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EXTENSIONS OF NORMAL IMMERSIONS OF S^1 INTO R^2

BY

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ABSTRACT. Suppose that $f: S^1 \to R^2$ is an *immersion*, i.e., a C^1 map such that f' is never zero. We call f *normal* if there are only finitely many self-intersections and these are transverse double points. A normal immersion f can be topologically determined by a finite number of combinatorial invariants. Using these invariants it is possible to give considerable information about extensions of f to D^2. In this paper we give a new set of invariants, inspired by the work of S. Blank, to solve several problems concerning the existence of certain kinds of extensions. The problems solved are as follows:

(1) When does f have a light open extension $F: D^2 \to R^2$? (Recall that *light* means $F^{-1}(y)$ is totally disconnected for all y and *open* means F maps open sets of the interior of D^2 to open sets of R^2.) Because of the work of Stoïlow, the question is equivalent to the following: when does there exist a homeomorphism $h: S^1 \to S^1$, such that fh has an analytic extension to D^2?

(2) Suppose that $F: D^2 \to R^2$ is light, open, sense preserving, and, at each point of S^1, F is a local homeomorphism. At each point of the interior of D^2, F is locally topologically equivalent to the power mapping z^m on D^2, $m \geq 1$. Points where $m > 1$ are called *branch points* and $m - 1$ is the *multiplicity* of the point. There are only a finite number of branch points. The problem is to discover the minimum number of branch points of any properly interior extension of f. Also we can ask what multiplicities can arise for extensions of a given f.

(3) Given a normal f, find the maximum number of properly interior extensions of f that are pairwise inequivalent. Since each immersion of the disk is equivalent to a local homeomorphism, the problem of immersion extensions is a special case of this. It is Blank's solution of the immersion problem that prompted this paper.

1. Introduction. Throughout the paper we shall let R^2 denote Euclidean two-space; S^1, the unit circle in R^2; and D^2, the unit disk in R^2.

Suppose that $f: S^1 \to R^2$ is a C^1 map such that f' is zero at only a finite number of points. We call f a *normal immersion* if there are only finitely many self-intersections and these are transverse double points. We insist that f' be

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nonzero at the preimages of the double points. A normal immersion f can be topologically determined by a finite number of combinatorial invariants. Using these invariants it is possible to give considerable information about extensions of f to D^2. In this paper we give a new set of invariants, inspired by the work of S. Blank [1], [6], to solve several problems concerning the existence of certain kinds of extensions. We list these problems and survey the work that has been done on them. In what follows $f: S^1 \to R^2$ is a normal immersion. All mappings are continuous.

I. When does f have a light open extension $F: D^2 \to R^2$? Recall that *light* means $F^{-1}(y)$ is totally disconnected for all y and *open* means F maps open sets of the interior of D^2 to open sets of R^2. Because of the work of Stoïlow [7], the question is equivalent to the following: when does there exist a homeomorphism $h: S^1 \to S^1$, such that fh has an analytic extension to D^2?

Titus [9] traces work on problems of this nature back to Picard. The problem in its present form was first stated circa 1948 by C. Loewner. The solution was given by Titus in [9] in the form of an algorithm.

II. Suppose that $F: D^2 \to R^2$ is light, open, sense preserving, and F is a local homeomorphism at each point of $F^{-1}(F(S^1))$. We call such a map *properly interior*. Again appealing to the work of Stoïlow (see [7]), we discover that at each point of the interior of D^2, F is locally topologically equivalent to the power mapping z^m on D^2, $m \geq 1$. Points where $m > 1$ are called *branch points* and $m - 1$ is the *multiplicity* of the point. There are only a finite number of branch points. If $m_1, m_2, \cdots, m_k$ are the multiplicities of the branch points of F, then $1 + m_1 + m_2 + \cdots + m_k = \mathrm{TWN}(f)$. (Recall that $\mathrm{TWN}(f)$ is the Brouwer degree of the mapping $f'/\|f'\|: S^1 \to S^1$.) This last result is standard for analytic functions and, hence, is also true for properly interior mappings, since they are topologically equivalent to analytic mappings. The problem here is to discover the minimum number of branch points of any properly interior extension of f. Also we can ask what multiplicities can arise for extensions of a given f.

The author gave an algorithm in [4] to solve the above for normal f.

III. Oral tradition has it that John Milnor discovered a normal $f: S^1 \to R^2$ with two extensions F and G, both immersions, but not equivalent. (Maps ϕ: $X \to Y$ and $\psi: Z \to Y$ are *equivalent* if there exists a homeomorphism $h: X \to Z$ with $\phi = \psi h$.) The problem then is to discover the maximum number of immersion extensions that are pairwise inequivalent. It is enough to solve the problem for extensions that are local homeomorphisms instead of immersions, since each local homeomorphism is equivalent to an immersion (see Jewitt [2, Theorem 3, p. 111]).

Although nowhere stated in the paper, Titus' solution to I also solves this problem. A second and very interesting solution was given by S. Blank in his dissertation [1].

IV. Having restated problem III in terms of local homeomorphisms instead of immersions, we give a straightforward generalization: Given a normal f, discover the maximum number of properly *interior* extensions of f that are pairwise inequivalent. In this context, problem I asks whether that number is zero. If we have solved problem IV and either know that $\mathrm{TWN}(f) = 1$ or solve problem II, then we obtain a solution to problem III.

In this paper a solution is given to the preceding four problems. First to each normal f a word is assigned (§3). This word is a finite string of symbols. Then we define a collection of subsets of the symbols in the word called a maximal assemblage (§5). Theorems 6, 7, and 8 establish a one-to-one correspondence between equivalence classes of certain light open extensions of f and maximal assemblages. Thus, problems I, III, and IV are solved. Further, every maximal assemblage $\mathfrak{A}$ contains a subcollection $\mathfrak{B}$ (empty in the case of local homeomorphism extensions) such that there is a one-to-one correspondence between branch points of the extension of f corresponding to $\mathfrak{A}$ and the sets of $\mathfrak{B}$. This correspondence is such that the multiplicity of the branch point associated with $B \in \mathfrak{B}$ is $(-1) +$ (cardinality of B). Hence, problem II is solved.

2. **Definitions and conventions.** We have already defined normal mapping in the introduction. The points of the image that are self-intersections of such a mapping are called *vertices*. We can also consider mappings $f: I \to R^2$, I the unit interval, as being normal. For any normal f mapping from either I or S^1, we use the symbol $[f]$ to denote the image of f. Any mapping equivalent to a normal mapping that has a properly interior extension to D^2 is called an *interior boundary*.

The problems investigated in this paper are of such a nature that any mapping can be replaced with one equivalent to it. Actually, we are working with equivalence classes of maps. The notation and language become too cumbersome with classes, so we use maps and replace them when desired.

Suppose f and g are mappings on I (they may be defined on S^1 and the modifications for this case are easy). Let $a < b$ and $a' < b'$ be such that $f(b) = g(a')$. By $f(a)f(b)(f) + g(a')g(b')(g)$ we mean the mapping ϕ defined by $\phi(t) = f(2(b-a)t + a)$ for $0 \le t \le \frac{1}{2}$ and $\phi(t) = g(2(b'-a')(t-1) + b')$ for $\frac{1}{2} \le t \le 1$, i.e. the path composition of $f|[a, b]$ and $g|[a', b']$. If $f^{-1}(y) = \{a\}$, there is no ambiguity in writing $yf(b)(f)$. As usual, we will substitute for the given parametrization ϕ any one that is equivalent to it.

