EXTENSIONS OF NORMAL IMMERSIONS OF $S^1$ INTO $R^2$

BY

MORRIS L. MARX

ABSTRACT. Suppose that $f: S^1 \to R^2$ is an immersion, i.e., a $C^1$ map such that $f'$ is never zero. We call $f$ normal if there are only finitely many self-intersections and these are transverse double points. A normal immersion $f$ can be topologically determined by a finite number of combinatorial invariants. Using these invariants it is possible to give considerable information about extensions of $f$ to $D^2$. In this paper we give a new set of invariants, inspired by the work of S. Blank, to solve several problems concerning the existence of certain kinds of extensions. The problems solved are as follows:

1. When does $f$ have a light open extension $F: D^2 \to R^2$? (Recall that light means $F^{-1}(y)$ is totally disconnected for all $y$ and open means $F$ maps open sets of the interior of $D^2$ to open sets of $R^2$.) Because of the work of Stoilow, the question is equivalent to the following: when does there exist a homeomorphism $h: S^1 \to S^1$, such that $fh$ has an analytic extension to $D^2$?

2. Suppose that $F: D^2 \to R^2$ is light, open, sense preserving, and, at each point of $S^1$, $F$ is a local homeomorphism. At each point of the interior of $D^2$, $F$ is locally topologically equivalent to the power mapping $z^m$ on $D^2$, $m \geq 1$. Points where $m > 1$ are called branch points and $m - 1$ is the multiplicity of the point. There are only a finite number of branch points. The problem is to discover the minimum number of branch points of any properly interior extension of $f$. Also we can ask what multiplicities can arise for extensions of a given $f$.

3. Given a normal $f$, find the maximum number of properly interior extensions of $f$ that are pairwise inequivalent. Since each immersion of the disk is equivalent to a local homeomorphism, the problem of immersion extensions is a special case of this. It is Blank's solution of the immersion problem that prompted this paper.

1. Introduction. Throughout the paper we shall let $R^2$ denote Euclidean two-space; $S^1$, the unit circle in $R^2$; and $D^2$, the unit disk in $R^2$.

Suppose that $f: S^1 \to R^2$ is a $C^1$ map such that $f'$ is zero at only a finite number of points. We call $f$ a normal immersion if there are only finitely many self-intersections and these are transverse double points. We insist that $f'$ be
nonzero at the preimages of the double points. A normal immersion \( f \) can be topologically determined by a finite number of combinatorial invariants. Using these invariants it is possible to give considerable information about extensions of \( f \) to \( D^2 \). In this paper we give a new set of invariants, inspired by the work of S. Blank [1], [6], to solve several problems concerning the existence of certain kinds of extensions. We list these problems and survey the work that has been done on them. In what follows \( f: S^1 \to R^2 \) is a normal immersion. All mappings are continuous.

I. When does \( f \) have a light open extension \( F: D^2 \to R^2 \)? Recall that \( \text{light} \) means \( F^{-1}(y) \) is totally disconnected for all \( y \) and \( \text{open} \) means \( F \) maps open sets of the interior of \( D^2 \) to open sets of \( R^2 \). Because of the work of Stoïlow [7], the question is equivalent to the following: when does there exist a homeomorphism \( b: S^1 \to S^1 \), such that \( fb \) has an analytic extension to \( D^2 \)?

Titus [9] traces work on problems of this nature back to Picard. The problem in its present form was first stated circa 1948 by C. Loewner. The solution was given by Titus in [9] in the form of an algorithm.

II. Suppose that \( F: D^2 \to R^2 \) is light, open, sense preserving, and \( F \) is a local homeomorphism at each point of \( F^{-1}(F(S^1)) \). We call such a map properly interior. Again appealing to the work of Stoïlow (see [7]), we discover that at each point of the interior of \( D^2 \), \( F \) is locally topologically equivalent to the power mapping \( z^m \) on \( D^2 \), \( m \geq 1 \). Points where \( m > 1 \) are called branch points and \( m - 1 \) is the multiplicity of the point. There are only a finite number of branch points. If \( m_1, m_2, \ldots, m_k \) are the multiplicities of the branch points of \( F \), then \( 1 + m_1 + m_2 + \cdots + m_k = \text{TWN}(f) \). (Recall that \( \text{TWN}(f) \) is the Brouwer degree of the mapping \( f'/\|f'\|: S^1 \to S^1 \).) This last result is standard for analytic functions and, hence, is also true for properly interior mappings, since they are topologically equivalent to analytic mappings. The problem here is to discover the minimum number of branch points of any properly interior extension of \( f \). Also we can ask what multiplicities can arise for extensions of a given \( f \).

The author gave an algorithm in [4] to solve the above for normal \( f \).

III. Oral tradition has it that John Milnor discovered a normal \( f: S^1 \to R^2 \) with two extensions \( F \) and \( G \), both immersions, but not equivalent. (Maps \( \phi: X \to Y \) and \( \psi: Z \to Y \) are equivalent if there exists a homeomorphism \( h: X \to Z \) with \( \phi = \psi h \).) The problem then is to discover the maximum number of immersion extensions that are pairwise inequivalent. It is enough to solve the problem for extensions that are local homeomorphisms instead of immersions, since each local homeomorphism is equivalent to an immersion (see Jewitt [2, Theorem 3, p. 111]).

Although nowhere stated in the paper, Titus’ solution to I also solves this problem. A second and very interesting solution was given by S. Blank in his dissertation [1].
IV. Having restated problem III in terms of local homeomorphisms instead of immersions, we give a straightforward generalization: Given a normal \( f \), discover the maximum number of properly interior extensions of \( f \) that are pairwise inequivalent. In this context, problem I asks whether that number is zero. If we have solved problem IV and either know that \( \text{TWN}(f) = 1 \) or solve problem II, then we obtain a solution to problem III.

In this paper a solution is given to the preceding four problems. First to each normal \( f \) a word is assigned (§3). This word is a finite string of symbols. Then we define a collection of subsets of the symbols in the word called a maximal assemblage (§5). Theorems 6, 7, and 8 establish a one-to-one correspondence between equivalence classes of certain light open extensions of \( f \) and maximal assemblages. Thus, problems I, III, and IV are solved. Further, every maximal assemblage \( \mathcal{A} \) contains a subcollection \( \mathcal{B} \) (empty in the case of local homeomorphism extensions) such that there is a one-to-one correspondence between branch points of the extension of \( f \) corresponding to \( \mathcal{A} \) and the sets of \( \mathcal{B} \). This correspondence is such that the multiplicity of the branch point associated with \( B \in \mathcal{B} \) is \((-1) + (\text{cardinality of } B)\). Hence, problem II is solved.

2. Definitions and conventions. We have already defined normal mapping in the introduction. The points of the image that are self-intersections of such a mapping are called vertices. We can also consider mappings \( f: I \to R^2, I \) the unit interval, as being normal. For any normal \( f \) mapping from either \( I \) or \( S^1 \), we use the symbol \( [f] \) to denote the image of \( f \). Any mapping equivalent to a normal mapping that has a properly interior extension to \( D^2 \) is called an interior boundary.

