$K_1$ OF A CURVE OF GENUS ZERO$^{(1)}$

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ABSTRACT. We determine the structure of the vector bundles on a curve of genus zero and calculate the "universal determinant" $K_1$ of such a curve.

1. Introduction. Let $F$ be a field. Then there is a bijection between (non-singular projective irreducible) curves of genus zero over $F$ and central simple algebras of rank 4 over $F$. If $X$ is such a curve then $X$ is isomorphic to a plane curve of degree 2, and there is a separable extension $[K: F]$ of degree 2 such that $X \times_F K \cong P_1^1$, the projective line over $K$.

Let $\mathcal{O}$ be the category of vector bundles (= locally free sheaves of finite type) on $X$ and $\mathcal{M}$ the (abelian) category of coherent sheaves on $X$. Let $K_1$ be the "universal determinant" $K_1$ as defined in [3, Chapter VIII]. The groups $K_1(\mathcal{O})$ and $K_1(\mathcal{M})$ are both defined. Set $K_1(X) = K_1(\mathcal{O})$. In this paper we prove that if $X$ is the curve of genus zero over $F$ corresponding to the central simple algebra $A$ then $K_1(X) \cong K_1(F) \oplus K_1(A)$. At the end of the paper it is proved that the inclusion of categories $\mathcal{O} \rightarrow \mathcal{M}$ induces an isomorphism $K_1(\mathcal{O}) \rightarrow K_1(\mathcal{M})$ so $K_1(X)$ could have been defined with coherent sheaves instead of vector bundles. If $A$ is the ring of $2 \times 2$ matrices over $F$, then $X = P_1^1$ and the formula reads $K_1(X) \cong K_1(F) \oplus K_1(A) = F^* \oplus F^*$ (where $F^*$ denotes the nonzero elements of $F$). This has already been proved in [8] or [9] (working with coherent sheaves in the first case and vector bundles in the second) so we can confine ourselves to the case where $A$ does not split, i.e. is a division ring of rank 4.

Recently [7] Quillen has developed a theory of higher $K$'s for schemes, and in [7] he calculates the $K$-theory for Severi-Brauer schemes, the simplest example of which are the curves of genus zero. The result that I have obtained agrees with his, although Gersten in [6] has proved that if $X$ is a nonsingular elliptic curve over $C$ Quillen's $K_1$ is not the same as the "universal determinant" $K_1$.

This paper is based on the second half of [11]. Throughout $Z$ denotes the integers, $R$ the real numbers, and $C$ the complex numbers.

I am grateful to M. P. Murthy for pointing out reference [14] to me.

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2. The structure of vector bundles on $X$. The results in this section seem to be more or less "known" (see the discussion in [7, §8]). I include them here because I need the detailed description of the vector bundles for my calculation of $K_1$ and do not know a suitable reference.

Let $X$ be a curve of genus 0 over the field $F$, $X \not\cong P^1_F$, and let $K$ be a separable extension of degree 2 such that $X_K = X \times_F K \cong P^1_K$. If $f: X_K \to X$ is the morphism obtained from the change of field, then $f^*$ is an injection on isomorphism classes of vector bundles or coherent sheaves [12, remark at the end of §1]. We use this, together with the known structure of vector bundles on $P^1$ to determine the structure of vector bundles on $X$.

The Krull-Schmidt theorem holds for vector bundles on $X$ and $X_K$ because all Hom's are finite dimensional vector spaces over the ground field [2]. That is, every vector bundle can be written in a unique manner as the direct sum of indecomposable vector bundles. If we write $P^1_K = \text{Proj } K[T_0, T_1]$ and let $\mathcal{O}(1)$ be the canonical line bundle determined by this projective structure, then the indecomposable vector bundles on $X_K$ are just the line bundles $\mathcal{O}(n)$. Some discussion of this can be found in [10]. Furthermore $\Gamma(\mathcal{O}(n))$ is a vector space of dimension $n + 1$ over $K$ if $n \geq 0$ and is zero otherwise. Also $\text{Hom}(\mathcal{O}(m), \mathcal{O}(n)) = \Gamma(\mathcal{O}(-m) \otimes \mathcal{O}(n)) = \Gamma(\mathcal{O}(n-m))$. Therefore there are no nonzero morphisms from $\mathcal{O}(m)$ to $\mathcal{O}(n)$ unless $n \geq m$.