3. **Words.** In this section we describe a class of combinatorial invariants for a normal immersion. Suppose $f: S^1 \to R^2$ is normal. Let Θ be a subset of the positive integers. Let $\{\alpha_j \mid j \in \Theta\}$ be a collection of arc paths (homeomorphic mappings on the unit interval) in R^2 with the following properties.

(1) The initial point a_j of α_j is a point in some bounded component of $R^2 - [f]$. The terminal point b_j of α_j is in the unbounded component of $R^2 - [f]$.

(2) There is at least one a_j in each bounded component of $R^2 - [f]$.

(3) Each α_j is transverse to f.

(4) The α_j are pairwise disjoint.

Orient α_j from a_j to b_j. Then define a crossing point of f and α_j to be *positive* if f crosses α_j from right to left; define it *negative* if f crosses α_j from left to right. Index the crossing points of f and α_j by $\alpha(j, k)$ (if the crossing is positive) or $\alpha(j, k)^{-1}$ (if it is negative), where k ranges over some subset of the positive integers, say $\Delta(j)$.

Trace along $[f]$ using the positive orientation; i.e., that orientation induced by the counterclockwise orientation on S^1. Make a sequence or list of the crossing point indices (hereafter called *letters*) as they are encountered in the above tracing of $[f]$. Such a list will be called a *word* for f. We have not yet made use of the double indexing of the crossing points. We could assume that, for fixed j, $\alpha(j, k)$ precedes $\alpha(j, k')$ in the word if and only if $k < k'$. In this case, the second index carries no more information than is already present in the word. When this assumption is made we call the word a *Blank word* (see [1]). Suppose we require that the word contain more information by assuming that, for fixed j, $\alpha(j, k')$ follows $\alpha(j, k)$ in the orientation of α_j ($\alpha(j, k')$ is "closer to ∞") if and only if $k < k'$. Then we simply call the word a word. Hopefully, the reader will find the preceding understandable by looking at Figure 1.

A word or Blank word is said to be *reduced* if the following hold.

(1) It does not begin with $\alpha(j, k)$ and end with $\alpha(j, k')^{-1}$.

(2) It does not begin with $\alpha(j, k)^{-1}$ and end with $\alpha(j, k')$.

(3) In the word, $\alpha(j, k)$ is never next to $\alpha(j, k')^{-1}$.

Blank shows that the rays can always be drawn so that the word is reduced (see [1, p. 6]); we will assume this has been done.

A given f may have many words, depending on the starting point of the tracing of $[f]$ and, more importantly, on the choice of $\{\alpha_j\}$.

4. Cuts. We introduce here some methods of "cutting" a normal immersion into less complicated curves.

Definition. Suppose $f: S^1 \to R^2$ is normal. Let $v \in [f]$ be a *terminal vertex* of f; that is, there are points $t_1 = e^{i\theta_1}$ and $t_2 = e^{i\theta_2}$, with $0 \le \theta_1 < \theta_2 < 2\pi$, such that $f(t_1) = f(t_2) = v$, but f is one-to-one on $\{e^{i\theta} \mid \theta_1 < \theta < \theta_2\}$. Take f^* to be any regular parametrization of the curve $[f \mid T]$, where $T = \{e^{i\theta} \mid 0 \le \theta \le \theta_1, \theta_2 \le \theta < 2\pi\}$ (see Figure 2). We shall call f^* *a cut of Type* I *associated with* v. Since f is normal, it has only finitely many vertices, so such f^* always exist.

Definition. Let $f: S^1 \to R^2$ be normal. Suppose $a = \alpha(j, k)^e$ and $b =$

$\alpha(j, k')^{e'}$ are two letters in α with $k < k'$. Define f^* and f^{**} by

$$f^* = ab(f) + ba(-\alpha_j), \qquad f^{**} = ab(\alpha_j) + ba(f).$$

(See Figure 2.) If $e = -1$ and $e' = +1$, then we call f^* and f^{**} *cuts of Type* II associated with $\alpha(j, k)^e$ and $\alpha(j, k')^{e'}$; if $e = e' = +1$, *cuts of Type* III.

Now we investigate the ways in which the cuts are simpler than f.

Theorem 1.

(a) *Type* I. $\mathrm{TWN}(f) = \mathrm{TWN}(f^*) + \lambda(v)$, *where* $\lambda(v) = +1$ *if the loop at* v *has counterclockwise orientation and* $\lambda(v) = -1$ *otherwise.*

(b) *Type* II. $\mathrm{TWN}(f) = \mathrm{TWN}(f^*) + \mathrm{TWN}(f^{**}) - 1$.

(c) *Type* III. $\mathrm{TWN}(f) = \mathrm{TWN}(f^*) + \mathrm{TWN}(f^{**})$.

The proof of Theorem 1 uses standard arguments on winding number and will not be given.

Theorem 2. *Suppose* $f: S^1 \to R^2$ *is normal with word* α *and let* v *be a terminal vertex. Suppose* $\alpha = \beta x \zeta y \delta$, *where* x *is the last letter before* f *crosses* v *the first time and* y *is the first letter after* f *crosses* v *the second time. Then* f^*, *the cut of Type* I *associated with* v, *has a word which is a subword of* $\beta x y \delta$.

Proof. Since f^* does not trace along f from x to y, we may omit ζ as f^* does not cross those letters. Also $[f^*]$ removes a Jordan curve from $[f]$, so some of the components of $S^1 - [f]$ may come together in $S^1 - [f^*]$. Thus we may discard some of the α_j. However, what is crucial is that no new components are created.

Definition. Suppose $\alpha(j, k)^{-1}$ is a letter of α such that for any $\alpha(j, k')^e$ in α with $k < k'$, we have $e = +1$. Then we shall call $\alpha(j, k)^{-1}$ a *terminal* negative letter.

Theorem 3. *Let* $f: S^1 \to R^2$ *be normal with word* α. *Suppose* $\alpha(j, k)^{-1}$ *is a terminal negative letter. Let* f^* *and* f^{**} *be the Type* II *cuts associated with* $\alpha(j, k)^{-1}$ *and* $\alpha(j, k')$, $k < k'$. *Suppose* $\alpha = \beta\alpha(j, k)^{-1}\zeta\alpha(j, k')\delta$. *Then* f^* *has a word* α^* *of which* ζ *is a subword. Also,* f^{**} *has a word* α^{**} *of which* $\beta\delta$ *is a subword. Finally, both* α^* *and* α^{**} *have strictly less negative letters than* α.

Proof. First, we know that the letters of ζ are traced by f^* and these are the only letters of α that it traces. Note that every component of $R^2 - [f^*]$ is a subset of some component of $R^2 - [f]$. Notice that some of the components through which α_j passes have been subdivided into two components. One of the two pieces already has some α_j originating in that piece. We will need a new α_j

originating in the other piece, and it should be placed in the manner suggested by Figure 3.

Let $k < k_1 < \cdots < k_s < k' < k_{s+1} < \cdots < k_t$ be those points of $\Delta(j)$ such that f^* crosses α_j at only the $\alpha(j, k_i)$, $1 \le i \le t$. Again referring to Figure 3, we need an α_j for some of the components below $[f^*]$ and to the left of some $\alpha(j, k_i)$. Thus, if Θ had M elements, we define $\alpha_{M+1}, \cdots, \alpha_{M+s+1}$ as shown in Figure 3. Also, we need α_j for some of the components above $[f^*]$ and to the left of some $\alpha(j, k_i)$. So we define $\alpha_{M+s+2}, \cdots, \alpha_{M+t}$.

