in [5] that both G_1 and G_2 are homeomorphic to $\Gamma_0(\omega^\alpha)$. By Lemma 4.3, $\phi[\mathrm{Ker}\,(X) \cap \mathrm{Cl}[X^{(\gamma)} \sim \mathrm{Ker}\,(X)]] = \phi(\mathrm{Ker}\,X) = \mathrm{Ker}\,(Y) = \mathrm{Ker}\,(Y) \cap \mathrm{Cl}[Y^{(\gamma)} \sim \mathrm{Ker}\,(Y)]$ for each $\gamma < \alpha$. Since $\mathrm{Cl}[X^{(\gamma)} \sim \mathrm{Ker}\,(X)] = \mathrm{Cl}[Y^{(\gamma)} \sim \mathrm{Ker}\,(X)] = \emptyset$ for all $\gamma \geq \alpha$, it follows that $\phi(\mathrm{Ker}\,(X) \cap \mathrm{Cl}[X^{(\gamma)} \sim \mathrm{Ker}\,(X)]) = \mathrm{Ker}\,(Y) \cap \mathrm{Cl}[Y^{(\gamma)} \sim \mathrm{Ker}\,(Y)]$ for each ordinal number γ. By Theorem 1.1 in [23], ϕ can be extended to a homeomorphism of X onto Y.

It is well known that a nondispersed compact metric space X contains a subset K homeomorphic to the Cantor set. However, consideration of the case $X = \mathcal{C}_1$ indicates that even when X is 0-dimensional, it is sometimes impossible to select K to be open in X. The next lemma establishes that if X is 0-dimensional, then one can find a compact open subset K of X homeomorphic to $\mathcal{C}_\alpha$ for some α.

Lemma 4.5. *Let X be an uncountable 0-dimensional compact metric space. Then X contains an open-and-closed subset homeomorphic to $\mathcal{C}_\alpha$ for some countable ordinal number α.*

Proof. Let α be the least ordinal such that there exists an open-and-closed, nondispersed subset Y of X with $Y^{(\alpha)}$ perfect [25, Theorem 4.7]. Let Y be such a subspace of X. Then for each y in $\mathrm{Ker}\,(Y)$ and each open-and-closed neighborhood U of y included in Y, it follows by the minimality of α that $U^{(\gamma)}$ is perfect if and only if $\gamma \geq \alpha$. Thus, $[Y^{(\gamma)} \sim \mathrm{Ker}\,(Y)] \cap U$ is nonempty and $y \in \mathrm{Cl}[Y^{(\gamma)} \sim \mathrm{Ker}\,(Y)]$ for each $\gamma < \alpha$. Since this is true for each $y \in \mathrm{Ker}\,(Y)$,

$$\mathrm{Ker}(Y) \subset \mathrm{Cl}\,[Y^{(\gamma)} \sim \mathrm{Ker}(Y)]$$

for each $\gamma < \alpha$. Therefore, by Lemma 4.3, Y is a $\mathcal{C}_\alpha$-space.

The second "uncomplemented $C(X)$-subalgebra of $C(X)$" result is stated in the next theorem. In contrast to Theorem 4.2, this theorem is applicable to 0-dimensional compact metric spaces with a finite nonempty perfect derivative such as $\mathcal{C}$, the free union $\mathcal{C}_1 + \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ [13, p. 127], and $\Gamma(\Omega) \times \mathcal{C}$. However, Theorem 4.6 does not include Theorem 4.2, since it is easy to construct spaces which satisfy the hypothesis of Theorem 4.2 but not of Theorem 4.6 (e.g., the subset $(\mathcal{C}_{\omega^\omega} \times \{0\}) \cup (\mathcal{C} \times [0, 1])$ of the unit square).