The Picard group of $X_K$ is $\mathbb{Z}$, generated by $\mathcal{O}(1)$. The Picard group of $X$ is $\mathbb{Z}$ also, generated by $\mathcal{O}(1)$, where $\mathcal{O}(1)$ is defined by the projective structure of $X$ as a second degree curve. Also $f^*\mathcal{O}(1) = \mathcal{O}(2)$. (I will write simply $\mathcal{O}(n)$, it being clear from the context whether this is a bundle on $X$ or $X_K$.)

The following lemma will be used throughout this discussion:

**Lemma.** Let $Y$ be a scheme of finite type over a noetherian ring $A$. Let $B$ be a flat $A$-algebra, and let $U$ be an open subset of $Y$. Write $Y' = Y \otimes_A B$ and let $f: Y' \to Y$ be the morphism induced by change of base. Let $N$ be a quasicoherent sheaf on $Y$ and $N' = f^*(N)$. Then $\Gamma(f^{-1}(U), N') = B \otimes_A \Gamma(U, N)$.

**Proof.** First of all the lemma is true if $U$ is affine, by the construction of product in the category of schemes and the behavior of $f^*$ in the affine case. If $U$ is not affine then $U$ can be covered by a finite number of affine schemes $U_i$ ($1 \leq i \leq n$). If we write $U_{ij} = U_i \cap U_j$ then $U_{ij}$ will be affine also. We have an exact sequence

$$0 \to \Gamma(U, N) \to \prod_i \Gamma(U_i, N) \to \prod_{i,j} \Gamma(U_{ij}, N).$$
X, of a curve of genus zero since \( N \) is a sheaf. Tensoring with \( B \) and using the lemma for affine sets we get an exact sequence

\[
0 \rightarrow B \otimes_A \Gamma(U, N) \rightarrow \prod_i \Gamma(f^{-1}(U_i), N') \Rightarrow \prod_i \Gamma(f^{-1}(U_i), N').
\]

But \( f^{-1}(U) \) is covered by the \( f^{-1}(U_i) \) and \( f^{-1}(U_i) = f^{-1}(U_i) \cap f^{-1}(U_j) \). Therefore we have an exact sequence

\[
0 \rightarrow \Gamma(f^{-1}(U), N') \rightarrow \prod_i \Gamma(f^{-1}(U_i), N') \Rightarrow \prod_i \Gamma(f^{-1}(U_i), N').
\]

Therefore \( B \otimes_A \Gamma(U, N) = \Gamma(f^{-1}(U), N') \) as required.

In the application \( A = F, B = K \) and \( Y = U = X \). The lemma is false if \( B \) is not a flat \( A \)-module. For example, let \( A = K[T_1, T_2], Y = \text{Spec } A, U = \text{Spec } A - (\text{origin}) \) and \( B = K \), the homorphism \( A \rightarrow B \) given by sending \( T_1 \) and \( T_2 \) to 0. If \( N \) is the structure sheaf of \( Y \), then \( \Gamma(U, N) = A \), and \( B \otimes_A \Gamma(U, N) = B \). But \( f^{-1}(U) = \emptyset \), so \( \Gamma(f^{-1}(U), N') = 0 \). (This example was pointed out to me by Paul-Jean Cahen.)

The structure of the vector bundles on \( X \) is given by the following theorem:

**Theorem 1.** Let \( X \) be a curve of genus 0 over the field \( F, X \cong P^1_F \), and let \( K \) be a separable extension of degree 2 such that \( X_K = X \times_F K \cong P^1_K \). Let \( f: X_K \rightarrow X \) be the morphism obtained from change of base. Then the vector bundle \( E(n) = f_*O(n) \) (n odd) on \( X \) is indecomposable of rank 2. Every vector bundle on \( X \) can be written uniquely (up to order of summands) as the direct sum of line bundles \( O(n) \), and the bundles \( E(n) \).