To construct the word for f^*, first take $\alpha(M+s+1, 1)\alpha(M+s, 1) \cdots \alpha(M+1, 1)\zeta$. For each $\alpha(j, k_i)^{+1}$, insert $\alpha(M+i, 2)\alpha(M+i-1, 3) \cdots \alpha(M+1, i)$ into ζ immediately before $\alpha(j, k_i)^{+1}$. Also, for $i > 1$, insert $\alpha(M+s+i, 1) \cdot \alpha(M+s+i-1, 2) \cdots \alpha(M+s+2, i-1)$ into ζ immediately after $\alpha(j, k_i)^{+1}$. Recall that $\alpha(j, k)^{-1}$ is terminal, so we are justified in assuming that the exponent of $\alpha(j, k_i)^{+1}$ is indeed $+1$. Clearly ζ is a subword of this new word. Also, we have lost $\alpha(j, k)^{-1}$ and added no new negative letters; hence there are less negative letters than in α.

The proof for f^{**} is very similar and, in fact, can be reduced to the preceding case by reflecting the plane about a line through α_j and reversing the orientation of f.

We need a theorem like Theorem 3 for Type III cuts. However, it turns out that for the application we make of these, it is better to modify the cuts slightly (as opposed to moving α_j so that it will be transverse to the cut).

Consider first the cut f^* associated with $\alpha(j, k)$ and $\alpha(j, k')$, $k < k'$. Let β be an arc to the right of and parallel to α_j which is transverse to f^* (see Figure 4). Then define $\operatorname{mod} f^*$ to be the mapping $\alpha(j, k)u_1(f^*) + u_1v_1(\beta) + v_1\alpha(j, k)(f)$. We may think of this as moving the arc $[\alpha(j, k)\alpha(j, k')(f)]$ to the right of α_j. For f^{**}, let γ be an arc to the left of and parallel to α_j which is transverse to f^{**} (see Figure 4). Define $\operatorname{mod} f^{**}$ to be the mapping $\alpha(j, k)v_2(f) + v_2u_2(\gamma) + u_2\alpha(j,k)(f^{**})$.

It is apparent from Figure 4 that $\operatorname{mod} f^*$ and f^* have the same intersection sequence [8, p. 1084]; similarly, $\operatorname{mod} f^{**}$ and f^{**}. By [9, Theorem 3, p. 49] $\operatorname{mod} f^*$ and f^* are topologically equivalent, as are $\operatorname{mod} f^{**}$ and f^{**}. Thus, any result that follows, concerning a Type III cut remains true when it is modified as above.

Theorem 4. *Let $f: S^1 \to R^2$ be normal with word α and suppose α has all positive letters. Let f^* and f^{**} be the Type* III *cuts associated with $\alpha(j,k)$ and $\alpha(j, k')$, $k < k'$. Suppose $\alpha = \beta\alpha(j, k)\zeta\alpha(j, k')\delta$. Then $\operatorname{mod} f^*$ has a word α^* with all positive letters of which $\alpha(j, k)\zeta$ is a subword and $\operatorname{mod} f^{**}$ has a word α^{**} with all positive letters of which $\beta\alpha(j, k)\delta$ is a subword.*

Proof. The techniques of proof here are similar to those used in Theorem 3. Figure 5 should illustrate the differences between Theorems 3 and 4.

5. **Assemblages.** We now discuss the operations on the word that allow us to state the main theorems in §6.

Definition. Let $f: S^1 \to R^2$ be normal with word α. Suppose $\mathcal{G}$ and $\mathcal{B}$ are collections whose elements are sets of letters of α. Let $\mathcal{A} = \mathcal{G} \cup \mathcal{B}$. Suppose that:

(1) The sets of $\mathcal{A}$ are mutually disjoint.

(2) No two sets of $\mathcal{A}$ are linked, i.e. if $A, A' \in \mathcal{A}$, then $f^{-1}(A)$ does not separate $f^{-1}(A')$ in S^1.

(3) Every set in $\mathcal{G}$ is of the form $\{\alpha(j, k)^{-1}, \alpha(j, k')\}$ with $k < k'$.

(4) Every negative letter of α occurs in some set in $\mathcal{G}$.

(5) Let $B \in \mathcal{B}$. Then all the letters in B come from the same α_j, i.e., there exists $j \in \Theta$ such that every letter in B is of the form $\alpha(j, k)$ for some $k \in \Delta_j$. (Note that because of (1) and (4), no letter of B can be negative.)

Then we call $\mathcal{A}$ an *assemblage*, $\mathcal{G}$ a *grouping*, and $\mathcal{B}$ a *branching* for α.

We take $\mathcal{G} = \emptyset$ in case α has no negative letters; also we permit $\mathcal{B} = \emptyset$.

The main goal of this section is to investigate the relationship between assemblages and properly interior extensions of f. We first prove a theorem for the case $\mathcal{G} = \emptyset$.

Theorem 5. *Suppose $f: S^1 \to R^2$ is normal with word α and α has all positive letters.*

(1) *f is an interior boundary.*

(2) $\mathrm{TWN}(f) \geq 1$.

(3) *If* $\mathrm{TWN}(f) = 1$, *then f represents a positively oriented Jordan curve.*

Proof. We prove the theorem by induction on n, the number of vertices of f.

If $n = 0$, then f has no vertices and is a one-to-one mapping. Hence, $[f]$ is a Jordan curve. Since α has all positive letters, the orientation must be positive. Then (1), (2), (3) hold.

Suppose the theorem is true for any normal mapping with $n - 1$ vertices, $n \geq 1$. Let f have n vertices. Then f must have a terminal vertex v. This vertex must have $\lambda(v) = +1$. Otherwise, the loop determined by v would have negative orientation and any a_j originating inside this loop would give α a negative letter, a contradiction to hypothesis.

Let f^* be the cut of Type I associated with v. Clearly f^* has $n - 1$ vertices and it is not difficult to see that f^* has a word which is a subword of α. Thus, we can apply the induction hypothesis to f^* and conclude that it is an interior boundary. By Theorem 19.1 [5, p. 67], $\mathrm{TWN}(f^*) \geq 1$. But Theorem 1

implies that $\mathrm{TWN}(f) = \mathrm{TWN}(f^*) + 1 \geq 2$. Hence, (2) holds for f and (3) does not apply. Finally to see that f is an interior boundary refer to Theorem 4 [9, p. 54].

Definition. Let $f: S^1 \to R^2$ be normal with word α. Suppose $\mathfrak{A}$ is an assemblage for α. Let $F: D^2 \to R^2$ be a properly interior extension of f.

Let $P = \{\alpha(j, k)^{-1}, \alpha(j, k')\}$ be a set in $\mathfrak{G}$. Let $U = [\alpha(j, k)^{-1}\alpha(j, k')(\alpha_j)]$, i.e., the set of points on α_j between $\alpha(j, k)^{-1}$ and $\alpha(j, k')$. We say the arc $T \subset D^2$ *belongs to* P if the endpoints of T are $f^{-1}(P)$, if T intersects S^1 only at its endpoints, and if F maps T homeomorphically onto U.