Theorem 4.6. *If a topological space X contains an open, 0-dimensional compact metric subspace K with $K^{(\omega)} \neq \emptyset$, then there is a map ϕ of X onto itself such that $\phi^0[C(X)]$ is uncomplemented in $C(X)$.*

Proof. If $K^{(n)}$ contains an isolated point for each positive integer n, this result follows by Theorem 4.2. Therefore, we may suppose $K^{(n)}$ is nonempty and perfect for some n By Lemma 4.5, there is an open-and-closed subset Y of K homeomorphic to $\mathcal{C}_t$ for some integer t. Let $S = \mathrm{Ker}\,(Y)$. By Lemma 3.5 there is a decomposition D of X with each plural set of D a subset of S such that $C(X/D)$ is uncomplemented in $C(X)$.

Let q be the quotient map of D. We show $q(Y)$ is 0-dimensional by examining the construction of D in the proofs of Lemma 3.3 and Lemma 3.5. The notation used in these constructions will be preserved. Let $y \in Y$ and suppose V is a neighborhood of $q(y)$ in $q(Y)$. If $y = x$ where x is the point selected in the proof of Lemma 3.5, then there exists n such that O_n is included in the neighborhood $q^{-1}(V)$ of x. Then $q(O_n)$ is an open-and-closed neighborhood of $q(y)$ included in V.

Next, suppose y belongs to the complement of the closed set $F = (\bigcup_{n=1}^{\infty} E_n) \cup \{x\}$ where E_n and x are defined in the proof of Lemma 3.5. Then there is an open-and-closed neighborhood W of y contained in $q^{-1}(V)$ which does not intersect F. In this case, $q(W)$ is an open-and-closed subset of V in $q(Y)$ containing $q(y)$.

Finally, we suppose y belongs to E_n for some n and restrict our attention to the selection of M_n made in the proof of Lemma 3.3. This M_n was constructed so that each plural set of M_n would be a subset of $S_n = S \cap \operatorname{Int}(E_n)$. Suppose $q(y)$ belongs to one of the disjoint families C_m selected in the proof of Lemma 3.3. Then $q(y) = \{a_1, a_2, \cdots, a_z\}$ where $z = 1$ if $m = 1$ and $z = k$ if $m > 1$. We may assume that each set in the monotone neighborhood basis $\{V_{ij}\}$ of each a_i selected in that proof is open-and-closed. Recall the sets W_{ij} were selected so that $W_{ij} \subset V_{ij} \sim V_{i(j+1)}$. Since $q^{-1}(V)$ is a neighborhood of $q(y)$, there is a positive integer p such that $V_{ip} \subset q^{-1}(V)$ for $i = 1, 2, \cdots, z$. If $W = \bigcup_{i=1}^{z} V_{i(pk+1)}$, it is obvious from the construction of M_n in Lemma 3.3 that $q(W)$ is an open-and-closed neighborhood of $q(y)$ contained in V.

On the other hand, suppose $q(y)$ is not a set in any C_i. Then, by the construction of M_n, $q(y)$ is a nonlimit singleton set in D and there is an open-and-closed neighborhood W of y included in $q^{-1}(V)$ which does not intersect any plural set of D. In this case, $q(W)$ is an open-and-closed neighborhood of $q(y)$ included in V. This completes the proof that $q(Y)$ is 0-dimensional.

Since Y is compact and q is continuous, $q(Y)$ is compact. By a theorem of K. Morita and S. Hanai [19, Theorem 1] and also of A. H. Stone [26, Theorem 1], $q(Y)$ is metrizable. To establish $q(Y)$ is a $\mathcal{C}_t$-space, it remains to be shown that $q(Y)^{(t)}$ is a nonempty perfect set and each point in $q(Y)^{(t)}$ is the limit of a well-ordered sequence of points in $q(Y) \sim q(Y)^{(t)}$ homeomorphic to $\Gamma_0(\omega^t)$. Recall that each plural set in D is a subset of $\operatorname{Ker}(Y)$. Consequently,

$$\operatorname{Ker} q(Y) = q(\operatorname{Ker} Y) \tag{1}$$

and

$$q(Y) \sim \operatorname{Ker} q(Y) = q(Y \sim \operatorname{Ker} Y). \tag{2}$$

Since $\operatorname{Ker} q(Y)$ is perfect and the restriction of q to $Y \sim \operatorname{Ker}(Y)$ is a homeomorphism, it follows from the fact that topological derivatives are preserved by homeomorphisms [23, Lemma 2.1 (e)] and equalities (1) and (2) above that

$$\begin{aligned} q(Y)^{(t)} &= [q(Y) \sim \operatorname{Ker} q(Y)]^{(t)} \cup [\operatorname{Ker} q(Y)] \\ &= q([Y \sim \operatorname{Ker} q(Y)]^{(t)}) \cup [\operatorname{Ker} q(Y)] = \operatorname{Ker} q(Y). \end{aligned}$$

Thus, $q(Y)^{(t)}$ is perfect.