**Proof.** First we show that \( E(n) \) is indecomposable. If \( E(n) \) is decomposable, then \( E(n) = O(n_1) \oplus O(n_2) \) and \( f^*E(n) = O(n_1) \oplus O(n_2) \). The Galois group \( \mathbb{Z}/2\mathbb{Z} \) of \( K \) over \( F \) acts on the vector bundles on \( X_K \) in an obvious way (denoted by a \( \sim \)). By (2') of [12] we have \( f^*f_*O(n) = O(n) \oplus O(n) \). The equation \( f^*O(1) = O(2) \) proves that the Galois group acts trivially. Therefore we must have \( f^*(E(n)) = O(n) \oplus O(n) \). Hence \( E(n) \) must be indecomposable, otherwise the Krull-Schmidt theorem would be violated.

If \( n \) is even, \( f_*O(n) = O(n/2) \oplus O(n/2) \). For we get \( O(n) \oplus O(n) \) on both sides if we apply \( f^* \), and \( f^* \) is an injection on isomorphism classes. Now suppose that \( V \) is a vector bundle on \( X \). Then \( f^*(V) = \bigoplus_i O(n_i) \), so \( f_*f^*(V) = V \oplus V \) is the direct sum of the \( O(n_i) \) and \( E(n_i) \). By the Krull-Schmidt theorem so also is \( V \). This completes the proof of Theorem 1.

I will conclude this section by making some general remarks about the vector bundles on \( X \). Let \( \text{Hom}_F \) denote morphisms of vector bundles on \( X \), and \( \text{Hom}_K \) denote morphisms of vector bundles on \( X_K \). First of all, \( \text{Hom}_F (O(n), O(m)) = 0 \) if \( n > m \) and is nonzero if \( n \leq m \). Also \( \text{Hom}_F (O(n), E(m)) \otimes_K K = \).
Hom\(_K(\pi(n), \pi(E(m))) = Hom\_K(\pi(2n), \pi(m) \oplus \pi(m))\) so \(\text{Hom}_F(\pi(n), E(m)) = 0\) if \(2n > m\) and is nonzero if \(2n < m\). By applying \(\pi^*\) and using the fact that \(\pi^*\) is an injection on isomorphism classes we can prove that \(E(n) \otimes \pi(m) \cong E(n + 2m),\)
\(E(n)^* = E(-n)\) (*denotes dual) and \(E(n) \otimes E(m) \cong 4 \pi((m + n)/2).\) From the last isomorphism it follows that \(\text{Hom}_F(E(n), E(m)) = 0\) if \(n > m\) and is nonzero if \(n \leq m\). Hence we may linearly order the vector bundles \(\cdots E(-3), \pi(-1), E(-1), \pi, E(1), \pi(1), E(3), \pi(2), \cdots\) with nonzero morphisms going only to the right. One can also show that \(\Lambda^2E(1) = \pi(1)\) by applying \(\pi^*\) to both sides.

Now we consider \(\text{Hom}_F(E(n), E(n))\). From the above it is a 4 dimensional vector space over \(F\), and since \(E(n)\) is indecomposable, there are no nontrivial idempotents. Finally \(\text{Hom}_F(E(n), E(n)) \otimes_F K\) is the ring of \(2 \times 2\) matrices over \(K\). Thus \(\text{Hom}_F(E(n), E(n))\) is semisimple and therefore a division ring over \(F\). The \(\text{Hom}_F(E(n), E(n))\) are all isomorphic, since \(E(n) \otimes \pi(m) \cong E(n + 2m)\).