Suppose $T_1, T_2, \cdots, T_n$ are arcs and there exists a point x such that x is an endpoint of each of the arcs and such that $T_i \cap T_j = \{x\}$ for $1 \leq i < j \leq n$. Then we shall call $\bigcup_{i=1}^n T_i$ a *fan* and x the *hinge*.

Let $B \in \mathfrak{B}$. We say that the fan $V = \bigcup_{i=1}^n T_i \subset D^2$ *belongs* to B if the following conditions are satisfied:

(1) Let E be the set of endpoints of the T_i other than x. Then $V \cap S^1 = E = f^{-1}(B)$.

(2) For each T_i, there exists $\alpha(j, k) \in B$ such that F maps T_i homeomorphically onto $[a_j\alpha(j, k)(\alpha_j)]$ (the set of points on α_j that precede $\alpha(j, k)$).

Note that x is a branch point of multiplicity at least $-1 +$ cardinality B.

Theorem 6. *Let $f: S^1 \to R^2$ have assemblage $\mathfrak{A}$. There exists a properly interior extension F of f to D^2 such that the following hold:*

(1) *For each $G \in \mathfrak{G}$, there is an arc in D^2 belonging to G.*

(2) *For each $B \in \mathfrak{B}$, there is a branch point x in D^2 of multiplicity $-1 +$* cardinality B *and x is the hinge of a fan in D^2 belonging to B. Also, $F(x) = a_j$, where α_j is the ray associated with B.*

Proof. We prove the theorem by induction on n, the number of negative letters in α. For $n = 0$, α has all positive letters and $\mathfrak{G} = \emptyset$. Only (2) applies here. This is a subtheorem which we also prove by induction, but we induct on $\mathrm{TWN}(f)$. This is possible since α has all positive letters and, by Theorem 5, $\mathrm{TWN}(f) \geq 1$.

First consider the case $\mathrm{TWN}(f) = 1$. By Theorem 5, f represents a positively oriented Jordan curve and f is an interior boundary. The winding number of f about each a_j is $+1$. It then follows from [8, Lemma 2, p. 1085] that α has only one letter from each α_j, since it cannot contain a negative crossing. We conclude that $\mathfrak{B} = \emptyset$ or consists of one element sets. Hence (2) is vacuously true.

Now suppose the subtheorem is true for all normal mappings having a word with all positive letters and with $\mathrm{TWN} < \mathrm{TWN}(f)$. Prove the theorem for $\mathrm{TWN}(f) \geq 2$.

If $\mathfrak{B} = \emptyset$ or has only one-element sets, then (2) requires nothing more than that f be an interior boundary, which follows from Theorem 5. Now we give an argument in case $\mathfrak{B}$ has at least one set with two or more elements.

Hence suppose $B = \{\alpha(j, k_1), \cdots, \alpha(j, k_m)\}$ is in $\mathfrak{B}$. Assume without loss of generality that $k_1 < k_2$ and $\alpha = \beta\alpha(j, k_1)\zeta\alpha(j, k_2)\delta$ where $\alpha(j, k_r)$ is not a letter of ζ for any r, $1 \leq r \leq m$. Let f^* and f^{**} be the Type III cuts associated with $\alpha(j, k_1)$ and $\alpha(j, k_2)$. By Theorem 4, mod f^* has a word ζ^* with all positive letters of which $\alpha(j, k_1)\,\zeta$ is a subword; similarly mod f^{**} has word α^{**} with all positive letters of which $\beta\alpha(j, k_1)\delta$ is a subword. From part (c) of Theorem 1 and Theorem 5, we see that $\mathrm{TWN}(f^*) = \mathrm{TWN}(\text{mod } f^*)$ and $\mathrm{TWN}(f^{**}) = \mathrm{TWN}(\text{mod } f^{**})$ are both $< \mathrm{TWN}(f)$. We will be able to apply the induction hypothesis as soon as we have assemblages for mod f^* and mod f^{**}.

For any set $B' \in \mathfrak{B}$, $B' \neq B$, by the nonlinking property of assemblages, either the letters of B' all come from ζ or all from $\beta\delta$. Let $\mathfrak{B}^*$ be the collection of all sets B' that come from ζ together with the set $\{\alpha(j, k_1)\}$. Let $\mathfrak{B}^{**}$ be the collection of all sets B' that come from $\beta\delta$ together with $B - \{\alpha(j, k_2)\}$. Then $\mathfrak{B}^*$ is an assemblage for mod f^* and $\mathfrak{B}^{**}$ is an assemblage for mod f^{**}. Thus, there is a properly interior extension of mod f^* to D^2 that satisfies the conclusion of the theorem with respect to $\mathfrak{B}^*$; similarly F^{**}.

Consider the fan that belongs to $\{\alpha(j, k_1)\} \in \mathfrak{B}^*$. In this case, it is a single arc T^* from y^*, the preimage of $\alpha(j, k)$ under mod f^*, to x^*, some point in the interior of D^2. Let us assume without loss of generality that $T^* = \{re^{i\theta} \mid 0 \leq r \leq 1,\ \theta = 0\}$.

We are going to construct a topological disk D and a properly interior mapping of it which gives a mapping equivalent to f on boundary D. Let $D_1 = \{re^{i\theta} \mid 0 \leq \theta \leq \pi,\ 0 \leq r \leq 1\}$; on D_1, define G_1 by $G_1 = F^* \circ z^2$. Assume that $G_1(i) = u_1$. Now let $r(\theta)$ be some function of θ, $\pi/2 \leq \theta \leq \pi$, where $r(\theta)$ strictly increases in value from 1 to 2. Take $D_2 = \{re^{i\theta} \mid \pi/2 \leq \theta \leq \pi,\ 1 \leq r \leq r(\theta)\}$. Suppose that g_2 maps $\{r(\theta)e^{i\theta} \mid \pi/2 \leq \theta \leq \pi\}$ homeomorphically onto $[u_1\alpha(j, k_2)(f)]$ with $g_2(i) = u_1$ and $g_2(-2) = \alpha(j, k_2)$. Let h_2 map $\{re^{i\theta} \mid -2 \leq r \leq -1,\ \theta = \pi\}$ homeomorphically onto $[\alpha(j, k_2)\alpha(j, k_1)(-\alpha_j)]$ with $h_2(-2) = \alpha(j, k_2)$ and $h_2(-1) = \alpha(j, k_1)$. Then using G_1, g_2, h_2 on the three arcs that make up boundary D_2, we obtain a counterclockwise oriented homeomorphism h of boundary D_2 onto $\{\alpha(j, k_1)v_1(-f^*) + v_1u_1(-\beta) + u_1\alpha(j, k_2)(f) + \alpha(j, k_2)\alpha(j, k_1)(-\alpha_j)]$, a Jordan curve with a similar orientation. Thus there is a sense-preserving homeomorphism $G_2\colon D_2 \to R^2$ that extends h.