The problems investigated in this paper are of such a nature that any mapping can be replaced with one equivalent to it. Actually, we are working with equivalence classes of maps. The notation and language become too cumbersome with classes, so we use maps and replace them when desired.

Suppose \( f \) and \( g \) are mappings on \( I \) (they may be defined on \( S^1 \) and the modifications for this case are easy). Let \( a < b \) and \( a' < b' \) be such that \( f(b) = g(a') \). By \( f(a)(b)(f) + g(a')g(b')(g) \) we mean the mapping \( \phi \) defined by \( \phi(t) = f(2(b - a)t + a) \) for \( 0 \leq t \leq \frac{1}{2} \) and \( \phi(t) = g(2(b' - a')(t - 1) + b') \) for \( \frac{1}{2} \leq t \leq 1 \), i.e. the path composition of \( f \mid [a, b] \) and \( g \mid [a', b'] \). If \( f^{-1}(y) = \{a\} \), there is no ambiguity in writing \( y/(b)(f) \). As usual, we will substitute for the given parameterization \( \phi \) any one that is equivalent to it.

3. Words. In this section we describe a class of combinatorial invariants for a normal immersion. Suppose \( f: S^1 \to R^2 \) is normal. Let \( \Theta \) be a subset of the positive integers. Let \( \{a_j \mid j \in \Theta\} \) be a collection of arc paths (homeomorphic mappings on the unit interval) in \( R^2 \) with the following properties.
(1) The initial point \( a_j \) of \( a_j \) is a point in some bounded component of \( R^2 - [\gamma] \). The terminal point \( b_j \) of \( a_j \) is in the unbounded component of \( R^2 - [\gamma] \).

(2) There is at least one \( a_j \) in each bounded component of \( R^2 - [\gamma] \).

(3) Each \( a_j \) is transverse to \( \gamma \).

(4) The \( a_j \) are pairwise disjoint.

Orient \( a_j \) from \( a_j \) to \( b_j \). Then define a crossing point of \( \gamma \) and \( a_j \) to be positive if \( \gamma \) crosses \( a_j \) from right to left; define it negative if \( \gamma \) crosses \( a_j \) from left to right. Index the crossing points of \( \gamma \) and \( a_j \) by \( a(j, k) \) (if the crossing is positive) or \( a(j, k)^{-1} \) (if it is negative), where \( k \) ranges over some subset of the positive integers, say \( \Delta(j) \).

Trace along \([\gamma]\) using the positive orientation; i.e., that orientation induced by the counterclockwise orientation on \( S^1 \). Make a sequence or list of the crossing point indices (hereafter called letters) as they are encountered in the above tracing of \([\gamma]\). Such a list will be called a word for \( \gamma \). We have not yet made use of the double indexing of the crossing points. We could assume, that, for fixed \( j \), \( a(j, k) \) precedes \( a(j, k') \) in the word if and only if \( k < k' \). In this case, the second index carries no more information than is already present in the word. When this assumption is made we call the word a Blank word (see [1]). Suppose we require that the word contain more information by assuming that, for fixed \( j \), \( a(j, k') \) follows \( a(j, k) \) in the orientation of \( a_j \) (\( a(j, k') \) is "closer to \( -\infty \)") if and only if \( k < k' \). Then we simply call the word a word. Hopefully, the reader will find the preceding understandable by looking at Figure 1.

A word or Blank word is said to be reduced if the following hold.

1. It does not begin with \( a(j, k) \) and end with \( a(j, k')^{-1} \).
2. It does not begin with \( a(j, k)^{-1} \) and end with \( a(j, k') \).
3. In the word, \( a(j, k) \) is never next to \( a(j, k')^{-1} \).

Blank shows that the rays can always be drawn so that the word is reduced (see [1, p. 6]); we will assume this has been done.

A given \( \gamma \) may have many words, depending on the starting point of the tracing of \([\gamma]\) and, more importantly, on the choice of \( \{a_j\} \).

4. Cuts. We introduce here some methods of "cutting" a normal immersion into less complicated curves.

Definition. Suppose \( f: S^1 \to R^2 \) is normal. Let \( v \in [\gamma] \) be a terminal vertex of \( f \); that is, there are points \( t_1 = e^{i\theta_1} \) and \( t_2 = e^{i\theta_2} \), with \( 0 \leq \theta_1 < \theta_2 < 2\pi \), such that \( f(t_1) = f(t_2) = v \), but \( f \) is one-to-one on \( \{ e^{i\theta} \mid \theta_1 < \theta < \theta_2 \} \). Take \( f^* \) to be any regular parametrization of the curve \( [f \mid T] \), where \( T = \{ e^{i\theta} \mid 0 \leq \theta \leq \theta_1, \theta_2 \leq \theta < 2\pi \} \) (see Figure 2). We shall call \( f^* \) a cut of Type I associated with \( v \). Since \( f \) is normal, it has only finitely many vertices, so such \( f^* \) always exist.

Definition. Let \( f: S^1 \to R^2 \) be normal. Suppose \( a = a(j, k) \) and \( b = \)
\( \alpha(j, k') e' \) are two letters in \( \alpha \) with \( k < k' \). Define \( f^* \) and \( f^{**} \) by

\[ f^* = ab(f) + ba(-a_j), \quad f^{**} = ab(a_j) + ba(f). \]

(See Figure 2.) If \( e = -1 \) and \( e' = +1 \), then we call \( f^* \) and \( f^{**} \) cuts of Type II associated with \( \alpha(j, k)e \) and \( \alpha(j, k')e' \); if \( e = e' = +1 \), cuts of Type III.

Now we investigate the ways in which the cuts are simpler than \( f \).

**Theorem 1.**

(a) Type I. \( \text{TWN}(f) = \text{TWN}(f^*) + \lambda(v) \), where \( \lambda(v) = +1 \) if the loop at \( v \) has counterclockwise orientation and \( \lambda(v) = -1 \) otherwise.

(b) Type II. \( \text{TWN}(f) = \text{TWN}(f^*) + \text{TWN}(f^{**}) - 1 \).

(c) Type III. \( \text{TWN}(f) = \text{TWN}(f^*) + \text{TWN}(f^{**}) \).

The proof of Theorem 1 uses standard arguments on winding number and will not be given.

**Theorem 2.** Suppose \( f: S^1 \to R^2 \) is normal with word \( \alpha \) and let \( v \) be a terminal vertex. Suppose \( \alpha = \beta x \zeta y \delta \), where \( x \) is the last letter before \( f \) crosses \( v \) the first time and \( y \) is the first letter after \( f \) crosses \( v \) the second time. Then \( f^* \), the cut of Type I associated with \( v \), has a word which is a subword of \( \beta xy \delta \).

