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ON A COMPACTNESS PROPERTY OF TOPOLOGICAL GROUPS

BY

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ABSTRACT. A density theorem of semisimple analytic groups acting on locally compact groups is presented.

Let G and H be locally compact groups with G acting continuously on H as a group of automorphisms. An element h of H is said to be *G-bounded* if the orbit $Gh = \{g(h): g \in G\}$ has compact closure in H. We write $F_G(H)$ for the set of all G-bounded elements in H. It is very easy to verify that $F_G(H)$ is a G-invariant subgroup of H. However in general, $F_G(H)$ is not closed in H. In this paper, we shall study the group $F_G(H)$ for certain topological groups G. Our main result is the following

Theorem. *Let G be a semisimple analytic group without compact factors acting on a locally compact group H continuously as a group of automorphisms. If the set $F_G(H)$ is dense in H, then G acts trivially on H.*

The theorem generalizes some results in [2], [4] and is closely related to the density property of certain subgroups in semisimple analytic groups without compact factors. The result of Corollaries 4.1 and 4.2 is contained in [2], [4].

In the sequel, we shall use the term "G acts on H" for "G acts on H as a group of automorphisms".

1. Minimally almost periodic groups. Let G be a locally compact group. We recall that G is *minimally almost periodic* if there are no nontrivial continuous homomorphisms $f: G \longrightarrow G'$ of locally compact groups such that the closure $\mathrm{Cl}(f(G))$ of $f(G)$ in G' is compact. Minimally almost periodic groups have been widely studied. Yet for our need, we shall establish some lemmas concerning minimally almost periodic groups.

Lemma 1.1. *Let G be a minimally almost periodic group acting continuously on a locally compact group H, and N be a closed G-invariant normal subgroup of H.*

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If the set $F_G(H)$ *is dense in* H, *and* G *acts trivially on both* N *and* H/N, *then* G *acts trivially on* H.

Proof. Let h be any fixed element of $F_G(H)$. As G acts trivially on H/N, there is a continuous function $f: G \longrightarrow N$ such that $g(h) = hf(g)$, $g \in G$. Since G acts trivially on N, we have

$$\begin{aligned} hf(g'g) &= (g'g)h = g'(gh) = g'(hf(g)) \\ &= g'(h)f(g) = hf(g')f(g) \qquad (g',\ g \in G). \end{aligned}$$

Hence f is a continuous homomorphism. We know that $\mathrm{Cl}(Gh)$ is compact. $\mathrm{Cl}(f(G))$, being contained in $h^{-1}\mathrm{Cl}(Gh)$, is evidently compact. Therefore f has to be trivial; equivalently $g(h) = h$ for every g in G. Since the set $F_G(H)$ is dense in H, it follows readily that G acts trivially on H.

Lemma 1.2. *Let* G *be a connected minimally almost periodic group acting continuously on a locally compact abelian group* A. *If the set* $F_G(A)$ *is dense in* A, *then* G *acts trivially on* A.

Proof. First we assume that A is compactly generated. In this case, A has a unique maximal compact subgroup K. Obviously K is characteristic, hence G-invariant. By a well-known theorem of Iwasawa [3], the automorphism group $\mathrm{Aut}(K)$ of K with compact-open topology is totally disconnected, hence G acts trivially on K. By Lemma 1.1, we may assume that $K = \{e\}$ and A is an abelian Lie group. Let A° be the identity component of A. Since G is connected, G acts trivially on A/A°. Again by Lemma 1.1, we may even assume that A is connected. Under these additional assumptions, $A = R^l$ for some positive integer l. Now we pick out a basis $\{e_i, \cdots, e_l\}$ of R^l from $F_G(R^l)$. This is possible because $F_G(R^l)$ is dense in R^l. With respect to this basis, for every g in G, we write

$$g(e_i) = \sum_{j=1}^{l} g_{ji} e_j \qquad (1 \leq i \leq l),$$

with g_{ji} in R. It is easy to show that the map $g \longrightarrow (g_{ij})$ $(g \in G)$ is a continuous homomorphism f of G into $\mathrm{GL}(l, R)$. Since all the entries g_{ij} $(1 \leq i, j \leq l, g \in G)$ are bounded, we conclude $f(G)$ has compact closure in $\mathrm{GL}(l, R)$. Hence f has to be trivial, and the lemma is proved in case that A is compactly generated. For the general case, G acts trivially on A/A°, hence G leaves any open subgroup of A invariant. Let N be a compactly generated open subgroup of A. Clearly $F_G(N) = F_G(A) \cap N$ is still dense in N. By what we have just proved, G acts trivially on N and by Lemma 1.1, the proposition follows.

Remark. In the preceding lemma, we assume only that the set $F_G(A)$ is dense in A. In general we do not know whether the set $F_G(A^\circ) = F_G(A) \cap A^\circ$ is dense in A°. That is why we consider first compactly generated open subgroups of A rather than the subgroup A°.