More precisely, if \(F\) is a field of characteristic \(\neq 2\) then the equation for the plane curve \(X\) (in homogeneous co-ordinates) is (for suitable choice of variables) \(T^2_0 - aT^2_1 - bT^2_2 = 0, a, b \in F\) and \(\text{Hom}_F(E(-1), E(-1))\) is isomorphic to the quaternion algebra \((a, b)\) (as defined on p. 96 of [13]). If the characteristic \(F\) is 2, then \(X\) is given by the equation \(aT^2_1 + T_1T_2 + bT^2_2 + cT^2_0 = 0\) with \(a, b, c \in F,\) and \(\text{Hom}_F(E(-1), E(-1))\) is isomorphic to the Clifford algebra of the quadratic form \(acv^2 + cuv + bcv^2\) as defined in [1, p. 150]. The characteristic \(\neq 2\) case was proved by a straightforward but tedious calculation in [11] and the characteristic 2 case can be proved in a similar manner. I will omit these proofs because all we need to know for the calculation in §3 is that \(\text{Hom}_F(E(-1), E(-1))\) is a division ring of dimension 4 over its centre \(F\). In fact, \(\text{Hom}_F(E(-1), E(-1))\) is just the central simple algebra corresponding to \(X\) in the bijection mentioned at the beginning of the paper.

3. Calculation of \(K_1\). We now calculate the group \(K_1(X)\), where \(X\) is as in §2. Let \(V\) be a vector bundle on \(X\), with automorphism \(\alpha\). Let \(V = n_1V_1 \oplus n_2V_2 \oplus \cdots \oplus n_rV_r\) be an expression for \(V\) as the direct sum of indecomposable vector bundles \(V_i\) which are ordered so that there exist nonzero morphisms \(V_i \rightarrow V_j\) if and only if \(i \leq j\). Using this direct sum decomposition \(\alpha\) can be represented by a lower triangular matrix, with \(r n_i \times n_i\) blocks \(\alpha_i\) down the diagonal having entries in either \(F\) or \(A\) depending on whether \(V_i\) is of rank 1 or 2.

Here \(A = \text{Hom}_F(E(-1), E(-1))\), which is the quaternion algebra \((a, b)\) that determines the curve if the characteristic \(\neq 2\), or a certain Clifford algebra is characteristic = 2. In both cases \(A\) is a division ring. The \(\alpha_i\) are invertible. One of the defining relations of \(K_1\) is that if we have a short exact sequence \(0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0\) in \(\pi\) and \(\beta_1, \beta_2, \beta_3\) are automorphisms such that
is commutative, then \( \kappa_1(V_2, \beta_2) = \kappa_1(V_1, \beta_1) + \kappa_1(V_3, \beta_3) \), where \( \kappa_1 \) denotes the canonical image in \( K_1 \). Repeated application of this proves that \( \kappa_1(V, \alpha) = \sum_{i=1}^r \kappa_1(n_i V_i, \alpha_i) \). Write \( A^* \) and \( F^* \) for the nonzero elements of \( A \) and \( F \) respectively. One sees easily that \( \kappa_1(n_i V_i, \alpha_i) = \kappa_1(V_i, \alpha_i) \) for some \( a_i \in A^* \) or \( F^* \) (depending on whether rank \( V_i = 2 \) or 1). Thus we have found generators for \( K_1(X) \). We now try to reduce the number of generators by using exact sequences.