**Proof.** Since \( f^* \) does not trace along \( f \) from \( x \) to \( y \), we may omit \( \zeta \) as \( f^* \) does not cross those letters. Also \( [f^*] \) removes a Jordan curve from \( [f] \), so some of the components of \( S^1 - [f] \) may come together in \( S^1 - [f^*] \). Thus we may discard some of the \( a_j \). However, what is crucial is that no new components are created.

**Definition.** Suppose \( \alpha(j, k)^{-1} \) is a letter of \( \alpha \) such that for any \( \alpha(j, k') e \) in \( \alpha \) with \( k < k' \), we have \( e = +1 \). Then we shall call \( \alpha(j, k)^{-1} \) a terminal negative letter.

**Theorem 3.** Let \( f: S^1 \to R^2 \) be normal with word \( \alpha \). Suppose \( \alpha(j, k)^{-1} \) is a terminal negative letter. Let \( f^* \) and \( f^{**} \) be the Type II cuts associated with \( \alpha(j, k)^{-1} \) and \( \alpha(j, k') \), \( k < k' \). Suppose \( \alpha = \beta \alpha(j, k)^{-1} \zeta \alpha(j, k') \delta \). Then \( f^* \) has a word \( \alpha^* \) of which \( \zeta \) is a subword. Also, \( f^{**} \) has a word \( \alpha^{**} \) of which \( \beta \delta \) is a subword. Finally, both \( \alpha^* \) and \( \alpha^{**} \) have strictly less negative letters than \( \alpha \).

**Proof.** First, we know that the letters of \( \zeta \) are traced by \( f^* \) and these are the only letters of \( \alpha \) that it traces. Note that every component of \( R^2 - [f^*] \) is a subset of some component of \( R^2 - [f] \). Notice that some of the components through which \( \alpha_j \) passes have been subdivided into two components. One of the two pieces already has some \( \alpha_j \) originating in that piece. We will need a new \( \alpha_j \).
Let $k < k_1 < \ldots < k_s < k' < k_{s+1} < \ldots < k_t$ be those points of $\Delta(j)$ such that $f^*$ crosses $\alpha_j$ at only the $\alpha(j, k_i)$, $1 \leq i \leq t$. Again referring to Figure 3, we need an $\alpha_j$ for some of the components below $[f^*]$ and to the left of some $\alpha(j, k_i)$. Thus, if $\Theta$ had $M$ elements, we define $\alpha_{M+1}, \ldots, \alpha_{M+s+1}$ as shown in Figure 3. Also, we need $\alpha_j$ for some of the components above $[f^*]$ and to the left of some $\alpha(j, k_i)$. So we define $\alpha_{M+s+2}, \ldots, \alpha_{M+t}$.

To construct the word for $f^*$, first take $\alpha(M + s + 1, 1) \alpha(M + s, 1) \ldots \alpha(M + 1, 1) \zeta$. For each $\alpha(j, k_i)^{+1}$, insert $\alpha(M + i, 2) \alpha(M + i - 1, 3) \ldots \alpha(M + 1, i)$ into $\zeta$ immediately before $\alpha(j, k_i)^{+1}$. Also, for $i > 1$, insert $\alpha(M + s + i, 1) \ldots \alpha(M + s + i - 1, 2) \ldots \alpha(M + s + 2, i - 1)$ into $\zeta$ immediately after $\alpha(j, k_i)^{+1}$. Recall that $\alpha(j, k_i)^{-1}$ is terminal, so we are justified in assuming that the exponent of $\alpha(j, k_i)^{-1}$ is indeed $+1$. Clearly $\zeta$ is a subword of this new word. Also, we have lost $\alpha(j, k_i)^{-1}$ and added no new negative letters; hence there are less negative letters than in $\alpha$.

The proof for $f^{**}$ is very similar and, in fact, can be reduced to the preceding case by reflecting the plane about a line through $\alpha_j$ and reversing the orientation of $f$.

We need a theorem like Theorem 3 for Type III cuts. However, it turns out that for the application we make of these, it is better to modify the cuts slightly (as opposed to moving $\alpha_j$ so that it will be transverse to the cut).

Consider first the cut $f^*$ associated with $\alpha(j, k)$ and $\alpha(j, k')$, $k < k'$. Let $\beta$ be an arc to the right of and parallel to $\alpha_j$ which is transverse to $f^*$ (see Figure 4). Then define mod$f^*$ to be the mapping $\alpha(j, k)u_1(f^*) + u_1v_1(\beta) + v_1\alpha(j, k)/(f)$. We may think of this as moving the arc $[\alpha(j, k)\alpha(j, k')(f)]$ to the right of $\alpha_j$. For $f^{**}$, let $\gamma$ be an arc to the left of and parallel to $\alpha_j$ which is transverse to $f^{**}$ (see Figure 4). Define mod$f^{**}$ to be the mapping $\alpha(j, k)v_2(f) + v_2u_2(\gamma) + u_2\alpha(j, k)(f^{**})$.

It is apparent from Figure 4 that mod$f^*$ and $f^*$ have the same intersection sequence [8, p. 1084]; similarly, mod$f^{**}$ and $f^{**}$. By [9, Theorem 3, p. 49] mod$f^*$ and $f^*$ are topologically equivalent, as are mod$f^{**}$ and $f^{**}$. Thus, any result that follows, concerning a Type III cut remains true when it is modified as above.

**Theorem 4.** Let $f: S^1 \to \mathbb{R}^2$ be normal with word $\alpha$ and suppose $\alpha$ has all positive letters. Let $f^*$ and $f^{**}$ be the Type III cuts associated with $\alpha(j, k)$ and $\alpha(j, k')$, $k < k'$. Suppose $\alpha = \beta \alpha(j, k)\zeta \alpha(j, k')\delta$. Then mod$f^*$ has a word $\alpha^*$ with all positive letters of which $\alpha(j, k)\zeta$ is a subword and mod$f^{**}$ has a word $\alpha^{**}$ with all positive letters of which $\beta \alpha(j, k)\delta$ is a subword.
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Proof. The techniques of proof here are similar to those used in Theorem 3. Figure 5 should illustrate the differences between Theorems 3 and 4.

5. Assemblages. We now discuss the operations on the word that allow us to state the main theorems in §6.

Definition. Let $f: S^1 \to R^2$ be normal with word $\alpha$. Suppose $\mathfrak{G}$ and $\mathfrak{B}$ are collections whose elements are sets of letters of $\alpha$. Let $\mathfrak{A} = \mathfrak{G} \cup \mathfrak{B}$. Suppose that:

1. The sets of $\mathfrak{A}$ are mutually disjoint.
2. No two sets of $\mathfrak{A}$ are linked, i.e. if $A, A' \in \mathfrak{A}$, then $f^{-1}(A)$ does not separate $f^{-1}(A')$ in $S^1$.
3. Every set in $\mathfrak{G}$ is of the form $\{\alpha(j, k)^{-1}, \alpha(j, k')\}$ with $k < k'$.
4. Every negative letter of $\alpha$ occurs in some set in $\mathfrak{G}$.
5. Let $B \in \mathfrak{B}$. Then all the letters in $B$ come from the same $\alpha$, i.e., there exists $j \in \mathfrak{G}$ such that every letter in $B$ is of the form $\alpha(j, k)$ for some $k \in \Delta_j$. (Note that because of (1) and (4), no letter of $B$ can be negative.)