Corollary 1.3. *Let G be a connected minimally almost periodic group and L a closed subgroup of G with compact quotient G/L. Let A be a locally compact abelian group such that G acts continuously on A. If L leaves an element x of A fixed, then x is fixed by G.*

Proof. Consider the group $\mathrm{Cl}(F_G(A))$. By Lemma 1.2, G acts trivially on $\mathrm{Cl}(F_G(A))$. Clearly x lies in $F_G(A)$ and the corollary follows.

Corollary 1.3 reveals at least some density property of those subgroups L of G with compact quotient G/L. In general, the structure of minimally almost periodic groups is not entirely clear. However for connected groups, we have the following criterion. The result must be known but we offer a proof here for completeness.

Lemma 1.4. *Let G be a connected locally compact group. The following statements are equivalent:*

(i) *G is minimally almost periodic.*

(ii) *G is an analytic group such that $[G, G]$ is dense in G and $G/R(G)$ has no compact factors where $R(G)$ is the radical of G.*

Proof. (i) $\Longrightarrow$ (ii) Since G is a connected locally compact group, locally G is the direct product of a compact group and a local Lie group. However G is minimally almost periodic, hence G is a Lie group. Consider then the groups $G/\mathrm{Cl}([G, G])$ and $G/R(G)$. $G/R(G)$ (resp. $G/\mathrm{Cl}([G, G])$) is minimally almost periodic semisimple (resp. abelian minimally almost periodic). (ii) follows immediately.

(ii) $\Longrightarrow$ (i) By a well-known theorem of von Neumann, any topological group G contains a unique minimal closed normal subgroup N such that G/N is maximally almost periodic, i.e., there is a continuous injection of G/N into a compact group K. Hence it suffices to show that G/N is trivial in our case. Clearly G/N still satisfies all the assumptions in (ii). But, by a theorem of Freudenthal, a connected maximally almost periodic locally compact group is the direct product of a compact group and a vector group. Hence one concludes readily that G/N is trivial, i.e., $G = N$ is minimally almost periodic.

2. **Cross homomorphisms.** Let G be a locally compact group acting on a locally compact abelian group A continuously. A continuous map $f\colon G \longrightarrow A$ is called a *cross homomorphism* if f satisfies the condition

$$f(gg') = g(f(g')) + f(g)$$

for all g, g' in G. Given any v in A, the map $d_v\colon G \longrightarrow A$, defined by $d_v(g) = gv - v$ $(g \in G)$ clearly is a cross homomorphism. A cross homomorphism f is said *to be homologous to* 0 if $f = d_v$ for some v in A.

Lemma 2.1. *Let G be a semisimple analytic group acting on a locally compact abelian group A continuously. Then any cross homomorphism $f\colon G \longrightarrow A$ is homologous to 0.*

Proof. Let e be the identity element of G. Since f is a cross homomorphism, $f(e) = 0$. Hence $f(G)$ is contained in A° because f is continuous. Therefore we may even assume that A is connected. Let K be the unique maximal compact subgroup of A. Clearly K is G-invariant and G acts trivially on K. f induces then a cross homomorphism $\bar{f}: G \longrightarrow A/K$. A/K is isomorphic to R^l for some positive integer l. It is well known that $\bar{f}$ is homologous to 0. Hence there exists an element v in A such that

$$f(g) \equiv gv - v \pmod{K}, \qquad g \in G.$$

Let $f_1: G \longrightarrow K$ be the map defined by

$$f_1(g) = f(g) - gv + v, \qquad g \in G.$$

One verifies readily that f_1 is a cross homomorphism. Since G acts trivially on K, f_1 is a homomorphism, hence $f_1(G) = \{0\}$. Thus $f = d_v$ is homologous to 0.

3. **Linear Lie groups.** Let $GL(n, C)$ (resp. $\mathfrak{gl}(n, C)$) be the group of all n by n nonsingular complex matrices (resp. the Lie algebra of all n by n complex matrices). Clearly $\mathfrak{gl}(n, C)$ is the Lie algebra of $GL(n, C)$ and the exponential map $\exp: \mathfrak{gl}(n, C) \longrightarrow GL(n, C)$ is just the usual one. Let λ be any positive number. We denote by $\mathfrak{gl}(n, C; \lambda)$ the set of all elements X in $\mathfrak{gl}(n, C)$ such that the imaginary parts of all the eigenvalues of X lie in the open interval $(-\lambda, \lambda)$. Let G be any Lie subgroup of $GL(n, C)$ and $\mathfrak{g}$ its Lie algebra. We write $\mathfrak{g}_\lambda$, G_λ and $\exp_\lambda$ for $\mathfrak{g} \cap \mathfrak{gl}(n, C; \lambda)$, $\exp(\mathfrak{g}_\lambda)$ and the restriction of $\exp_\lambda$ on $\mathfrak{g}_\lambda$ respectively.

Lemma 3.1 [4]. *The maps* $\exp_\lambda$ $(0 < \lambda \leq \pi)$ *are diffeomorphisms.*

Proposition 3.2. *Let G be a semisimple analytic subgroup of* $GL(n, C)$ *and H a Lie subgroup of* $GL(n, C)$. *Suppose that*

(i) *G has no compact factors,*

(ii) *G normalizes H, and*

(iii) *$F_G(H)$ is dense in H where G acts on H through conjugation.*

Then G centralizes H.

