B-CONVEXITY AND REFLEXIVITY IN BANACH SPACES

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ABSTRACT. A proof of James that uniformly nonsquare spaces are reflexive is extended in part to B-convex spaces. A condition sufficient for non-B-convexity and related conditions equivalent to non-B-convexity are given. The following theorem is proved: A Banach space is B-convex if each subspace with basis is B-convex.

0. Introduction. The notion of a B-convex Banach space was introduced by A. Beck [1], [2] as a characterization of those Banach spaces X having the property that a certain strong law of large numbers holds for X valued random variables.

Definition. Let k be a positive integer and ε a positive number. X is said to be k, ε-convex if for any \( \{x_1, \ldots, x_k\} \), \( \|x_i\| \leq 1, \ i = 1, \ldots, k \), there is some choice of signs \( \xi_1, \ldots, \xi_k \) so that \( \|\sum_{i=1}^{k} \xi_i x_i\| \leq k(1 - \varepsilon) \). X is said to be B-convex if it is k, ε-convex for some k and ε.

Further study of B-convex spaces has been done by R. C. James [6], [7], D. P. Giesy [5] and C. A. Kottman [8]. Giesy showed that B-convex spaces have many of the properties of reflexive spaces. James conjectured that all B-convex spaces are reflexive, and proved the conjecture true for 2, ε-convex spaces. Both James and Giesy proved the conjecture true for B-convex spaces having an unconditional basis. Kottman extended James' 2, ε-convex proof to a larger subclass, P-convex spaces. Examples are known of spaces which are reflexive but not B-convex.

§1 of this paper adopts a part of James' 2, ε-convex theorem to all non-B-convex spaces, presents a condition sufficient for non-B-convexity, and gives related characterizations of non-B-convex spaces, though the conjecture of James remains open. §2 proves a theorem on B-convexity and subspaces with basis analogous to a theorem of Pelczyński on reflexivity and subspaces with basis.

For a Banach space X, \( U(X) \) will denote the closed unit ball \( \{x: \|x\| \leq 1\} \) of X.

I. Non-B-convexity. In James' proof [6] that 2, ε-convex spaces are reflexive, he defines for a Banach space X a sequence of numbers \( K_n \), and shows that if X

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is not reflexive then $K'_n \leq 2n$, and in that case $X$ cannot be $2$, $\varepsilon$-convex. We will extend the second step of this result to show that if $K'_{n'}$ is a bounded sequence then $X$ cannot be $B$-convex. The numbers $K'_n$, to be defined almost the same as James' $K_n$, will be used instead of $K_n$. Another condition, which implies that $\{K'_n\}$ is bounded, is introduced and is shown to be sufficient for non-$B$-convexity by a much simpler proof.

Let $X$ be a Banach space. For each sequence $\{f_i\}$ of continuous linear functionals with unit norms and each increasing sequence of integers $\{p_1, \ldots, p_{2n}\}$, let $S(p_1, \ldots, p_{2n}; \{f_i\})$ denote the set of all $x$ such that, for all $k$ and $i$, $3/4 \leq (-1)^{i-1}k(x)$ if $p_{2i-1} \leq k \leq p_{2i}$, and $1 \leq i \leq n$. Let

$$K(n, \{f_i\}) = \liminf_{p_1 \to \infty} \liminf_{p_2 \to \infty} \cdots \liminf_{p_{2n} \to \infty} \inf \{\|z\| : z \in S(p_1, \ldots, p_{2n}; \{f_i\})\}$$

and

$$K'_{n'} = \inf \{K(n, \{f_i\}) : \|f_i\| = 1 \text{ for all } f\}.$$

James' definition of $K_n$ is similar. It follows from the definitions that $K'_{n'} \leq K_n$ and $K'_{n'} \leq K'_{n+1}$ for all $n$.

