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REPRESENTATION OF FUNCTIONS AS LIMITS OF MARTINGALES(1)

BY

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ABSTRACT. In this paper we show that if $(\Omega, \mathcal{F}, P)$ is a probability space and if $\{\mathcal{F}_n\}_{n\geq 1}$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$ which satisfy an additional condition, then every real valued, $\mathcal{F}_\infty$-measurable function f can be written as the a.e. limit of a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$. The case where f takes values in the extended real line is also studied. A construction is given of a "universal" martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ such that any $\mathcal{F}_\infty$-measurable function is the a.e. limit of a suitably chosen subsequence.

1. **Introduction.** Throughout this paper $\{\mathcal{F}_n\}_{n\geq 1}$ will denote an increasing sequence of σ-fields on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_\infty$ be the smallest σ-field containing every $\mathcal{F}_n$. Assume $\mathcal{F}_\infty \subset \mathcal{F}$. A d-sequence of σ-fields is defined in §2. The terminology d-sequence is used to indicate that the sequence is "disintegrating". If every $\mathcal{F}_n$ is atomic, then $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence if and only if no set in $\bigcup_{n\geq 1}\mathcal{F}_n$ is an atom of $\mathcal{F}_\infty$.

In §3 we prove (Theorem 3.2) that if $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence and $f: \Omega \to (-\infty, \infty)$ is $\mathcal{F}_\infty$-measurable, then there exists a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ such that $\lim_n f_n = f$ a.e. If f is P-integrable, then we may take $f_n = E\{f \mid \mathcal{F}_n\}$ and the assumption that $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence is superfluous. The case of interest here is when f is not P-integrable. The case where $\mathcal{F}_n = \mathcal{F}_1$ for all $n \geq 1$ shows that if f is not P-integrable, then some additional assumptions are necessary. The above representation theorem was motivated by the work of R. F. Gundy [5] on orthogonal series and martingales and we obtain some of Gundy's results as special cases. Using our representation theorem, we prove a martingale version of a theorem of Marcinkiewicz [9] on the existence of "universal" antiderivatives (Theorem 3.3). §3 concludes with a few comments on continuous parameter martingale representations and stochastic integrals.

In §4 the case where f has values in the extended real line $[-\infty, \infty]$ is considered. In [1] Chow has studied the concept of a regular sequence of atomic

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σ-fields. We show how to define the subset $\Omega_r \subset \Omega$ on which an arbitrary increasing sequence of σ-fields is regular. It turns out (Theorem 4.4) that if $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence, then an $\mathcal{F}_\infty$-measurable function $f: \Omega \to [-\infty, \infty]$ can be written as the almost everywhere limit of a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ if and only if f is finite a.e. on Ω_r.

In this paper the notation A' is used for the complement of a set A and $1[A]$ for the indicator function of A. The reader is referred to [3] or [10] for background material on martingales.

2. Measure-theoretic preliminaries. In order to define a d-sequence of σ-fields, we first prove a result on the existence of certain maximal δ-splitting sets.

Proposition 2.1. *Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$ and $0 < \delta \leq \frac{1}{2}$. For $B \in \mathcal{H}$, let $S(B, \delta) = \{0 < P\{B \mid \mathcal{G}\} \leq \delta\}$. There exists a set $A \in \mathcal{H}$ such that*

(i) $P\{A \mid \mathcal{G}\} \leq \delta$ *a.e.,*

(ii) $S(B, \delta) \subset S(A, \delta)$ *a.e. for every* $B \in \mathcal{H}$.

The set $S(A, \delta)$ is uniquely determined up to equivalence and $S(A, \delta) \in \mathcal{G}$.

Proof. Let $0 < \delta \leq \frac{1}{2}$ be fixed. We will write $S(B)$ for $S(B, \delta)$ in the remainder of the proof. For $B \in \mathcal{H}$, it is convenient to let $B^* = B \cap S(B)$. Note that $S(B) = S(B^*)$, $(B^*)^* = B^*$, and $P\{B^* \mid \mathcal{G}\} \leq \delta$.

If $B_i \in \mathcal{H}$ and $B_i = B_i^*$ for $i = 1$ and 2, then $B_3 = B_1 \cup (B_2 \cap S(B_1)')$ satisfies

(a) $B_3 \in \mathcal{H}$,

(b) $B_1 \subset B_3$,

(c)
$$P\{B_3 \mid \mathcal{G}\} = \begin{cases} P\{B_1 \mid \mathcal{G}\} & \text{on } S(B_1), \\ P\{B_2 \mid \mathcal{G}\} & \text{on } S(B_1)' \cap S(B_2), \\ 0 & \text{on } S(B_1)' \cap S(B_2)', \end{cases}$$

(d) $S(B_1) \cup S(B_2) = S(B_3)$,

(e) $B_3 = B_3^*$.