Then we call $\mathfrak{A}$ an assemblage, $\mathfrak{G}$ a grouping, and $\mathfrak{B}$ a branching for $\alpha$.

We take $\mathfrak{G} = \emptyset$ in case $\alpha$ has no negative letters; also we permit $\mathfrak{B} = \emptyset$.

The main goal of this section is to investigate the relationship between assemblages and properly interior extensions of $f$. We first prove a theorem for the case $\mathfrak{G} = \emptyset$.

Theorem 5. Suppose $f: S^1 \to R^2$ is normal with word $\alpha$ and $\alpha$ has all positive letters.

1. $f$ is a boundary
2. $\text{TWN}(f) \geq 1$.
3. If $\text{TWN}(f) = 1$, then $f$ represents a positively oriented Jordan curve.

Proof. We prove the theorem by induction on $n$, the number of vertices of $f$. If $n = 0$, then $f$ has no vertices and is a one-to-one mapping. Hence, $[f]$ is a Jordan curve. Since $\alpha$ has all positive letters, the orientation must be positive. Then (1), (2), (3) hold.

Suppose the theorem is true for any normal mapping with $n - 1$ vertices, $n \geq 1$. Let $f$ have $n$ vertices. Then $f$ must have a terminal vertex $v$. This vertex must have $\lambda(v) = +1$. Otherwise, the loop determined by $v$ would have negative orientation and any $\alpha_j$ originating inside this loop would give $\alpha$ a negative letter, a contradiction to hypothesis.

Let $f^*$ be the cut of Type I associated with $v$. Clearly $f^*$ has $n - 1$ vertices and it is not difficult to see that $f^*$ has a word which is a subword of $\alpha$. Thus, we can apply the induction hypothesis to $f^*$ and conclude that it is an interior boundary. By Theorem 19.1 [5, p. 67], $\text{TWN}(f^*) \geq 1$. But Theorem 1
implies that \(\text{TWN}(f) = \text{TWN}(f^*) + 1 \geq 2\). Hence, (2) holds for \(f\) and (3) does not apply. Finally to see that \(f\) is an interior boundary refer to Theorem 4 [9, p. 54].

Definition. Let \(f: S^1 \to \mathbb{R}^2\) be normal with word \(a\). Suppose \(\mathcal{G}\) is an assemblage for \(a\). Let \(F: D^2 \to \mathbb{R}^2\) be a properly interior extension of \(f\).

Let \(P = \{a(j, k)^{-1}, a(j, k')\}\) be a set in \(\mathcal{G}\). Let \(U = \{a(i, k)^{-1}a(j, k')(a_j)\}\), i.e., the set of points on \(a_j\) between \(a(j, k)^{-1}\) and \(a(j, k')\). We say the arc \(T \subset D^2\) belongs to \(P\) if the endpoints of \(T\) are \(f^{-1}(P)\), if \(T\) intersects \(S^1\) only at its endpoints, and if \(F\) maps \(T\) homeomorphically onto \(U\).

Suppose \(T_1, T_2, \ldots, T_n\) are arcs and there exists a point \(x\) such that \(x\) is an endpoint of each of the arcs and such that \(T_i \cap T_j = \{x\}\) for \(1 \leq i < j \leq n\). Then we shall call \(U_{i=1}^n T_i\) a fan and \(x\) the hinge.

Let \(B \in \mathcal{B}\). We say that the fan \(V = U_{i=1}^n T_i \subset D^2\) belongs to \(B\) if the following conditions are satisfied:

1. Let \(E\) be the set of endpoints of the \(T_i\) other than \(x\). Then \(V \cap S^1 = E = f^{-1}(B)\).
2. For each \(T_i\), there exists \(a(j, k) \in B\) such that \(F\) maps \(T_i\) homeomorphically onto \([a_i, a(j, k)(a_j)]\) (the set of points on \(a_j\) that precede \(a(j, k)\)).

Note that \(x\) is a branch point of multiplicity at least \(-1 + \text{cardinality } B\).

Theorem 6. Let \(f: S^1 \to \mathbb{R}^2\) have assemblage \(\mathcal{A}\). There exists a properly interior extension \(F\) of \(f\) to \(D^2\) such that the following hold:

1. For each \(G \in \mathcal{G}\), there is an arc in \(D^2\) belonging to \(G\).
2. For each \(B \in \mathcal{B}\), there is a branch point \(x\) in \(D^2\) of multiplicity \(-1 + \text{cardinality } B\) and \(x\) is the hinge of a fan in \(D^2\) belonging to \(B\). Also, \(F(x) = a_i\), where \(a_i\) is the ray associated with \(B\).

Proof. We prove the theorem by induction on \(n\), the number of negative letters in \(a\). For \(n = 0\), \(a\) has all positive letters and \(\mathcal{G} = \emptyset\). Only (2) applies here. This is a subtheorem which we also prove by induction, but we induct on \(\text{TWN}(f)\). This is possible since \(a\) has all positive letters and, by Theorem 5, \(\text{TWN}(f) \geq 1\).

First consider the case \(\text{TWN}(f) = 1\). By Theorem 5, \(f\) represents a positively oriented Jordan curve and \(f\) is an interior boundary. The winding number of \(f\) about each \(a_j\) is \(+1\). It then follows from [8, Lemma 2, p. 1085] that \(a\) has only one letter from each \(a_j\), since it cannot contain a negative crossing. We conclude that \(\mathcal{B} = \emptyset\) or consists of one element sets. Hence (2) is vacuously true.

Now suppose the subtheorem is true for all normal mappings having a word with all positive letters and with \(\text{TWN} < \text{TWN}(f)\). Prove the theorem for \(\text{TWN}(f) \geq 2\).

If \(\mathcal{B} = \emptyset\) or has only one-element sets, then (2) requires nothing more than that \(f\) be an interior boundary, which follows from Theorem 5. Now we give an argument in case \(\mathcal{B}\) has at least one set with two or more elements.
Hence suppose \( B = \{ \alpha(j, k_1), \ldots, \alpha(j, k_m) \} \) is in \( \mathcal{B} \). Assume without loss of generality that \( k_1 < k_2 \) and \( \alpha = \beta \alpha(j, k_1) \zeta \alpha(j, k_2) \delta \) where \( \alpha(j, k) \) is not a letter of \( \zeta \) for any \( r, 1 \leq r \leq m \). Let \( f^* \) and \( f^{**} \) be the Type III cuts associated with \( \alpha(j, k_1) \) and \( \alpha(j, k_2) \). By Theorem 4, \( \text{mod } f^* \) has a word \( \zeta^* \) with all positive letters of which \( \alpha(j, k_1) \zeta \) is a subword; similarly \( \text{mod } f^{**} \) has word \( \alpha^{**} \) with all positive letters of which \( \beta \alpha(j, k_1) \delta \) is a subword. From part (c) of Theorem 1 and Theorem 5, we see that \( \text{TWN}(*^t) = \text{TWN}(\text{mod } f^*) \) and \( \text{TWN}(**^t) = \text{TWN}(\text{mod } f^{**}) \) are both \( < \text{TWN}(\beta) \). We will be able to apply the induction hypothesis as soon as we have assemblages for \( \text{mod } f^* \) and \( \text{mod } f^{**} \).