**Theorem 1.1.** If the sequence $\{K'_n\}$ for a Banach space $X$ is bounded, then $X$ is not $B$-convex.

**Proof.** For any positive integer $k$ and any $0 < \delta < 2$ we will show $X$ is not $k$, $\delta$-convex by showing there are $x_1, \ldots, x_k \in U(X)$ such that for any choice of signs $\xi_1, \ldots, \xi_k$ we have $\|\sum_{i=1}^k \xi_i x_i\| > k(1 - \delta)$. Since the sequence $\{K'_n\}$ is bounded, and monotone, we can choose $m$ such that $K'_{2m}/K'_{3m2k} > 1 - \delta/3$. Let $3m2^k = M$. Choose $\mu, \{f_i\}$ where $\|f_i\| = 1$, and $\varepsilon$ such that $0 < \mu < (K'_M)^2\delta/3K'_{2m}$, $K'_M + \mu > K(M, \{f_i\})$ and $0 < \varepsilon < (K'_M)^2\delta/3(K'_{2m} + K'_M)$. From these inequalities, it follows that

$$(K(2m, \{f_i\}) - \varepsilon)/(K(M, \{f_i\}) + \varepsilon) > 1 - \delta.$$

As will be described below, it is possible to choose an increasing set of integers

$P = \{p_{i,j} : i = 1, \ldots, k; j = 1, \ldots, 2M\}$ having the following properties:

(1) For each $i = 1, \ldots, k$ there is $u_i \in S(p_{i,1}, \ldots, p_{i,2M}; \{f_i\})$ such that $\|u_i\| \leq K(M, \{f_i\}) + \varepsilon$.

(2) For each choice of signs $\xi_1, \ldots, \xi_k$ there is an increasing set of integers $\{\sigma_1, \ldots, \sigma_{4m}\} \subset P$ such that

(2a) $(1/k) \sum_{i=1}^k \xi_i u_i \in S(\sigma_1, \ldots, \sigma_{4m}; \{f_i\})$, and

(2b) any element of $S(\sigma_1, \ldots, \sigma_{4m}; \{f_i\})$ has norm greater than or equal to $K(2m, \{f_i\}) - \varepsilon$. 

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Let \( x_i = u_i / K(M, \{ f_i \}) + \epsilon \) for \( i = 1, \ldots, k \). From property (1), \( \|x_i\| \leq 1 \). From property (2), for any choice of signs \( \xi_1, \ldots, \xi_k \) we have
\[
\left\| \frac{1}{k} \sum_{i=1}^{k} \xi_i x_i \right\| \geq \frac{K(2m, \{ f_i \}) - \epsilon}{K(M, \{ f_i \}) + \epsilon} > 1 - \delta
\]
which completes the proof except for the choice of \( P \).

The choice of \( P \) is rather tedious. Integers are chosen successively in \( m \) blocks of increasing integers:
\[
P_1 = \{ p_{i,j} : i = j, \ldots, k; j = 1, \ldots, 2M/m \}
\]
\[
P_2 = \{ p_{i,j} : i = 1, \ldots, k; j = (2M/m) + 1, \ldots, 4M/m \}
\]
\[
\vdots
\]
\[
P_m
\]
Let the \( k \)-tuples of signs \( (\xi_1, \ldots, \xi_k) \) be denoted \( \Xi_1, \ldots, \Xi_{2^k} \). The integers of \( P_1 \) are chosen successively in \( 2^k \) sets of increasing integers \( P_1(\Xi_1), \ldots, P_1(\Xi_{2^k}) \). The number of integers in \( P_1(\Xi_n) \) depends on \( \Xi_n \) as will be shown, four are chosen for each plus sign in \( \Xi_n \) and eight for each minus sign, so that \( P_1 \) has \( 6k2^k = 2km/m \) integers.

Property (2a) is provided by the order of choice of the integers. This order may be illustrated by supposing \( \Xi_n = (\xi_1, \ldots, \xi_k) \) where \( \xi_\eta = -1 \) (\( \eta \neq 1 \) or \( k \)) and \( \xi_i = +1 \) for \( i \neq \eta \). Suppose for each \( i = 1, \ldots, k \), the last integers of \( P_1(\Xi_{n-1}) \) are \( p_{i,j}(i) \). The order of choice of integers of \( P_1(\Xi_n) \) is shown in Figure 1. The integers \( \sigma_1, \ldots, \sigma_4 \) of property (2) for this choice of signs are \( p_{k-1}(k+1), p_1(1+1), p_{k-1}(k+1) \), \( p_{k-1}(k+1) \), \( p_1(1+1) + 1 \). The sets \( P_2, P_3, \ldots, P_m \) are chosen in succession by the same method. The integers \( \sigma_1, \ldots, \sigma_4m \) for a choice of signs \( \Xi_n \) are in the set \( \{ P_1(\Xi_1), \ldots, P_m(\Xi_{2^k}) \} \). Properties (1) and (2b) are provided by requiring that the integers chosen satisfy appropriate inequalities at each step.