Next, suppose z is in $q(Y)^{(t)}$. By equality (1) above, there exists $y \in \operatorname{Ker}(Y)$ with $q(y) = z$. Let $\{x_\mu\}_{\mu<\omega^t}$ be a well-ordered sequence in $Y \sim \operatorname{Ker}(Y)$ homeomorphic to $\Gamma_0(\omega^t)$ which converges to z. Since the restriction of q to $Y \sim \operatorname{Ker}(Y)$ is a homeomorphism, it follows from line (2) that $\{q(y_\mu)\}_{\mu<\omega^t}$ is a well-ordered sequence in $q(Y) \sim \operatorname{Ker} q(Y)$ homeomorphic to $\Gamma_0(\omega^t)$ which converges to z. This completes the proof that $q(Y)$ is homeomorphic to $q(Y)$ (see Proposition 4.4). Since Y is an open-and-closed set and q is a homeomorphism of $X \sim Y$ onto $q(X) \sim q(Y)$, X is homeomorphic to $q(X)$.

Corollary 4.7. *If X is a 0-dimensional compact metric space with $X^{(\omega)} \neq \emptyset$, then there exists a map ϕ onto itself such that $\phi^0[C(X)]$ is an uncomplemented subalgebra of $C(X)$.*

5. Applications of uncomplemented $C(X)$-subspaces of $C(X)$. The next two theorems are the uncomplemented analogues to Theorem 1 and Theorem 1a, respectively, in [22] as they essentially replace "complemented" in the two theorems by A. Pełczyński with "uncomplemented".

Theorem 5.1. *Let S be a compact metric space with $S^{(\omega)} \neq \emptyset$. If a Banach space X contains a subspace Y isomorphic to $C(S)$, then there is a subspace Z of Y such that Z is isomorphic to $C(S)$ and Z is not complemented in X.*

Proof. Let μ be an isomorphism of $C(S)$ onto Y. First, suppose S is countable. Then S is dispersed, and, by Theorem 4.2, there is a map ϕ of X onto itself such that $\phi^0[C(S)]$ is uncomplemented in $C(S)$. In this case, $\mu\phi^0[C(S)]$ is a subset of Y isomorphic to $C(S)$ which is not complemented in X.

Next suppose S is uncountable. By Milutin's Theorem (see [21, Theorem 8.5] or [11]) there is an isomorphism ν of $C(\mathcal{C})$ onto $C(S)$. By Corollary 4.7 there is a map ϕ of $\mathcal{C}$ onto itself such that $\phi^0[C(\mathcal{C})]$ is uncomplemented in $C(\mathcal{C})$. Then $\mu\nu\phi^0[C(\mathcal{C})]$ is an uncomplemented subspace of X included in Y.

Theorem 5.2. *Let S be a 0-dimensional compact metric space with $S^{(\omega)} \neq \emptyset$. If a Banach space X contains a subspace Y (algebraically) isometrically isomorphic to $C(S)$, then there is a subspace Z of Y such that Z is (algebraically) isometrically isomorphic to $C(S)$ and Z is not complemented in $C(S)$.*

Proof. Let μ be an (algebraic) isometric isomorphism of $C(S)$ onto Y. By Corollary 4.7 there is a map ϕ of S onto itself such that $\phi^0[C(S)]$ is uncomplemented in $C(Y)$. Then $\mu\phi^0[C(S)]$ is an uncomplemented subspace of $C(X)$ contained in Y which is (algebraically) isometrically isomorphic to $C(Y)$.

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