For any set \( B' \in \mathcal{B}, B' \neq B \), by the nonlinking property of assemblages, either the letters of \( B' \) all come from \( \zeta \) or all from \( \beta \delta \). Let \( \mathcal{B}^* \) be the collection of all sets \( B' \) that come from \( \zeta \) together with the set \( \{ \alpha(j, k_1) \} \). Let \( \mathcal{B}^{**} \) be the collection of all sets \( B' \) that come from \( \beta \delta \) together with \( B - \{ \alpha(j, k_2) \} \). Then \( \mathcal{B}^* \) is an assemblage for \( \text{mod } f^* \) and \( \mathcal{B}^{**} \) is an assemblage for \( \text{mod } f^{**} \). Thus, there is a properly interior extension of \( \text{mod } f^* \) to \( D^2 \) that satisfies the conclusion of the theorem with respect to \( \mathcal{B}^* \); similarly \( F^{**} \).

Consider the fan that belongs to \( \{ \alpha(j, k_1) \} \in \mathcal{B}^* \). In this case, it is a single arc \( T^* \) from \( y^* \), the preimage of \( \alpha(j, k) \) under \( f^* \), to \( x^* \), some point in the interior of \( D^2 \). Let us assume without loss of generality that \( T^* = \{ re^{i\theta} \mid 0 \leq r \leq 1, \theta = 0 \} \).

We are going to construct a topological disk \( D \) and a properly interior mapping of it which gives a mapping equivalent to \( f \) on boundary \( D \). Let \( D_1 = \{ re^{i\theta} \mid 0 \leq \theta \leq \pi, 0 \leq r \leq 1 \} \); on \( D_1 \), define \( G_1 \) by \( G_1 = f^* \circ z^2 \). Assume that \( G_1(1) = u_1 \).

Now let \( \tilde{\alpha} \) be some function of \( \theta, \pi/2 \leq \theta \leq \pi, 1 \leq r \leq \alpha(\theta) \). Suppose that \( g_2 \) maps \( \{ e^{i\theta} \mid \pi/2 \leq \theta \leq \pi \} \) homeomorphically onto \( \{ u_1, \alpha(j, k_1) \} \) with \( g_2(z) = u_1 \) and \( g_2(-2) = \alpha(j, k_1) \). Let \( h_2 \) map \( \{ e^{i\theta} \mid -2 \leq r \leq -1, \pi = \theta = 0 \} \) homeomorphically onto \( \{ \alpha(j, k_1), \alpha(j, k_1)(-\alpha_j) \} \) with \( h_2(-2) = \alpha(j, k_1) \) and \( h_2(-1) = \alpha(j, k_1) \). Then using \( G_1, g_2, h_2 \) on the three arcs that make up boundary \( D_2 \), we obtain a counterclockwise oriented homeomorphism \( h \) of boundary \( D_2 \) onto \( \{ \alpha(j, k_1)u_1(-1)^* + u_1(-1) + \alpha(j, k_1)(-\alpha_j) \} \), a Jordan curve with a similar orientation. Thus there is a sense-preserving homeomorphism \( G_2 : D_2 \rightarrow R^2 \) that extends \( h \).

Now recall there is a fan in \( D^2 \) associated with \( B - \{ \alpha(j, k_2) \} \) with regard to \( f^{**} \). One arc \( T^{**} \) of this fan maps onto \( \{ a, \alpha(j, k_2)(a) \} \) under \( F^{**} \). There is no loss of generality in assuming that \( T^{**} = T^* = \{ re^{i\theta} \mid 0 \leq r \leq 1, \theta = 0 \} \) and further that \( F^{**}(t) = f^*(t) \) for all \( t \in T^* = T^{**} \). Take \( D_3 = \{ re^{i\theta} \mid \pi \leq \theta \leq 2\pi, 0 \leq r \leq 1 \} \); on \( D_3 \), define \( G_3 \) by \( G_3 = F \circ z^2 \). Assume \( G_3(-1) = u_2 \). Note that \( G_1 \) and \( G_3 \) agree where their domains overlap, i.e. on the real axis between -1 and 1. Now let
$s(\theta)$ be a function of $\theta$, $\pi \leq \theta \leq 3\pi/2$, that strictly decreases from 2 to 1. Take $D_4 = \{r e^{i\theta} : \pi \leq \theta \leq 3\pi/2, 1 \leq r \leq s(\theta)\}$. Suppose $g_4$ maps $\{s(\theta)e^{i\theta} : \pi \leq \theta \leq 3\pi/2\}$ homeomorphically onto $[\alpha(j, k_2)x_2(f)]$ with $g_4(-2) = \alpha(j, k_2)$ and $g_4(-i) = u_2$.

Using $g_3, g_4, b_2$ on the boundary of $D_4$ we obtain a counterclockwise oriented homeomorphism $b'$ of the boundary of $D_4$ onto the similarly oriented Jordan curve $\{\alpha(j, k_1)x_1(f) + u_2v_2(-v) + v_2\alpha(j, k_1)(-f)]\}$. Thus there exists a sense-preserving homeomorphism $G_4 : D_4 \to \mathbb{R}^2$ that extends $b'$.

Let $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Note that each of $G_1, G_2, G_3, G_4$ agree where their domains overlap. Thus we can define a mapping $G$ on $D$ by $G|D_i = G_i, 1 \leq i \leq 4$. By [10, Theorem 9, p. 336], $G$ is properly interior. Also $G$ restricted to the boundary of $D$ is a local homeomorphism that parametrizes $\alpha$ and so gives a mapping equivalent to $\alpha$. Thus, $\alpha$ is an interior boundary.

We also note that there is the desired fan belonging to $B$. One arc of the fan is the real axis between 0 and 1, one arc is the real axis between $-1$ and 0, the remaining $(2 + \text{cardinality } B)$ arcs are in $D_3$ and were produced by the induction hypothesis. There are no other arcs at 0 that map into $[\alpha_j]$ except the aforementioned. Thus $0$ is a branch point of multiplicity $1 + \text{cardinality } B$.

This completes the proof of the subtheorem—the case when $\sigma = 0$. Recall that this is the induction step $\sigma = 0$, where $\sigma$ is the number of negative letters in $\alpha$.

Now let us suppose the theorem is true for any normal mapping with word with $< n$ negative letters and let $f : S^1 \to \mathbb{R}^2$ have word $\alpha$ with $n > 0$ negative letters.