The proof of Theorem 1.1 is rather tedious. We now present a strengthening of the condition "\( \|K'\| \) bounded" which will be sufficient for non-B-convexity in a much simpler way.

**Condition 1.** For some \( 0 < \epsilon < 1 \), \( U(X^*) \) contains a sequence \( \{ f_j \} \) such that for any \( m \geq 1 \) and any choice of signs \( \xi_1, \ldots, \xi_m \) there is \( x \in U(X) \) satisfying \( f_j(\xi_j x) > \epsilon, \ j = 1, \ldots, m \).

**Theorem 1.2.** If a Banach space satisfies Condition 1, then the sequence \( \|K'\| \) for that space is bounded.

**Proof.** We will show \( K' \leq 3/4\epsilon \) by showing that for \( \|f_j\| \) as asserted in Con-
Figure 1. Order of choice of $P_1(\Xi_n)$

For order of choice, read down first column, then down second column, etc.

\[
\begin{array}{cccc}
P_{1, j(1)+1} & P_{1, j(1)+2} & P_{1, j(1)+3} & P_{1, j(1)+4} \\
\vdots & \vdots & \vdots & \vdots \\
P_{\eta-1, j(\eta-1)+1} & P_{\eta-1, j(\eta-1)+2} & P_{\eta-1, j(\eta-1)+3} & P_{\eta-1, j(\eta-1)+4} \\
P_{\eta, j(\eta)+1} & P_{\eta, j(\eta)+2} & P_{\eta, j(\eta)+3} & P_{\eta, j(\eta)+4} \\
P_{\eta+1, j(\eta+1)+1} & P_{\eta+1, j(\eta+1)+2} & P_{\eta+1, j(\eta+1)+3} & P_{\eta+1, j(\eta+1)+4} \\
P_{k, j(k)+1} & P_{k, j(k)+2} & P_{k, j(k)+3} & P_{k, j(k)+4} \\
\end{array}
\]

Condition 1, and any $p_1, \ldots, p_{2n}$ there is $y \in S(p_1, \ldots, p_{2n} ; \|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/\|/}\]
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\[ a_k(X^*) = \sup \left\{ \inf \left\{ \frac{1}{k} \sum_{i=1}^{k} \xi_i x_i : \xi_1, \ldots, \xi_k = \pm 1 \right\} : f_1, \ldots, f_k \in U(X^*) \right\} \]

and we may choose \( k \) so that \( a_k(X^*) < \epsilon \). Therefore, for \( f_1, \ldots, f_k \) asserted by Condition 2 there is some choice of signs such that \( \| \sum_{j=1}^{k} \xi_j f_j \| < k \epsilon \). But by Condition 2, \( f_j(\xi_j x) > \epsilon \), so that \( \sum_{j=1}^{k} \xi_j f_j(x) > \epsilon k \) and \( \| \sum_{j=1}^{k} \xi_j f_j \| > k \epsilon \) which is a contradiction.

Condition 3 may be shown equivalent to non-B-convexity by similar proofs.

Corollary 1.4. If a Banach space satisfies Condition 1 it is not B-convex.

We do not know whether nonreflexivity implies \( \| K \| \) bounded or Condition 1. There is however a large class of nonreflexive, non-B-convex spaces satisfying Condition 1, listed in the following easily proved proposition.

Proposition 1.5. The Banach spaces \( c_0, 1_1 \) and all spaces containing \( c_0 \) or \( 1_1 \) satisfy Condition 1.

2. B-convexity and basis. It is known that a Banach space is reflexive if each subspace with basis is reflexive [9]. In this section we show that a Banach space is B-convex if each subspace with basis is B-convex.