Now recall there is a fan in D^2 associated with $B - \{\alpha(j, k_2)\}$ with regard to f^{**}. One arc T^{**} of this fan maps onto $[a_j\alpha(j, k_1)(\alpha_j)]$ under F^{**}. There is no loss of generality in assuming that $T^{**} = T^* = \{re^{i\theta} \mid 0 \leq r \leq 1,\ \theta = 0\}$ and further that $F^{**}(t) = F^*(t)$ for all $t \in T^* = T^{**}$. Take $D_3 = \{re^{i\theta} \mid \pi \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}$; on D_3 define G_3 by $G_3 = F \circ z^2$. Assume $G_3(-i) = u_2$. Note that G_1 and G_3 agree where their domains overlap, i.e. on the real axis between -1 and 1. Now let

$s(\theta)$ be a function of θ, $\pi \leq \theta \leq 3\pi/2$, that strictly decreases from 2 to 1. Take $D_4 = \{re^{i\theta} \mid \pi \leq \theta \leq 3\pi/2,\ 1 \leq r \leq s(\theta)\}$. Suppose g_4 maps $\{s(\theta)e^{i\theta} \mid \pi \leq \theta \leq 3\pi/2\}$ homeomorphically onto $[\alpha(j, k_2)u_2(f)]$ with $g_4(-2) = \alpha(j, k_2)$ and $g_4(-i) = u_2$. Using G_3, g_4, h_2 on the boundary of D_4 we obtain a counterclockwise oriented homeomorphism h' of the boundary of D_4 onto the similarly oriented Jordan curve $[\alpha(j, k_1)\alpha(j, k_2)(\alpha_j) + \alpha(j, k_2)u_2(f) + u_2v_2(-\gamma) + v_2\alpha(j, k_1)(-f)]$. Thus there exists a sense-preserving homeomorphism $G_4: D_4 \to R^2$ that extends h'. Let $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Note that each of G_1, G_2, G_3, G_4 agree where their domains overlap. Thus we can define a mapping G on D by $G \mid D_i = G_i$, $1 \leq i \leq 4$. By [10, Theorem 9, p. 336], G is properly interior. Also G restricted to the boundary of D is a local homeomorphism that parametrizes $[f]$ and so gives a mapping equivalent to f. Thus, f is an interior boundary.

We also note that there is the desired fan belonging to B. One arc of the fan is the real axis between 0 and 1, one arc is the real axis between -1 and 0, the remaining (-2 + cardinality B) arcs are in D_3 and were produced by the induction hypothesis. There are no other arcs at 0 that map into $[\alpha_j]$ except the aforementioned. Thus 0 is a branch point of multiplicity -1 + cardinality B. This completes the proof of the subtheorem—the case when $\mathcal{G} = \emptyset$. Recall that this is the induction step $n = 0$, where n is the number of negative letters in α.

Now let us suppose the theorem is true for any normal mapping with word with $< n$ negative letters and let $f: S^1 \to R^2$ have word α with $n > 0$ negative letters.

Pick $\alpha(j, k)^{-1}$ some terminal negative letter and then there must exist $P = \{\alpha(j, k)^{-1}, \alpha(j, k')\}$ in $\mathcal{G}$. Let f^* and f^{**} be the Type II cuts associated with P. By Theorem 3, assuming $\alpha = \beta\alpha(j, k)^{-1}\zeta\,\alpha(j, k')\delta$, then f^* has a word α^* of which ζ is a subword; similarly f^{**} has a word α^{**} of which $\beta\delta$ is a subword. Both α^* and α^{**} have less than n negative letters. It follows from the nonlinking property of assemblages that any $A \in \mathcal{A}$ either comes from ζ or $\beta\delta$. Let $\mathcal{A}^*$ be the collection of the former; $\mathcal{A}^{**}$, the latter. Then $\mathcal{A}^*$ is an assemblage for f^* and f^{**} for $\mathcal{A}^{**}$.

Applying the induction hypothesis, there are extensions F^* of f^* and F^{**} of f^{**} that satisfy the conclusions of the theorem. Suppose we take D_1 to be the upper half of the unit disk in the plane and D_2, the lower. We may assume without loss of generality that D_1 is the domain of F^* and D_2, of F^{**}. Further, we may assume that for any point z between -1 and 1 on the real axis, i.e. on $D_1 \cap D_2$, $f^*(z) = F^*(z) = F^{**}(z) = f^{**}(z)$. Thus, by [10, Theorem 9, p. 336] we can define a properly interior mapping F on $D^2 = D_1 \cup D_2$ by $F \mid D_1 = F^*$ and $F \mid D_2 = F^{**}$. For any $A \in \mathcal{A}$, $A \neq P$, the desired arc (for conclusion (1)) or fan and branch point (for conclusion (2)) are in D_1 or D_2. The required arc belong-

ing to P is the real axis between -1 and 1. This concludes the proof of the theorem.

Theorem 7. *Let $f: S^1 \to R^2$ be normal with word α. Suppose that F is a properly interior extension of f to D^2 such that each branch point maps onto an a_j. Then there exists an assemblage $\mathfrak{B} \cup \mathfrak{G}$ such that;*

(1) *For each $G \in \mathfrak{G}$, there is an arc in D^2 that belongs to G.*

(2) *There is a one-to-one correspondence between branch points of F and sets in $\mathfrak{B}$ such that if x and B correspond, then x is the hinge of a fan in D^2 that belongs to B and x has multiplicity $-1 +$* cardinality *B.*

Call this assemblage that relates to F in the above manner $\mathfrak{A}(F)$.

Proof. Let $\alpha(j, k)^{-1}$ be some negative letter of α. Consider the arc $U = [\alpha(j, k)^{-1} b_j(\alpha_j)]$ (recall that b_j is the endpoint of α_j in the unbounded component of $R^2 - [f]$). By [3, Theorem 1, p. 49] there is an arc T in D^2 with one endpoint at $f^{-1}(\alpha(j, k)^{-1})$ that maps homeomorphically into $[\alpha_j]$. The other endpoint is either in S^1 or maps onto b_j. However a properly interior map cannot map any points into the unbounded component of $R^2 - [f]$, so we conclude that the other endpoint is in S^1. Since F is locally one-to-one at each point of S^1, this endpoint must be the preimage of some letter of α. Further, this letter must be positive because F is orientation preserving. The arc U was lifted from $\alpha(j, k)^{-1}$ outward, so this letter, say $\alpha(j, k')$, has the property that $k < k'$. Thus, $G = \{\alpha(j, k)^{-1}, \alpha(j, k')\}$ meets the criteria for being in some grouping. By construction, the arc T belongs to G. Each negative letter yields one and only one G in this fashion. Take $\mathfrak{G}$ to be the collection of all such G. Since the arcs belonging to different G's are disjoint, no two G's link.

Pick $j \in \Theta$. Suppose $\{\alpha(j, k_i) \mid 1 \le i \le r\}$ are those positive letters on α_j that do not belong to any $G \in \mathfrak{G}$. Applying [3, Theorem 1, p. 49] to the arc $Q_i = [a_j \alpha(j, k_i)(\alpha_j)]$, we produce an arc T_i in D_2 with one endpoint at $f^{-1}(\alpha(j, k_i))$. The other endpoint is either in S^1 or maps onto a_j. Since F is orientation preserving, the other endpoint can be in S^1 only if it is the preimage of a negative letter. But this would mean that $\alpha(j, k_i) \in G$ for some $G \in \mathfrak{G}$. We conclude that the other endpoint maps onto a_j. Any point where two of the T_i intersect is a branch point. We have hypothesized that such a branch point would map onto a_j. Hence, if the T_i intersect, they intersect at the endpoint that maps onto a_j. Let $K_1, K_2, \cdots, K_s$ be the components of $\bigcup_{i=1}^r T_i$; each K_i is a fan. Take $B_i = F(K_i \cap S^1)$. Then take $\mathfrak{B}$ to be the collection of all such B_i as j ranges over Θ. Since no two of the fans can intersect, the sets of $\mathfrak{B}$ are disjoint and do not link.