Pick $\alpha(j, k)^{-1}$ some terminal negative letter and then there must exist $P = \{\alpha(j, k)^{-1}, \alpha(j, k')\}$ in $\mathcal{G}$. Let $f^*$ and $f^{**}$ be the Type II cuts associated with $P$. By Theorem 3, assuming $\alpha = \beta\alpha(j, k)^{-1}\zeta\alpha(j, k')\delta$, then $f^*$ has a word $\alpha^*$ of which $\zeta$ is a subword; similarly $f^{**}$ has a word $\alpha^{**}$ of which $\beta\delta$ is a subword. Both $\alpha^*$ and $\alpha^{**}$ have less than $n$ negative letters. It follows from the nonlinking property of assemblages that any $A \in \mathcal{A}$ either comes from $\zeta$ or $\beta\delta$.

Let $\mathcal{A}^*$ be the collection of the former; $\mathcal{A}^{**}$, the latter. Then $\mathcal{A}^*$ is an assemblage for $f^*$ and $f^{**}$ for $\mathcal{A}^{**}$.

Applying the induction hypothesis, there are extensions $F^*$ of $f^*$ and $F^{**}$ of $f^{**}$ that satisfy the conclusions of the theorem. Suppose we take $D_1$ to be the upper half of the unit disk in the plane and $D_2$, the lower. We may assume without loss of generality that $D_1$ is the domain of $F^*$ and $D_2$, of $F^{**}$. Further, we may assume that for any point $z$ between $-1$ and 1 on the real axis, i.e. on $D_1 \cap D_2$, $f^*(z) = F^*(z) = F^{**}(z) = f^{**}(z)$. Thus, by [10, Theorem 9, p. 336] we can define a properly interior mapping $F$ on $D^2 = D_1 \cup D_2$ by $F|D_1 = F^*$ and $F|D_2 = F^{**}$. For any $A \in \mathcal{A}, A \neq P$, the desired arc (for conclusion (1)) or fan and branch point (for conclusion (2)) are in $D_1$ or $D_2$. The required arc belong-
Theorem 7. Let \( f: S^1 \to R^2 \) be normal with word \( \alpha \). Suppose that \( F \) is a properly interior extension of \( f \) to \( D^2 \) such that each branch point maps onto an \( a_i \). Then there exists an assemblage \( B \cup G \) such that:

1. For each \( G \in \mathcal{G} \), there is an arc in \( D^2 \) that belongs to \( G \).
2. There is a one-to-one correspondence between branch points of \( F \) and sets in \( B \) such that if \( x \) and \( B \) correspond, then \( x \) is the hinge of a fan in \( D^2 \) that belongs to \( B \) and \( x \) has multiplicity \(-1 + \text{cardinality } B\).

Call this assemblage that relates to \( F \) in the above manner \( \mathcal{A}(F) \).

Proof. Let \( \alpha(i, k)^{-1} \) be some negative letter of \( \alpha \). Consider the arc \( U = [\alpha(i, k)^{-1}b_i(\alpha_i)] \) (recall that \( b_i \) is the endpoint of \( \alpha_i \) in the unbounded component of \( R^2 - \{f\} \)). By [3, Theorem 1, p. 49] there is an arc \( T \) in \( D^2 \) with one endpoint at \( f^{-1}(\alpha(i, k)^{-1}) \) that maps homeomorphically into \([\alpha_i]\). The other endpoint is either in \( S^1 \) or maps onto \( b_i \). However, a properly interior map cannot map any points into the unbounded component of \( R^2 - \{f\} \), so we conclude that the other endpoint is in \( S^1 \). Since \( F \) is locally one-to-one at each point of \( S^1 \), this endpoint must be the preimage of some letter of \( \alpha \). Further, this letter must be positive because \( F \) is orientation preserving. The arc \( U \) was lifted from \( \alpha(i, k)^{-1} \) outward, so this letter, say \( \alpha(i, k') \), has the property that \( k < k' \). Thus, \( G = \{\alpha(i, k)^{-1}, \alpha(i, k')\} \) meets the criteria for being in some grouping. By construction, the arc \( T \) belongs to \( G \). Each negative letter yields one and only one \( G \) in this fashion. Take \( \mathcal{G} \) to be the collection of all such \( G \). Since the arcs belonging to different \( G \)'s are disjoint, no two \( G \)'s link.

Pick \( j \in \Theta \). Suppose \( \{\alpha(i, k)\mid 1 \leq i \leq r\} \) are those positive letters on \( \alpha \) that do not belong to any \( G \in \mathcal{G} \). Applying [3, Theorem 1, p. 49] to the arc \( Q_i = [\alpha(i, k_i)(\alpha_i)] \), we produce an arc \( T_i \) in \( D^2 \) with one endpoint at \( f^{-1}(\alpha(i, k_i)) \). The other endpoint is either in \( S^1 \) or maps onto \( \alpha_i \). Since \( F \) is orientation preserving, the other endpoint can be in \( S^1 \) only if it is the preimage of a negative letter. But this would mean that \( \alpha(j, k_i) \in G \) for some \( G \in \mathcal{G} \). We conclude that the other endpoint maps onto \( \alpha_j \). Any point where two of the \( T_i \) intersect is a branch point. We have hypothesized that such a branch point would map onto \( a_j \). Hence, if the \( T_i \) intersect, they intersect at the endpoint that maps onto \( a_j \). Let \( K_1, K_2, \ldots, K_s \) be the components of \( \bigcup_{i=1}^r T_i \); each \( K_i \) is a fan. Take \( B_i = F(K_i \cap S^1) \). Then take \( \mathcal{B} \) to be the collection of all such \( B_i \) as \( i \) ranges over \( \Theta \). Since no two of the fans can intersect, the sets of \( \mathcal{B} \) are disjoint and do not link.

Suppose \( B_i \) and \( K_i \) are as in the previous paragraph. Let \( x_i \) be the hinge of the fan. As remarked previously, \( x_i \) is a branch point of multiplicity \( M_i \geq -1 + \text{cardinality } B_i \). We show that equality holds. There must be \( 1 + M_i \) arcs at \( x_i \).
Each of these arcs can be extended until it intersects \( S^1 \). No two intersect except at \( x \). The point of intersection of one of these arcs and \( S^1 \) must be the preimage of a positive letter of \( a_j \), say \( \alpha(j, t) \). Since \( \alpha(j, t) \notin G \) for any \( G \in \mathcal{G} \), \( \alpha(j, t) \in B_j \). This proves that \( M_i \leq -1 + \text{cardinality } B_j \) and completes the proof of the theorem.

**Definition.** Let \( f: S^1 \to \mathbb{R}^2 \) be normal with word \( \alpha \). Suppose \( \mathcal{A} \) is an assemblage for \( f \). Define \( r(\mathcal{A}) = 1 + \sum B \epsilon \mathcal{G} (-1 + \text{cardinality } B) \). We shall call an assemblage maximal if it maximizes \( r(\mathcal{A}) \).