As usual we say that \( \{ x_i \} \subset X \) is a basis for \( X \) if for each \( x \in X \) there is a unique sequence of numbers \( \{ a_i \} \) so that \( \lim_{n \to \infty} \| \sum_{i=1}^{n} a_i x_i - x \| = 0 \). It is well known that \( \{ x_i \} \) is a basis for \( X \) if there is some number \( k \) so that for any integers \( n \) and \( q \) and any sequence of numbers \( \{ a_i \} \) we have

\[ \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq k \sum_{i=1}^{n+q} a_i x_i \]

We will prove the main theorem of this section by constructing a basic sequence in an arbitrary non-B-convex space in such a way that the span of this sequence is not B-convex. The technique for construction of this basic sequence is an adaptation of the method of Day [3] and Gelbaum [4]. It relies on the following lemmas:

Lemma 2.1. If \( X \) is finite dimensional, for any \( \epsilon > 0 \) there is \( \| f \|_i \in M \subset U(X^*) \) such that, for any \( x \),

\[ \| x \| \leq (1 + \epsilon) \max_{i=1}^{n} \| f_i \|_i(x) : i = 1, \ldots, n \].

Lemma 2.2. If \( \{ f_i \}_{i=1}^{n} \subset X^* \) and \( Y = \bigcap_{i=1}^{n} \ker f_i(0) \), then \( Y \) is a space of finite codimension in \( X \); that is, there is a finite dimensional subspace \( Z \) so that \( X = Y \oplus Z \).

Theorem 2.3. If \( X \) is not B-convex it contains a subspace with basis not B-convex.
Proof. Let $\{|\delta_i|\}$ be a sequence of positive numbers less than one tending to zero. Let $\{|k_i|\}$ be a sequence of integers tending to infinity. Let $p(0) = 0$, $p(m) = \sum_{i=1}^{m} k_i$, $m = 1, 2, \ldots$.

The subspace to be constructed will be the closed span of a sequence $\{|x_i|\}$ having the following properties:

1. there is a sequence $\{|k_i|\}$ tending to zero so that for each $m = 1, 2, \ldots$, the space $[x_i]_{i=p(m-1)+1}^{p(m)}$ is an $\epsilon_m$ isometric image of $\ell^p_m$, in particular,

$$(1 - \epsilon_m) \sum_{i=p(m-1)+1}^{p(m)} |a_i| \leq \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i|.$$ 

2. For any $\{|\alpha_i|\}$, $\{n_i\}$, $\{q_i\}$,

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq (3 + \delta_n) \left\| \sum_{i=1}^{n+q} a_i x_i \right\|.$$ 

Property (1) shows that the subspace is not $B$-convex, and property (2) shows that $\{|x_i|\}$ is a basis.

For $\{|\delta_i|\}$, $\{|k_i|\}$ as above there are $\{|\epsilon_i|\}$, $\{|\eta_i|\}$ such that

3. $\epsilon_i \rightarrow 0$.

4. If $1 < n \leq p(1)$ then $(1 + \eta_1)/(1 - \epsilon_1) \leq 1 + \delta_n$.

5. If $p(m) < n \leq p(m + 1)$ then

$$1 + \eta_m + (2 + \eta_m + \eta_{m+1})/(1 - \epsilon_{m+1}) \leq 3 + \delta_n.$$ 

$[x_i]_{i=1}^{p(m)}$ will be denoted by $L_{p(m)}$. $\{|x_i|\}$ will be constructed in blocks by induction on $m$. In the induction step, from a previously constructed subspace $\Lambda_{m-1}$, $\{|x_i|_{i=p(m-1)+1}^{p(m)}\}$ will be chosen satisfying (1). Then a subspace $\Lambda_m$ of $\Lambda_{m-1}$, of finite codimension in $X$, will be constructed so that $\Lambda_m \cap L_{p(m)} = 0$ and the projection $P_m : \Lambda_m \oplus L_{p(m)} \rightarrow L_{p(m)}$ satisfies $\|P_m\| < 1 + \eta_m$. Finally (2) will be proved.