Suppose B_i and K_i are as in the previous paragraph. Let x_i be the hinge of the fan. As remarked previously, x_i is a branch point of multiplicity $M_i \ge -1 +$ cardinality B_i. We show that equality holds. There must be $1 + M_i$ arcs at x_i

mapping into $[a_j]$. Each of these arcs can be extended until it intersects S^1. No two intersect except at x_i. The point of intersection of one of these arcs and S^1 must be the preimage of a positive letter of α_j, say $\alpha(j, t)$. Since $\alpha(j, t) \notin G$ for any $G \in \mathcal{G}$, $\alpha(j, t) \in B_i$. This proves that $M_i \leq -1 +$ cardinality B_i and completes the proof of the theorem.

Definition. Let $f: S^1 \to R^2$ be normal with word α. Suppose $\mathfrak{A}$ is an assemblage for f. Define $\tau(\mathfrak{A}) = 1 + \Sigma_{B \in \mathfrak{B}}(-1 +$ cardinality $B)$. We shall call an assemblage *maximal* if it maximizes $\tau(\mathfrak{A})$.

Proposition. *$\mathfrak{A}$ is maximal if and only if $\tau(\mathfrak{A}) = \mathrm{TWN}(f)$.*

Proof. Let $\mathfrak{A}$ be any assemblage of α. By Theorem 6, $\tau(\mathfrak{A}) \leq 1 + \Sigma_{i=1}^m \mu_i$, where the μ_i are the branch point multiplicities of some properly interior extension of f. But $1 + \Sigma_{i=1}^m \mu_i = \mathrm{TWN}(f)$ [5, Theorem 19.1, p. 67]. We need only to observe that Theorem 7 guarantees that there is some assemblage $\mathfrak{A}$ for which $\tau(\mathfrak{A}) = \mathrm{TWN}(f)$.

Theorem 8. *Let $f: S^1 \to R^2$ have word α. Suppose F and G are properly interior extensions of f to D^2 such that each branch point maps onto an a_j. Then F and G are equivalent if and only if $\mathfrak{A}(F) = \mathfrak{A}(G)$.*

Proof. To dispose of the converse first, note that the proof of Theorem 7 is purely topological. Composing F with a homeomorphism will not change the relationships between the arcs, fans, linkages, etc.

Now suppose $\mathfrak{A}(F) = \mathfrak{A}(G) = \mathfrak{A} = \mathfrak{B} \cup \mathcal{G}$. We give a proof along the lines of the proof of Theorem 6. We induct on n, the number of negative letters in α.

First suppose $n = 0$. Then $\mathcal{G} = \emptyset$. We handle this case by induction on $\tau(\mathfrak{A}) = \mathrm{TWN}(f)$. If $\mathrm{TWN}(f) = 1$, then Theorem 5 implies that f is a positively oriented Jordan curve. Both F and G must be orientation preserving homeomorphisms and are equivalent.

Now suppose this case holds for all normal mappings with word of all positive letters and with $\mathrm{TWN} < \mathrm{TWN}(f)$. In view of the previous paragraph, we may assume $\mathrm{TWN}(f) \geq 2$. Thus $\tau(\mathfrak{A}) \geq 2$ and there is some set $B = \{\alpha(j, k_1), \cdots, \alpha(j, k_m)\} \in \mathfrak{B}$, $m \geq 2$. Assume without loss of generality that $k_1 < k_2$ and $\alpha = \beta\alpha(j, k_1)\zeta\alpha(j, k_2)\delta$ where $\alpha(j, k_r)$ is not a letter of ζ for any r, $1 \leq r \leq m$. Let f^* and f^{**} be the Type III cuts associated with $\alpha(j, k_1)$ and $\alpha(j, k_2)$. We are going to apply the induction hypothesis to f^* and f^{**}. However, they are not transverse to α_j. Select an arc α_j' that coincides with α_j from a_j to slightly before $\alpha(j, k_1)$, then runs parallel to α_j until slightly after $\alpha(j, k_2)$, where it coincides with α_j again. Make α_j' transverse to f (see Figure 6). If we label

the crossings of f and α_j' with the obvious choices of $\alpha(j, k)$'s, then we do not change the word α. Also, by making α_j' close enough to α_j, the fans of F and G can be "moved" slightly to yield similar fans involving α_j'. It follows from Theorem 4 that f^* has word ζ^* with all positive letters of which $\alpha(j, k_1)\zeta$ is a subword; similarly f^{**} has a word α^{**} with all positive letters of which $\beta\alpha(j, k_1)\delta$ is a subword. For any set $B' \in \mathfrak{B}$, $B' \neq B$, by the nonlinking property of assemblages, either the letters of B' all come from ζ or all from $\beta\delta$. Let $\mathfrak{B}^*$ be the collection of all sets B' that come from ζ together with the set $\{\alpha(j,k_1)\}$. Let $\mathfrak{B}^{**}$ be the collection of all sets B' that come from $\beta\delta$ together with $B - \{\alpha(j, k_2)\}$. Then $\mathfrak{B}^*$ is an assemblage for f^*, and $\mathfrak{B}^{**}$, for f^{**}.

Let $\bigcup_{i=1}^m T_i$ be the fan belonging to B with respect to F and let x be the hinge. Suppose that T_1 has x and $f^{-1}(\alpha(j, k_1))$ as its endpoints and T_2 has x and $f^{-1}(\alpha(j, k_2))$. Then $T_1 \cup T_2$ splits D^2 into two disks, D^* and D^{**}. Let $\mathfrak{D}^*$ be the decomposition of D^* whose nondegenerate elements are all sets of the form $F^{-1}(y)$, where y is a point of α_j' between a_j and $\alpha(j, k_1)$. Then $D_1 = D^*/\mathfrak{D}^*$ is a topological two-disk. Thus, there is no loss of generality in assuming $D_1 = D^2$. Let p_1 be the natural map from D^* to D_1. The mapping $F^* = Fp_1^{-1}$ is a properly interior extension of f^*.

In a similar way, take $\bigcup_{i=1}^s S_i$ to be the fan belonging to B with respect to G and let x' be the hinge. Suppose that S_1 has endpoints x' and $f^{-1}(\alpha(j, k_1))$ and S_2 has x' and $f^{-1}(\alpha(j, k_2))$. Then $S_1 \cup S_2$ splits D^2 into E^* and E^{**}. Let $\mathfrak{E}^*$ be the decomposition of E^* whose nondegenerate elements are all sets of the form $G^{-1}(y)$, where y is a point of α_j' between a and $\alpha(j, k_1)$. Let q_1 be the natural map from D^* to $E_1 = E^*/\mathfrak{E}^*$ and again we take $E_1 = D^2$. The mapping $G^* = Gq_1^{-1}$ is a properly interior extension of f^*. Both F^* and G^* have fan structure that corresponds to $\mathfrak{B}^*$. Since $\mathrm{TWN}(f^*) < \mathrm{TWN}(f)$ (Theorem 1 and Theorem 5), the induction hypothesis provides us with a homeomorphism $H_1: D^2 \to D^2$ such that $F^* = G^*H_1$.