**Proposition.** \( \mathcal{A} \) is maximal if and only if \( r(\mathcal{A}) = \text{TWN}(f) \).

**Proof.** Let \( \mathcal{A} \) be any assemblage of \( \alpha \). By Theorem 6, \( r(\mathcal{A}) \leq 1 + \sum_{i=1}^m \mu_i \) where the \( \mu_i \) are the branch point multiplicities of some properly interior extension of \( f \). But \( 1 + \sum_{i=1}^m \mu_i = \text{TWN}(f) \) [5, Theorem 19.1, p. 67]. We need only to observe that Theorem 7 guarantees that there is some assemblage \( \mathcal{A} \) for which \( r(\mathcal{A}) = \text{TWN}(f) \).

**Theorem 8.** Let \( f: S^1 \to \mathbb{R}^2 \) have word \( \alpha \). Suppose \( F \) and \( G \) are properly interior extensions of \( f \) to \( D^2 \) such that each branch point maps onto an \( a_j \).

Then \( F \) and \( G \) are equivalent if and only if \( \mathcal{A}(F) = \mathcal{A}(G) \).

**Proof.** To dispose of the converse first, note that the proof of Theorem 7 is purely topological. Composing \( F \) with a homeomorphism will not change the relationships between the arcs, fans, linkages, etc.

Now suppose \( \mathcal{A}(F) = \mathcal{A}(G) = \mathcal{A} = \mathcal{B} \cup \mathcal{G} \). We give a proof along the lines of the proof of Theorem 6. We induct on \( n \), the number of negative letters in \( \alpha \).

First suppose \( n = 0 \). Then \( \mathcal{G} = \emptyset \). We handle this case by induction on \( r(\mathcal{A}) = \text{TWN}(f) \). If \( \text{TWN}(f) = 1 \), then Theorem 5 implies that \( f \) is a positively oriented Jordan curve. Both \( F \) and \( G \) must be orientation preserving homeomorphisms and are equivalent.

Now suppose this case holds for all normal mappings with word of all positive letters and with \( \text{TWN} < \text{TWN}(f) \). In view of the previous paragraph, we may assume \( \text{TWN}(f) \geq 2 \). Thus \( r(\mathcal{A}) \geq 2 \) and there is some set \( B = \{ \alpha(j, k_1), \ldots, \alpha(j, k_m) \} \in \mathcal{B}, m \geq 2 \). Assume without loss of generality that \( k_1 < k_2 \) and \( \alpha = \beta \alpha(j, k_1) \zeta \alpha(j, k_2) \delta \) where \( \alpha(j, k_r) \) is not a letter of \( \zeta \) for any \( r, 1 \leq r \leq m \). Let \( f^* \) and \( f^{**} \) be the Type III cuts associated with \( \alpha(j, k_1) \) and \( \alpha(j, k_2) \). We are going to apply the induction hypothesis to \( f^* \) and \( f^{**} \). However, they are not transverse to \( a_j \). Select an arc \( a_j' \) that coincides with \( a_j \) from \( a_j \) to slightly before \( \alpha(j, k_1) \), then runs parallel to \( a_j \) until slightly after \( \alpha(j, k_2) \), where it coincides with \( a_j \) again. Make \( a_j' \) transverse to \( f \) (see Figure 6). If we label
the crossings of \( f \) and \( a_j' \) with the obvious choices of \( a(j, k)'s \), then we do not change the word \( a \). Also, by making \( a_j' \) close enough to \( a_j \), the fans of \( F \) and \( G \) can be "moved" slightly to yield similar fans involving \( a_j' \). It follows from Theorem 4 that \( f^* \) has word \( \zeta^* \) with all positive letters of which \( a(j, k_1)\zeta \) is a subword; similarly \( f^{**} \) has a word \( \alpha^{**} \) with all positive letters of which \( \beta a(j, k_1)\beta \) is a subword. For any set \( B' \in \mathcal{B}, B' \neq B \), by the nonlinking property of assemblages, either the letters of \( B' \) all come from \( \zeta \) or all from \( \beta \beta \). Let \( \mathcal{B}^* \) be the collection of all sets \( B' \) that come from \( \zeta \) together with the set \( \{a(j, k_1)\} \).

Let \( \mathcal{B}^{**} \) be the collection of all sets \( B' \) that come from \( \beta \beta \) together with \( B - \{a(j, k_2)\} \). Then \( \mathcal{B}^* \) is an assemblage for \( f^* \) and \( \mathcal{B}^{**} \), for \( f^{**} \).

Let \( \bigcup_{i=1}^{n} T_i \) be the fan belonging to \( B \) with respect to \( F \) and let \( x \) be the hinge. Suppose that \( T_1 \) has \( x \) and \( f^{-1}(a(j, k_1)) \) as its endpoints and \( T_2 \) has \( x \) and \( f^{-1}(a(j, k_2)) \). Then \( T_1 \cup T_2 \) splits \( D^2 \) into two disks, \( D^* \) and \( D^{**} \). Let \( \mathcal{D}^* \) be the decomposition of \( D^* \) whose nondegenerate elements are all sets of the form \( F^{-1}(y) \), where \( y \) is a point of \( a_j' \) between \( a_j \) and \( a(j, k_1) \). Then \( D_1 = D\cap \mathcal{D}^* \) is a topological two-disk. Thus, there is no loss of generality in assuming \( D_1 = D^2 \). Let \( p_1 \) be the natural map from \( D^* \) to \( D_1 \). The mapping \( F^* = Fp_1^{-1} \) is a properly interior extension of \( f^* \).

In a similar way, take \( \bigcup_{i=1}^{n} S_i \) to be the fan belonging to \( B \) with respect to \( G \) and let \( x' \) be the hinge. Suppose that \( S_1 \) has endpoints \( x' \) and \( f^{-1}(a(j, k_1)) \) and \( S_2 \) has \( x' \) and \( f^{-1}(a(j, k_2)) \). Then \( S_1 \cup S_2 \) splits \( D^2 \) into \( E^* \) and \( E^{**} \). Let \( \mathcal{E}^* \) be the decomposition of \( E^* \) whose nondegenerate elements are all sets of the form \( G^{-1}(y) \), where \( y \) is a point of \( a_j' \) between \( a \) and \( a(j, k_1) \). Let \( q_1 \) be the natural map from \( D^* \) to \( E_1 = E\cap \mathcal{E}^* \) and again we take \( E_1 = D^2 \). The mapping \( G^* = q_1^{-1} \) is a properly interior extension of \( f^* \). Both \( F^* \) and \( G^* \) have fan structure that corresponds to \( \mathcal{B}^* \). Since \( TWN(f^*) < TWN(f) \) (Theorem 1 and Theorem 5), the induction hypothesis provides us with a homeomorphism \( H_1 : D^2 \to D^2 \) such that \( F^* = G^*H_1 \).