Properties (a) and (b) are obvious and (c) follows by a routine calculation; (d) and (e) are implied by (c).

Let $\mathcal{S} = \{S(B): B \in \mathcal{H}\} = \{S(B): B \in \mathcal{H}, B = B^*\}$. Property (d) implies that $\mathcal{S}$ is a directed set under inclusion. Let $\alpha = \sup\{P(S): S \in \mathcal{S}\}$. An induction argument using properties (a) through (e) implies that there is an increasing sequence of sets $\{A_n\}_{n\geq 1}$ in $\mathcal{H}$ with $A_n = A_n^*$ and $\lim_n P\{S(A_n)\} = \alpha$. The set $A = \bigcup_{n\geq 1} A_n$ satisfies $P\{A \mid \mathcal{G}\} \leq \delta$ and $P\{S(A)\} = \alpha$. Therefore $S(A)$ is the unique maximal element of $\mathcal{S}$, and the proof is complete.

Definition 2.2. The set $S(A, \delta)$ of Proposition 2.1 is called the δ-splitting set of $\mathcal{G}$ with respect to $\mathcal{H}$ and we write $S(A, \delta) = S(\delta)$. In addition, $S(\frac{1}{2})$ is called the splitting set of $\mathcal{G}$ with respect to $\mathcal{H}$ and we write $S(\frac{1}{2}) = S$.

This terminology is justified by the fact that if $B \in \mathcal{H}$ and

$$\hat{B} = [B \cap \{0 < P\{B|\mathcal{G}\} \leq \tfrac{1}{2}\}] \cup [B' \cap \{\tfrac{1}{2} < P\{B|\mathcal{G}\} < 1\}],$$

then $\{0 < P\{B \mid \mathcal{G}\} < 1\} = \{0 < P\{\hat{B} \mid \mathcal{G}\} \leq \frac{1}{2}\} \subset S$. If $\mathcal{H}$ is atomic, then S is simply the union of all atoms of $\mathcal{G}$ which split into two or more atoms of $\mathcal{H}$. More generally, if $\mathcal{H}$ is countably generated and $(\omega, B) \to p(\omega, B)$ is a regular version of the conditional probability of $\mathcal{H}$ given $\mathcal{G}$ (see [6]), then the complement of S is equivalent to $\{\omega : p(\omega, B) = 1 \text{ if } \omega \in B, = 0 \text{ if } \omega \in B' \text{ for every } B \in \mathcal{H}\}$. No use will be made of this last fact and its verification is left to the reader.

Definition 2.3. For $1 \leq m < n < \infty$, let $S(m, n)$ be the splitting set of $\mathcal{F}_m$ with respect to $\mathcal{F}_n$. If $\bigcup_{n:n>m} S(m, n) = \Omega$ a.e. for every $m \geq 1$, then $\{\mathcal{F}_n\}_{n\geq 1}$ is called a d-sequence.

Since $S(m, n) \subset S(m, n+1)$ for $1 \leq m < n < \infty$, $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence if and only if $\lim_n P\{S(m, n)\} = 1$ for every $m \geq 1$. A sequence of atomic σ-fields $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence if and only if no set in $\bigcup_{n\geq 1} \mathcal{F}_n$ is an atom of $\mathcal{F}_\infty$.

Proposition 2.4. *For $1 \leq m < n < \infty$, let $S(m, n, \delta)$ be the δ-splitting set of $\mathcal{F}_m$ with respect to $\mathcal{F}_n$. If $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence, then $\bigcup_{n:n\geq m} S(m, n, \delta) = \Omega$ a.e. for every $m \geq 1$ and $0 < \delta \leq \frac{1}{2}$.*

Proof. It suffices to prove the proposition when $\delta = (\frac{1}{2})^i$ for $i \geq 1$. The case $i = 1$ follows from the definition of a d-sequence. Assume the result is true for $\delta = (\frac{1}{2})^k$. Let $\epsilon > 0$ and $m \geq 1$ be given. Choose an integer $n > m$ and a set $A_0 \in \mathcal{F}_n$ such that $P\{A_0 \mid \mathcal{F}_m\} \leq \frac{1}{2}$ and $P\{0 < P\{A_0 \mid \mathcal{F}_m\} \leq \frac{1}{2}\} > 1 - \epsilon$. Using the induction hypothesis and arguing as in the proof of Proposition 2.1, we choose an increasing sequence of sets $\{A_j\}_{j\geq 1}$ such that $A_j \in \mathcal{F}_{n+j}$, $P\{A_j \mid \mathcal{F}_n\} \leq (\frac{1}{2})^k$, and