In a similar way, there are natural maps $p_2: D^{**} \to D^2$ and $q_2: E^{**} \to D^2$, such that $F^{**} = Fp_2^{-1}$ and $G^{**} = Gq_2^{-1}$ are equivalent. Let H_2 be the homeomorphism from D^2 to D^2 such that $F^{**} = G^{**}H_2$.

Now we define a homeomorphism $H: D^2 \to D^2$ by $H(z) = q_1^{-1}H_1p_1(z)$ for $z \in D^* - D^{**}$, $H(z) = q_2^{-1}H_2p_2(z)$ for $z \in D^{**} - D^*$, $H(z) = S_1 \cap q_1^{-1}H_1p_1(z)$ for $z \in T_1$, and $H(z) = S_2 \cap q_1^{-1}H_1p_1(z)$ for $z \in T_2$. To see that H is well defined (from which it follows readily that H is a homeomorphism), we must show that $H(z) = S_1 \cap q_2^{-1}H_2p_2(z)$ for $z \in T_1$ and similarly for $z \in T_2$. Note then that $F^*p_1(z) = F^{**}p_2(z) = F(z)$. But $F^*(p_1(z)) = G^*H_1p_1(z)$ and $F^{**}p_2(z) = G^{**}H_2p_2(z)$. Hence, $Gq_1^{-1}H_1p_1(z) = F^*(p_1(z)) = F(z)$ and also $Gq_2^{-1}H_2p_1(z) = F(z)$. There is exactly one point u on S_1 such that $G(u) = F(z)$. Thus, $q_1^{-1}H_1p_1(z) \cap S_1 = u = q_2^{-1}H_2p_2(z) \cap S_1$. The proof for T_2 is similar.

Finally $GH = Gq_1^{-1}H_1p_1 = G^*H_1p_1 = F^*p_1 = F$ on $GH = Gq_2^{-1}H_2p_2 = G^*H_2p_2 = F^*p_2 = F$ and we have proved the theorem for the case the α has no negative letters.

Now suppose the theorem true for any normal mapping having word with $< n$ negative letters. Let $f: S^1 \to R^2$ have word α with $n > 0$ negative letters.

Select a terminal negative letter $\alpha(j, k)^{-1}$. Then $P = \{\alpha(j, k)^{-1}, \alpha(j, k')\} \in \mathcal{G}$ for some $k' > k$. Let f^* and f^{**} be the associated Type II cuts. Assume $\alpha = \beta\alpha(j,k)^{-1}\zeta\alpha(j, k')$. By Theorem 3, f^* has word α^* of which ζ is a subword; similarly f^{**} has a word α^{**} of which $\beta\delta$ is a subword. Both α^* and α^{**} have $< n$ negative letters. Any $A \in \mathcal{A}$ either comes from ζ or $\beta\delta$. Let $\mathcal{A}^*$ be the collection of the former; $\mathcal{A}^{**}$, the latter. Then $\mathcal{A}^*$ is an assemblage for f^*, and $\mathcal{A}^{**}$, for f^{**}.

Let T be the arc in D^2 that belongs to P with respect to F and let S be the arc that belongs to P with respect to G. Note that T divides D^2 into two disks D_1 and D_2; similarly S divides D^2 into E_1 and E_2. Since $\mathcal{A}(F \mid D_1) = \mathcal{A}^* = \mathcal{A}(G \mid D_1)$ and $\mathcal{A}(F \mid D_2) = \mathcal{A}^{**} = \mathcal{A}(G \mid E_2)$, the induction hypothesis gives that $F \mid D_1$ is equivalent to $G \mid E_1$ and $F \mid D_2$ is equivalent to $G \mid D_2$. Because F maps T homeomorphically onto $[\alpha(j, k)^{-1}\alpha(j, k')(\alpha)]$ and so does G map S, we can apply a glueing argument similar to the one used in the case $n = 0$, and obtain that F and G are equivalent.

6. **Conclusions.** We have now established the following theorems for $f: S^1 \to R^2$ normal:

I. The mapping f is an interior boundary if and only if it has an assemblage.

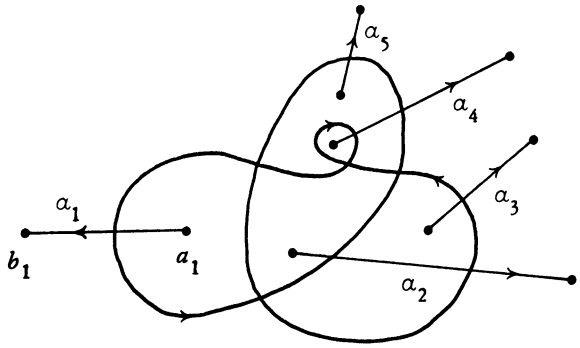
II. There is a one-to-one correspondence between maximal assemblages of some word for f and the equivalence classes of properly interior extensions of f.

III. The above correspondence is such that if the equivalence class of F is associated with $\mathcal{A} = \mathcal{B} \cup \mathcal{G}$, then the branch points of F have multiplicities $-1 +$ cardinality B, where B ranges over $\mathcal{B}$.

IV. The mapping f has an extension to D^2 that is an immersion if and only if f has a maximal assemblage $\mathcal{A} = \mathcal{B} \cup \mathcal{G}$ with $\mathcal{B} = \emptyset$. The maximal number of mutually inequivalent such extensions is given by II above.

The reader should check the Introduction to see that all the questions and problems stated there have been dealt with.

Those familiar with Blank's work may question the relationship between the words used here and Blank's words (see §3). Our hypotheses are (perhaps) stronger in that our word carries more information via the second index on each letter. However, we can deal with a wider variety of problems. Also, to apply Blank's theorem one must know that TWN = 1. There are normal mappings with TWN = 2 which having groupings in the sense of Blank.



Blank word: $\alpha(1,1)\alpha(2,1)\alpha(4,1)\alpha(5,1)\alpha(2,2)\alpha(3,1)\alpha(4,2)^{-1}$

word: $\alpha(1,1)\alpha(2,1)\alpha(4,2)\alpha(5,1)\alpha(2,2)\alpha(3,1)\alpha(4,1)^{-1}$