Proof. Let λ be any positive number smaller than π. By Lemma 3.1, $\exp_\lambda: \mathfrak{h} \longrightarrow H_\lambda$ is a diffeomorphism. Clearly $\mathfrak{h}_\lambda$ is G-invariant under conjugation. Since $F_G(H)$ is dense in H, there is a basis $\{X_1, \cdots, X_r\}$ of $\mathfrak{h}$ such that $X_i \in \mathfrak{h}_\lambda$ and $\exp X_i \in F_G(H)$ $(1 \leq i \leq r)$. Let Ad be the adjoint representation of $GL(n, C)$ on $\mathfrak{gl}(n, C)$. Then with respect to this basis, all elements in the group $\mathrm{Ad}(G)|_\mathfrak{h}$ have bounded entries because $\exp X_i \in F_G(H)$ $(1 \leq i \leq n)$ and $\exp_\lambda$ is a diffeomorphism. Hence $\mathrm{Ad}(G)|_\mathfrak{h}$ has compact closure. By (i) and Lemma 1.4, G centralizes H°. Clearly G acts trivially on H/H° for H/H° is discrete and G is connected. By Lemma 1.1, G acts trivially on H, therefore G centralizes H.

4. **Proof of the theorem.** We prove the theorem in several steps.

(i) G leaves invariant any open subgroup of H. Since H/H° is discrete and G is connected, G acts trivially on H/H°. Clearly H° is contained in any open subgroup of H. Hence (i) follows easily.

(ii) We may assume that H is an analytic group. Let H_1 be an open subgroup of H such that H_1 is a projective limit of Lie groups. Let K be a normal compact subgroup of H_1 such that H_1/K is a Lie group. Then consider $H_2 = H_1^\circ K$. H_2 is again an open subgroup of H. It is well known that a connected locally compact group has a unique maximal compact normal subgroup. It follows that H_2 also has a unique maximal normal compact subgroup L. By (i) H_2 is G-invariant, hence L is also G-invariant. Since L is compact, $\mathrm{Aut}(L)^\circ =$ the inner automorphism group by a theorem of Iwasawa [3]. Therefore $\mathrm{Aut}(L)^\circ$ is compact. The action of G on L is induced by a continuous homomorphism $f\colon G \longrightarrow \mathrm{Aut}(L)$. Clearly $f(G)$, being contained in $\mathrm{Aut}(L)^\circ$, has compact closure. By Lemma 1.4, $f(G)$ is trivial, i.e., G acts trivially on L. Therefore by Lemma 1.1, we may even assume that $H = H_2/L$ is an analytic group.

(iii) By (ii) we assume further that H is an analytic group. Let $M = G \cdot H$ be the semidirect product of G and H. Let Ad be the adjoint representation of M on its Lie algebra. Passing over to $\mathrm{Ad}(M)$, by Proposition 3.2, one concludes that given any $h \in H$

$$g(h) = hs(g), \qquad g \in G,$$

where $s(g)$ is in the center $Z(H)$ of H. By a direct calculation, $s\colon G \longrightarrow Z(H)$ is a cross homomorphism. By Lemma 2.1, s is homologous to 0. Hence there is $z \in Z(H)$ with $s(g) = g(z^{-1})z$ for all $g \in G$. Now consider the element hz. Clearly $g(hz) = hz$ for all $g \in G$. Let F be the set of all fixed points of H. Clearly F is a closed subgroup of H. By what we have just proved, $F \cdot Z(H) = H$. Hence F is normal and H/F is abelian. By Proposition 1.2, G acts trivially on H/F. By Lemma 1.1, G acts trivially on H. Therefore the proof of the theorem is hereby completed.

Corollary 4.1. *Let G be analytic semisimple group without compact factors, and g an element of G. If the conjugacy class $\{xgx^{-1}: x \in G\}$ has compact closure in G, g is in the center $Z(G)$ of G.*

Proof. G acts on G through conjugation. By the theorem $F_G(G) = Z(G)$. Clearly g is in $F_G(G)$.

Corollary 4.2. *Let G be an analytic semisimple group without compact factors and α an automorphism of G. If the subset $\{\alpha(g)g^{-1}: g \in G\}$ has compact closure then α is the identity map.*

Proof. Let $\omega: G \longrightarrow \mathrm{Aut}(G)$ be the homomorphism defined by $\omega(g)(x) = g x g^{-1}$, $(g, x \in G)$. Clearly G acts on $\mathrm{Aut}(G)$ through ω and conjugation, and $\alpha \in F_G(\mathrm{Aut}(G))$. By the theorem, G leaves α fixed, i.e. $\omega(\alpha(g)) = \alpha\omega(g)\alpha^{-1} = \omega(g)$ for all g in G. It follows then $\alpha(g)g^{-1}$ is in the center $Z(G)$ of G and the map $g \longrightarrow \alpha(g)g^{-1}$ $(g \in G)$ is a homomorphism of G into $Z(G)$. Since G is semi-simple, this map has to be trivial. Therefore $\alpha(g) = g$ for all g in G, i.e., α is the identity map of G.

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