First of all, there is an exact sequence \( 0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \) on \( X \) since \( \mathcal{O}(1) \) is generated by two global sections. (The kernel of the resulting map \( \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \) is a line bundle and must be \( \mathcal{O}(-1) \) because the degree is additive.) Tensoring with \( \mathcal{O}(n) \) we get exact sequences of the form \( 0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \oplus \mathcal{O}(n) \to \mathcal{O}(n+1) \to 0 \) and these enable us to replace all the generators of the form \( \kappa_1(\mathcal{O}(n), \lambda) \) by those of the form \( \kappa_1(\mathcal{O}(1), \lambda) \) and \( \kappa_1(\mathcal{O}(1), \lambda), \lambda \in F^* \). Tensoring with \( E(n) \) yields exact sequences of the form \( 0 \to E(n-2) \to E(n) \oplus E(n) \to E(n+2) \to 0 \) and these enable us to replace the generators \( \kappa_1(E(n), \mu), \mu \in A^* \), by those of the form \( \kappa_1(E(-1), \mu) \) and \( \kappa_1(E(1), \mu) \). There is a nonzero morphism \( E(1) \to \mathcal{O}(1) \) which must be onto, otherwise the image would be isomorphic to \( \mathcal{O}(n) \) for some \( n \leq 0 \) and there are no nonzero maps \( E(1) \to \mathcal{O}(n), n \leq 0 \). The kernel is a line bundle, which must be isomorphic to \( \mathcal{O} \) since we have seen that \( A^2 E(1) \cong \mathcal{O}(1) \). Therefore we have an exact sequence \( 0 \to \mathcal{O} \to E(1) \to \mathcal{O}(1) \to 0 \). This enables us to get rid of the generators of the form \( \kappa_1(\mathcal{O}(1), \lambda), \lambda \in F^* \). Finally, on \( X_K \) we have an exact sequence \( 0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \). If we apply \( f_\mu \) we get an exact sequence \( 0 \to E(-1) \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to E(1) \to 0 \). The map \( \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \) on \( X_K \) is onto on global sections. Therefore so also is the map \( 4 \mathcal{O} \to E(1) \) on \( X_P \) (since the global sections remain the same). If \( \mu \in \text{Aut} E(1) = A^* \) then \( \mu \) induces an automorphism of \( E(1) \) which is a 4 dimensional vector space over \( F \) (as can be seen by applying \( f^* \)). Therefore \( \mu \) can be lifted to an automorphism of \( 4 \mathcal{O}_i \) and hence to an automorphism of the whole exact sequence \( 0 \to E(-1) \to 4 \mathcal{O} \to E(1) \to 0 \). By taking duals any automorphism of \( E(-1) \) also extends to an automorphism of the sequence. The generators of \( K_1(X) \) are finally reduced to elements of the form \( \kappa_1(\mathcal{O}, \lambda) \) and \( \kappa_1(E(1), \mu) \) for \( \lambda \in F^*, \mu \in A^* \). This can be rephrased by saying there is a surjection \( \phi: F^* \oplus H^* \to K_1(X) \) defined by \( \phi(\lambda) = \kappa_1(\mathcal{O}, \lambda) (\lambda \in F^*) \) and \( \phi(\mu) = \kappa_1(E(1), \mu) (\mu \in H^*). \)
Now consider the reduced norm $N: A^* \rightarrow F^*$. It is proved in [14, Corollary p. 334], that the kernel of $N$ is the commutator subgroup of $A^*$ (the index being 2 which is square free). Let $NA^*$ denote the image of $N$. The abelianized group of $A^*$ is therefore $NA^*$. Therefore $\phi$ induces a surjection (also denoted $\phi$) $\phi: F^* \oplus NA^* \rightarrow K_1(X)$.

We now have homomorphisms $\det: K_1(X) \rightarrow F^*$ and $\chi: K_1(X) \rightarrow F^*$. The map $\det$ is defined by taking exterior powers. If $V$ is a vector bundle of rank $r$, then $a \in \text{Aut } V$ induces an automorphism $\det a$ of $\Lambda^r V$. But $\Lambda^r V$ is a line bundle so $\text{Aut } \Lambda^r V \cong F^*$ (canonically). Then $\det K_1(V, a) = \det a$. The vector spaces $H^0(X, V)$ and $H^1(X, V)$ are finite dimensional, and $a \in \text{Aut } V$ induces automorphisms of these vector spaces. These automorphisms will be denoted $\alpha_0$ and $\alpha_1$ respectively. Then $\chi(V, a) = (\det a_0)(\det a_1)^{-1}$. If $0 \rightarrow (V_1, a_1) \rightarrow (V, a) \rightarrow (V_2, a_2) \rightarrow 0$ is exact, then $\chi(V, a) = \chi(V_1, a_1)\chi(V_2, a_2)$ by the exact sequence of cohomology. That $\chi(V, a\beta) = \chi(V, a)\chi(V, \beta)$ is obvious, so $\chi$ defines a homomorphism $\chi: K_1(X) \rightarrow F^*$.