Figure 1

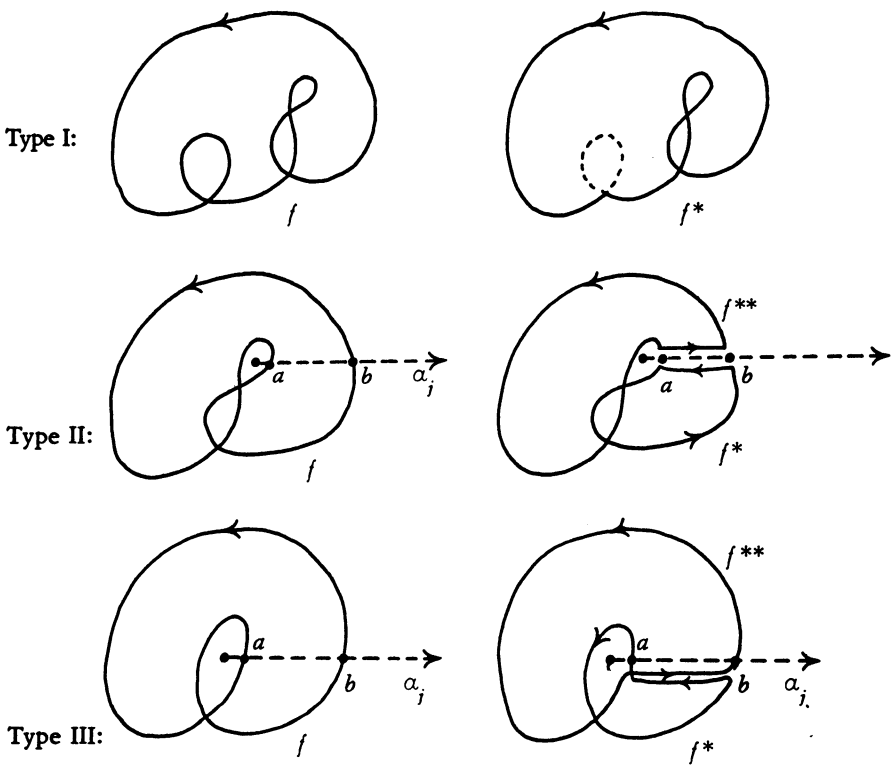
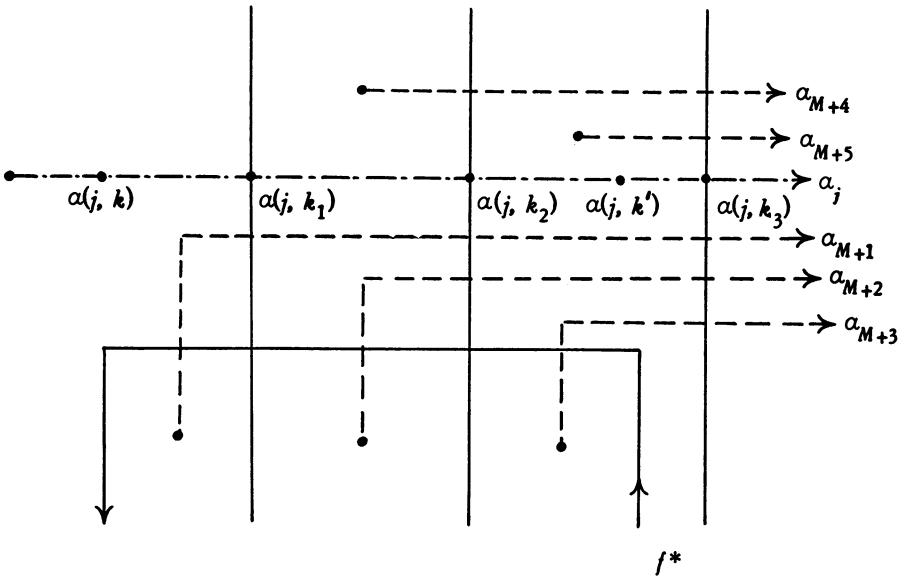


Figure 2



$(s = 2, t = 3)$

Figure 3

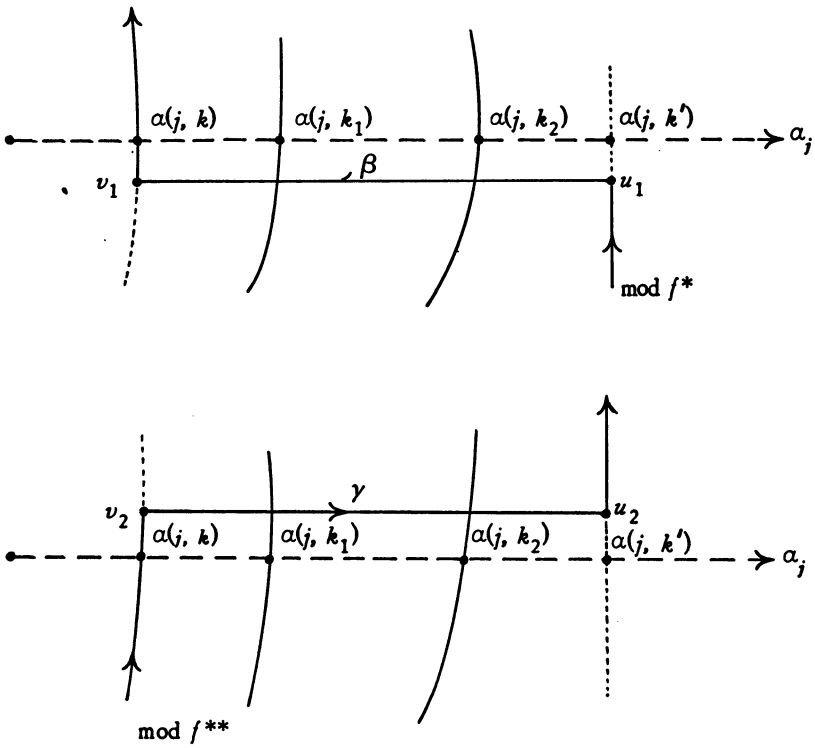


Figure 4

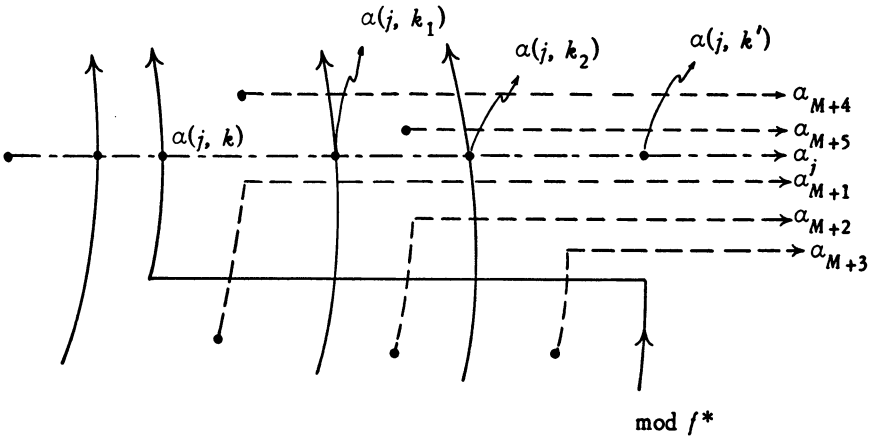


Figure 5

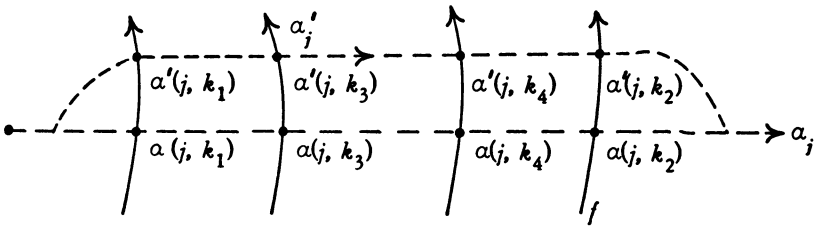


Figure 6

We only allow sets $\{\alpha(j, k)^{-1}, \alpha(j, k')\}$ in $\mathcal{G}$ when $k < k'$; Blank has no way of knowing whether $\alpha(j, k)^{-1}$ precedes $\alpha(j, k')$ as he does not have the second index. However, he cleverly proves that if a grouping exists, the sets in it have the property that the negative letter must preceed the positive (see [1, p. 21]). That one fact added to the theorems of this paper will give a proof of Blank's theorem.

There are many interesting questions concerning the relationship between Titus intersection sequences [8, p. 1084], Blank words, and the words used here. The author hopes to have a paper forthcoming on this subject.(1)

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