In a similar way, there are natural maps \( p_2 : D^{**} \to D^2 \) and \( q_2 : E^{**} \to D^2 \), such that \( F^{**} = Fp_2^{-1} \) and \( G^{**} = q_2^{-1} \) are equivalent. Let \( H_2 \) be the homeomorphism from \( D^2 \) to \( D^2 \) such that \( F^{**} = G^{**}H_2 \).

Now we define a homeomorphism \( H : D^2 \to D^2 \) by \( H(z) = q_1^{-1}H_1p_1(z) \) for \( z \in D^* - D^{**}, H(z) = q_2^{-1}H_2p_2(z) \) for \( z \in D^{**} - D^*, H(z) = S_1 \cap q_1^{-1}H_1p_1(z) \) for \( z \in T_1 \), and \( H(z) = S_2 \cap q_1^{-1}H_1p_1(z) \) for \( z \in T_2 \). To see that \( H \) is well defined (from which it follows readily that \( H \) is a homeomorphism), we must show that \( H(z) = S_1 \cap q_2^{-1}H_2p_2(z) \) for \( z \in T_1 \) and similarly for \( z \in T_2 \). Note then that \( F^*p_1(z) = F^{**}p_2(z) = F(z) \). But \( F^*(p_1(z)) = G^*H_1p_1(z) \) and \( F^{**}p_2(z) = G^{**}H_2p_2(z) \). Hence, \( Gq_1^{-1}H_1p_1(z) = F(p_1(z)) = F(z) \) and also \( Gq_2^{-1}H_2p_2(z) = F(z) \). There is exactly one point \( u \) on \( S_1 \) such that \( G(u) = F(z) \). Thus, \( q_1^{-1}H_1p_1(z) \cap S_1 = u = q_2^{-1}H_2p_2(z) \cap S_1 \). The proof for \( T_2 \) is similar.
Finally \( GH = Gq'_{1}H_{1}p_{1} = G^{*}H_{1}p_{1} = F^{*}p_{1} = F \) on \( GH = Gq'_{2}H_{2}p_{2} = G^{*}H_{2}p_{2} = F^{*}p_{2} = F \) and we have proved the theorem for the case the \( \alpha \) has no negative letters.

Now suppose the theorem true for any normal mapping having word with \( < n \) negative letters. Let \( f: S^{1} \to R^{2} \) have word \( \alpha \) with \( n > 0 \) negative letters.

Select a terminal negative letter \( \alpha(j, k)^{-1} \). Then \( P = \{ \alpha(j, k)^{-1}, \alpha(j, k') \} \in \mathcal{J} \) for some \( k' > k \). Let \( f^{*} \) and \( f^{**} \) be the associated Type II cuts. Assume \( \alpha = \beta\alpha(j, k)^{-1}\zeta\alpha(j, k') \). By Theorem 3, \( f^{*} \) has word \( \alpha^{*} \) of which \( \zeta \) is a subword; similarly \( f^{**} \) has a word \( \alpha^{**} \) of which \( \beta\beta \) is a subword. Both \( \alpha^{*} \) and \( \alpha^{**} \) have \( < n \) negative letters. Any \( A \in \mathcal{A}^{*} \) either comes from \( \zeta \) or \( \beta\beta \). Let \( \mathcal{A}^{*} \) be the collection of the former; \( \mathcal{A}^{**} \), the latter. Then \( \mathcal{A}^{*} \) is an assemblage for \( f^{*} \), and \( \mathcal{A}^{**} \), for \( f^{**} \).

Let \( T \) be the arc in \( D^{2} \) that belongs to \( P \) with respect to \( F \) and let \( S \) be the arc that belongs to \( P \) with respect to \( G \). Note that \( T \) divides \( D^{2} \) into two disks \( D_{1} \) and \( D_{2} \); similarly \( S \) divides \( D^{2} \) into \( E_{1} \) and \( E_{2} \). Since \( \mathcal{A}(F|D_{1}) = \mathcal{A}^{*} = \mathcal{A}(G|D_{1}) \) and \( \mathcal{A}(F|D_{2}) = \mathcal{A}^{**} = \mathcal{A}(G|E_{2}) \), the induction hypothesis gives that \( F|D_{1} \) is equivalent to \( G|E_{1} \) and \( F|D_{2} \) is equivalent to \( G|E_{2} \). Because \( F \) maps \( T \) homeomorphically onto \( [\alpha(j, k)^{-1}\alpha(j, k')(\alpha)] \) and so does \( G \) map \( S \), we can apply a glueing argument similar to the one used in the case \( n = 0 \), and obtain that \( F \) and \( G \) are equivalent.

6. Conclusions. We have now established the following theorems for \( f: S^{1} \to R^{2} \) normal:

I. The mapping \( f \) is an interior boundary if and only if it has an assemblage.

II. There is a one-to-one correspondence between maximal assemblages of some word for \( f \) and the equivalence classes of properly interior extensions of \( f \).

III. The above correspondence is such that if the equivalence class of \( F \) is associated with \( \mathcal{A} = \mathcal{B} \cup \mathcal{G} \), then the branch points of \( F \) have multiplicities \( -1 + \) cardinality \( B \), where \( B \) ranges over \( \mathcal{B} \).

IV. The mapping \( f \) has an extension to \( D^{2} \) that is an immersion if and only if \( f \) has a maximal assemblage \( \mathcal{A} = \mathcal{B} \cup \mathcal{G} \) with \( \mathcal{B} = \emptyset \). The maximal number of mutually inequivalent such extensions is given by II above.

The reader should check the Introduction to see that all the questions and problems stated there have been dealt with.

Those familiar with Blank's work may question the relationship between the words used here and Blank's words (see §3). Our hypotheses are (perhaps) stronger in that our word carries more information via the second index on each letter. However, we can deal with a wider variety of problems. Also, to apply Blank's theorem one must know that \( \text{TWN} = 1 \). There are normal mappings with \( \text{TWN} = 2 \) which having groupings in the sense of Blank.
Blank word: $a(1, 1)a(2, 1)a(4, 1)a(5, 1)a(2, 2)a(3, 1)a(4, 2)^{-1}$
word: $a(1, 1)a(2, 1)a(4, 2)a(5, 1)a(2, 2)a(3, 1)a(4, 1)^{-1}$

Figure 1

Type I:

Type II:

Type III:

Figure 2
Figure 3

$s = 2, \ t = 3$

Figure 4
We only allow sets \( \{ \alpha(j, k)^{-1}, \alpha(j, k') \} \) in \( \mathcal{G} \) when \( k < k' \); Blank has no way of knowing whether \( \alpha(j, k)^{-1} \) precedes \( \alpha(j, k') \) as he does not have the second index. However, he cleverly proves that if a grouping exists, the sets in it have the property that the negative letter must proceed the positive (see [1, p. 21]). That one fact added to the theorems of this paper will give a proof of Blank's theorem.

There are many interesting questions concerning the relationship between Titus intersection sequences [8, p. 1084], Blank words, and the words used here. The author hopes to have a paper forthcoming on this subject.\(^{(1)}\)

REFERENCES


DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 32735