We now examine what these homomorphisms do to the generators of $K_1(X)$. It is clear that $\det (\mathcal{O}, \lambda) = \lambda, \lambda \in F^*$. End $E(1) = A$ is a subalgebra of $\text{End } E(1) \otimes_F K = \text{End}_K (\mathcal{O}(1) \oplus \mathcal{O}(1))$ which is the ring of $2 \times 2$ matrices over $K$.

Det commutes with base change. Therefore by the definition of reduced norm [5, p. 142], we have that $\det (E(1), \mu) = N\mu(\mu \in A^*)$. The Riemann-Roch theorem says that $\dim_F H^0(X, V) - \dim_F H^1(X, V) = \text{degree } (\Lambda^r V) + r$, where rank $V = r$. If we take $V = \mathcal{O}$, then degree $\mathcal{O} = 0, r = 1$, so we get $\dim H^1(X, \mathcal{O}) = 0$. Therefore $\chi(\mathcal{O}, \lambda) = \lambda$ also. If we take $V = E(1)$, then $\dim H^0(X, E(1)) = 4$, degree $(\Lambda^2 E(1)) = \text{degree } \mathcal{O}(1) = 2$, and $r = 2$. Therefore $\dim H^1(X, E(1)) = 0$.

Thus $\chi(E(1), \mu) = \det \mu_0$. But $H^0(X, E(1)) = \Gamma(E(1))$ is a one dimensional vector space over $A$, so $\det \mu_0$ is the usual norm, which is the square of the reduced norm. That is, $\det \mu_0 = (N\mu)^2$.

Now define a homomorphism $\psi: K_1(X) \rightarrow F^* \oplus F^*$ by $\psi = ((\det)^2\chi^{-1}, \chi(\det)^{-1})$. Then $\psi_2(\mathcal{O}, \lambda) = (\lambda, 1)$ and $\psi_1(E(1), \mu) = (1, N\mu)$. Therefore the image of $\psi$ is $F^* \oplus NA^*$ and $\psi: F^* \oplus NA^* \rightarrow F^* \oplus NA^*$ is the identity. We have already seen that $\phi$ is onto. Therefore $\phi$ is an isomorphism. This proves

Theorem 2. Let $F$ be a field and let $X$ be a nonsingular curve of genus 0 which is not isomorphic to $P^1_F$. Let the division algebra $A$ be the endomorphism ring of the indecomposable vector bundle $E(1)$ of rank 2 on $X$. Let $NA^*$ denote the image of the reduced norm $N: A^* \rightarrow F^*$. Then there is an isomorphism $\phi: F^* \oplus NA^* \rightarrow K_1(X)$. (Note that $F^* = K_1(F)$ and $NA^* = K_1(A)$.)

4. Further remarks. We first consider the homomorphism $\Phi: K_0(X) \otimes_F F^* \rightarrow K_1(X)$ defined by $\Phi K_0(V) \otimes \lambda = K_1(V, \lambda)$. Here $K_0$ denotes the Grothendieck
group of vector bundles with relations coming from short exact sequences, and
\( \kappa_0(V) \) denotes the image of \( V \) in \( K_0 \). Then \( K_0(X) = \mathbb{Z} \oplus \mathbb{Z} \) with the second copy of \( \mathbb{Z} \) being the Picard group. If we use this to identify \( K_0(X) \oplus \mathbb{Z} F^* \) with \( F^* \oplus F^* \), then \( \psi\Phi(\lambda, \mu) = \psi\kappa_1(1, \lambda) + \psi\kappa_1(1, \mu) = (\lambda, 1) + (\mu^{-1}, \mu^2) = (\lambda\mu^{-1}, \mu^2) \).

If \( F = \mathbb{R} \), since \( \psi \) is an isomorphism, we see that \( \Phi \) is onto but has nontrivial kernel \((-1, -1)\). We note that in general \( NA^* \) will be bigger than \((F^*)^2\). If this is the case \( \psi\Phi \) will not be onto, so neither is \( \Phi \).