$$\lim_j P\{0 < P\{A_j|\mathcal{F}_n\} \leq (\tfrac{1}{2})^k\} = 1.$$

We have

$$P\{A_0 \cap A_j|\mathcal{F}_m\} = E\{P\{A_0 \cap A_j|\mathcal{F}_n\}|\mathcal{F}_m\} = E\{1[A_0]P\{A_j|\mathcal{F}_n\}|\mathcal{F}_m\}$$

$$\leq (\tfrac{1}{2})^k E\{1[A_0]|\mathcal{F}_m\} \leq (\tfrac{1}{2})^{k+1}.$$

The set where $P\{A_0 \cap A_j \mid \mathcal{F}_m\} > 0$ increases to the set where

$$E\left\{1[A_0] \lim_j P\{A_j \mid \mathcal{F}_n\} \middle| \mathcal{F}_m\right\} > 0,$$

and this latter set is equivalent to the set where $E\{1[A_0] \mid \mathcal{F}_m\} > 0$. Hence

$$\lim_j P\{S(m, n+j, (\tfrac{1}{2})^{k+1})\} \geq \lim_j P\{0 < P\{A_0 \cap A_j \mid \mathcal{F}_m\} \leq (\tfrac{1}{2})^{k+1}\}$$

$$= P\{0 < P\{A_0 \mid \mathcal{F}_m\} \leq (\tfrac{1}{2})\} > 1 - \epsilon,$$

and the proof is complete.

3. Representation of real valued functions.

Proposition 3.1. *Let $\{\mathcal{F}_n\}_{n\geq 1}$ be a d-sequence. Let $f: \Omega \to (-\infty, \infty)$ be $\mathcal{F}_\infty$-measurable and P-integrable. If $n_0 \geq 1$ is an integer, then there exists a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ such that*

(i) *$f_n = 0$ a.e. for $1 \leq n \leq n_0$,*

(ii) *$\lim_n f_n = f$ a.e.*

If, in addition, $|f| \leq M$ a.e., then there exists a sequence of constants $\{M_n\}_{n\geq 1}$ such that

(iii) *$|f_n| \leq M_n$ a.e. for every $n \geq 1$.*

Proof. We first construct a martingale $\{g_n, \mathcal{F}_n\}_{n\geq 1}$ such that $g_n = E\{f \mid \mathcal{F}_n\}$ for $1 \leq n \leq n_0$, and $\lim_{n\to\infty} g_n = 0$ a.e. Once this is done, the martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ is defined by $f_n = E\{f \mid \mathcal{F}_n\} - g_n$.

Choose $\{\delta_k\}_{k\geq 1}$ such that $\delta_k > 0$ and $\Sigma_{k\geq 1} \delta_k < \infty$. Using Proposition 2.3, we may choose $\{m_k, A_k, S_k\}_{k\geq 0}$ by induction such that $m_0 = n_0$, $m_k < m_{k+1}$, S_k is the δ_k-splitting set of $\mathcal{F}_{m_k}$ with respect to $\mathcal{F}_{m_k+1}$, $A_k \in \mathcal{F}_{m_k+1}$ corresponds to S_k as in Proposition 2.1, and

$$P\{S_k\} > 1 - \delta_k. \tag{3.1}$$

Define a sequence of functions $\{g_{m_k}\}_{k\geq 1}$ by induction as follows:

$$g_{m_0} = E\{f \mid \mathcal{F}_{m_0}\}, \qquad g_{m_{k+1}} = \begin{cases} g_{m_k} 1[A_k]/P\{A_k \mid \mathcal{F}_{m_k}\} & \text{on } S_k, \\ g_{m_k} & \text{on } S_k'. \end{cases} \tag{3.2}$$

An induction argument shows that $E\{|g_{m_k}|\} = E\{|g_{m_0}|\} \leq E\{|f|\} < \infty$, and a direct calculation yields $E\{g_{m_k+1} \mid \mathcal{F}_{m_k}\} = g_{m_k}$. For $m_k < n < m_{k+1}$, let $g_n = E\{g_{m_k+1} \mid \mathcal{F}_n\}$. It follows that $\{g_n\}_{n\geq 1}$ is an L^1-bounded martingale and $g_n = E\{f \mid \mathcal{F}_n\}$ for $1 \leq n \leq m_0 = n_0$.

We now show that g_n is eventually equal to 0 a.e. It suffices to show that g_{m_k} is eventually equal to 0, since $g_{m_k+1} = \cdots = g_{m_k+1} = 0$ on the set where $g_{m_k} = 0$. Equation (3.1) and the Borel-Cantelli lemma imply that

$$P\left\{\limsup_k S'_k\right\} = 0. \tag{3.3}$$

According to (3.3) and (3.2), it suffices to show that $P\{\limsup_k A_k\} = 0$, and this follows from $P\{A_k\} = E\{P\{A_k \mid \mathcal{F}_{m_k}\}\} \le \delta_k$. Parts (i) and (ii) are now proved.