Let $m = 1$. Since $X$ is not $B$-convex, we can choose $\{|x_i|_{i=1}^{k_1} \in U(X)\}$ satisfying (1). To construct $\Lambda_1$ choose $\{|x_i|_{i=1}^{q(1)} \in U(L^*_p(1))\}$ by Lemma 2.1 and extend them to $X$ without increase of norm so that if $x \in L_{p(1)}$, 

$$\|x\| \leq (1 + \eta_1) \max \|x_i: i = 1, \ldots, q(1)\|.$$ 

Let $\Lambda_1 = \bigcap_{i=1}^{q(1)} f_i^{-1}(0)$. By Lemma 2.2, $\Lambda_1$ is of finite codimension in $X$, and $L_{p(1)} \cap \Lambda_1 = 0$, so there is a projection $P_1 : L_{p(1)} \oplus \Lambda_1 \rightarrow L_{p(1)}$. To see that $\|P_1\| \leq 1 + \eta_1$ we have for any $x \in L_{p(1)}$, $\lambda \in \Lambda_1$, and some $i = 1, \ldots, q(1)$,

$$\|P(x + \lambda)\| = \|x\| \leq (1 + \eta_1) f_i(x) = (1 + \eta_1) f_i(x + \lambda) \leq (1 + \eta_1) \|x + \lambda\|.$$ 

Now suppose we have $\{|x_i|_{i=1}^{p(1)} \}$ satisfying (1), $\{|x_i|_{i=1}^{q(m)} \in U(X^*)\}$, and for
$n = 1, 2, \ldots, m - 1$, $\Lambda_n = \bigcap_{i=1}^{n} f_i^{-1}(0)$, such that $\|P_n\| \leq 1 + \eta_n$ where $P_n : \Lambda_n \oplus L_{p(n)} \to L_{p(n)}$. Since $\Lambda_{m-1}$ is of finite codimension in $X$, by [5, Theorem 2.12], it is not $B$-convex so that there are $x_i \in \bigcap_{i=m-1+1}^{p(m)} U(\Lambda_{m-1})$ satisfying (1). By Lemma 2.1 choose $f_i |_{i=\min(m)+1}^{\max(m)}$ so that if $x \in L_{p(m)}$, 
\[ \|x\| \leq (1 + \eta_m) \max_{i,q} \{f_i(x): i = q(m - 1) + 1, \ldots, q(m)\}. \]

Let $\Lambda_m = \bigcap_{i=1}^{p(m)} f_i^{-1}(0)$. Then exactly as in the $m = 1$ case, $L_{p(m)} \cap \Lambda_m = 0$ and $\|P_m\| \leq 1 + \eta_m$.

To show (2) holds we first observe, for any $a_i, i = 1, 2, \ldots$

A. $\|\sum_{i=1}^{p(m)} a_i x_i\| \leq (1 + \eta_m) \|\sum_{i=1}^{p(m)} a_i x_i\|$ for any $q = 1, 2, \ldots$, since $\|P_m\| \leq 1 + \eta_m$.

Further, we observe

B. If $p(m - 1) < n \leq p(m)$ then

\[ \left\| \sum_{i=p(m-1)+1}^{n} a_i x_i \right\| \leq \frac{1}{1 - \epsilon_m} \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\|, \]

since

\[ \left\| \sum_{i=p(m-1)+1}^{n} a_i x_i \right\| \leq \sum_{i=p(m-1)+1}^{n} |a_i| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i| \leq \frac{1}{1 - \epsilon_m} \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\|, \]

using the inequalities of (1).

(2) can be proved in four cases as follows:

Case 1. $1 \leq n < n + q < p(1)$. Using B, with $m = 1$, and (4) we obtain

\[ \|\sum_{i=1}^{n} a_i x_i\| \leq (1 + \delta_n) \left\| \sum_{i=1}^{n+q} a_i x_i \right\|. \]

Case 2. $1 \leq n \leq p(1) < n + q$. The above inequality can be proved using B and A, where $m = 1, \epsilon = 1, (4)$.

Case 3. There is $m$ so that $p(m) < n < n + q \leq p(m + 1)$. Inequality (2) can be proved by using A, B with $m$ replaced by $m + 1$, and (5).

Case 4. There is $m$ so that $p(m) < n \leq p(m + 1) < n + q$. Inequality (2) can be proved by using A, B with $m$ replaced by $m + 1$, A with $m$ replaced with $m + 1$, and (5).

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