In [8] it was proved that \( \Phi \) is an isomorphism for \( X \) a projective nonsingular variety over an algebraically closed field.

We now consider the homomorphism \( f^*: K_1(X) \to K_1(X_K) \) induced by the change of base. \( K_1(X_K) = K^* \oplus K^* \) generated by \( \kappa_1(1, \lambda) \) and \( \kappa_1(1, \mu) \), \( \lambda \in K^* \). The action of the Galois group \( G = \mathbb{Z}/2\mathbb{Z} \) of \( X \) over \( F \) on \( K_1(X_K) \) is just the obvious action on each copy of \( K^* \). The image of \( f^* \) is contained in \( K_1(X_K)^G \) (the fixed subgroup under the action of \( G \)) and by §2 of [12], the kernel and cokernel of the map \( f^*: K_1(X) \to K_1(X_K)^G = F^* \oplus F^* \) are both killed by 2. One can check (using the above identifications of \( K_1(X_K) \) with \( K^* \oplus K^* \) and \( K_1(X) \) with \( F^* \oplus NA^* \)) that \( f^*(\lambda, \mu) = (\lambda, \mu), \lambda \in F^*, \mu \in NA^* \). Therefore \( f^* \) is injective, and the cokernel of \( f^*: K_1(X) \to K_1(X_K)^G \) is \( F^* / NA^* \) which is indeed killed by 2 because every square is a reduced norm.

We conclude by proving that coherent sheaves and vector bundles give the same \( K_1 \).

**Theorem 3.** Let \( Y \) be a regular projective scheme of finite type over a field \( F \). Let \( \mathcal{O} \) be the category of vector bundles and \( \mathfrak{M} \) the category of coherent sheaves on \( Y \). Then the homomorphism \( K_1(\mathcal{O}) \to K_1(\mathfrak{M}) \) induced by the inclusion of categories is an isomorphism.

**Proof.** This follows from Theorem 5, p. 72 of [4]. The hypotheses are all immediate except (c). Let \( N \) be an object in \( \mathfrak{M} \). If \( n \) is sufficiently large then \( N \otimes \mathcal{O}(n) \) will be generated by its global sections (which form a finite dimensional vector space over \( F \) since \( Y \) is projective). If \( \alpha \) is an endomorphism of \( N \), then \( \alpha \otimes 1 \) induces an endomorphism of the global sections of \( N \otimes \mathcal{O}(n) \). Suppose \( \dim \Gamma(N \otimes \mathcal{O}(n)) = m \). Choose a basis for \( \Gamma(N \otimes \mathcal{O}(n)) \). We have a surjection \( m\mathcal{O}_Y \to N \otimes \mathcal{O}(n) \to 0 \) by mapping the unit sections of the copies of \( \mathcal{O}_Y \) to the corresponding basis vectors for \( \Gamma(N \otimes \mathcal{O}(n)) \). If \( \Gamma(\alpha \otimes 1) \) has matrix \( A \), then the endomorphism of \( m\mathcal{O} \) given by the same matrix lifts \( \alpha \otimes 1 \). Tensoring with \( \mathcal{O}(-n) \) we see that \( \alpha \) can be lifted to an endomorphism of \( m\mathcal{O}(-n) \), which proves (c). Theorem 3 now follows.

If \( X \) were affine a similar result holds by [4, Theorem 3], but if \( Y \) is neither affine nor projective then I do not know if the corresponding result holds. If
$A^2_F = \text{Spec } F[T_0, T_1]$ (the affine plane) and $Y = A^2_F - \text{(origin)}$ then I suspect that $\kappa_1(H, \lambda)$ where $H$ the structure sheaf of $\text{Spec } F[T_0]$ restricted to $Y$ and $\lambda$ is multiplication by $T_0$ does not lie in the image of the homomorphism $K_1(\mathbb{C}) \to K_1(\mathbb{N})$ but I do not know how to prove it.

REFERENCES

11. ———, *Real quadrics and $K_1$ of a curve of genus zero*, Mathematical Preprint No. 1971–60, Queen's University at Kingston, Ont.

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