To prove (iii), we need only replace A_k by $\hat{A}_k = A_k \cap \{P\{A_k \mid \mathcal{F}_{m_k}\} \ge \epsilon_k\}$ where $\epsilon_k > 0$ is chosen small enough such that $S_k = \{0 < P\{\hat{A}_k \mid \mathcal{F}_{m_k}\} \le \delta_k\}$ satisfies (3.1). It follows that $P\{\hat{A}_k \mid \mathcal{F}_{m_k}\} \ge \epsilon_k$ on S_k and (3.2) implies that g_{m_k} is bounded for every $k \ge 1$. The proof is now complete.

We will say that the martingale $\{f_n, \mathcal{F}_n\}_{n\ge1}$ of Proposition 3.1 is associated with n_0 and f.

Theorem 3.2. *Let $\{\mathcal{F}_n\}_{n\ge1}$ be a d-sequence. Let $f: \Omega \to (-\infty, \infty)$ be $\mathcal{F}_\infty$-measurable. There exists a martingale $\{f_n, \mathcal{F}_n\}_{n\ge1}$ and a sequence of constants $\{M_n\}_{n\ge1}$ such that*

(i) $|f_n| \le M_n$ *a.e. for every* $n \ge 1$,

(ii) $\lim_n f_n = f$ *a.e.*

Proof. It suffices to assume that $f \ge 0$. Let $B_i = \{i-1 \le f < i\}$ for $i \ge 1$. Choose $\{\epsilon_{ij}\}_{i,j\ge1}$ such that $\epsilon_{ij} > 0$ and $\Sigma_{i,j\ge1}\, \epsilon_{ij} < \infty$. Choose $\{m_{ij}, C_{ij}\}_{i,j\ge1}$ such that $C_{ij} \in \mathcal{F}_{m_{ij}}$, and if $D_{ij} = \bigcup_{k=1}^{j} C_{ik}$, then

$$P\{(B_i \cap D'_{i(j-1)}) \,\Delta\, C_{ij}\} < \epsilon_{ij} \tag{3.4}$$

for $i, j \ge 1$. We may assume that $\{C_{ij}\}_{j\ge1}$ is a disjoint sequence of sets for every $i \ge 1$, and that

$$\lim_{i,j} m_{ij} = \infty. \tag{3.5}$$

Form a martingale $\{f_n^{ij}, \mathcal{F}_n\}_{n\ge1}$ as in Proposition 3.1 which is associated with m_{ij} and $f1[B_i \cap C_{ij}]$. It follows from the construction of $\{f^{ij}, \mathcal{F}_n\}_{n\ge1}$ that

$$f_n^{ij} = 0 \quad \text{on } C'_{ij} \text{ for every } n \ge 1. \tag{3.6}$$

Note also that $\lim_n f_n^{ij} = f$ on $B_i \cap C_{ij}$, and $\lim_n f_n^{ij} = 0$ otherwise.

We claim that

(a) $P\{\bigcup_{i,j\ge1} C_{ij}\} = 1$,

(b) $P\{\limsup_{i,j} C_{ij}\} = 0$.

Property (a) follows from $B_i \subset \bigcup_{j\ge1} C_{ij}$ a.e. Property (b) follows from (3.4) and

$$\limsup_{i,j} C_{ij} \subset \limsup_{i,j} (B_i \cap D_{i(j-1)}) \,\Delta\, C_{ij}.$$

Now consider the sum

$$f_n = \sum_{i,j \geq 1} f_n^{ij}.$$

Equation (3.5) and the fact that $f_n^{ij} = 0$ for $1 \leq n \leq m_{ij}$ imply that $\{f_n, \mathcal{F}_n\}_{n \geq 1}$ is a well-defined martingale which satisfies (i). Properties (a), (b) and (3.6) imply that (ii) holds and the proof is complete.

The motivation for Theorem 3.2 came from the work of R. F. Gundy [5] on orthogonal series and martingales. In fact, the proof of Theorem 3.2 was modelled after Gundy's proof that any Borel measurable function f: $[0, 1] \to (-\infty, \infty)$ can be written as the a.e. (Lebesgue) limit of an orthogonal series $\Sigma_{k \geq 1} c_k \phi_k$, where $\{\phi_k\}_{k \geq 1}$ is a complete H-system. Recall that an orthonormal system of functions $\{\phi_k\}_{k \geq 1}$ is an H-system if, for every square integrable function f on $[0, 1]$,

$$E\{f \mid \phi_1, \cdots, \phi_n\} = \sum_{k \geq 1} (f, \phi_k)\phi_k,$$

where $(f, \phi_k) = \int_0^1 f \phi_k \, dx$. Equivalently, an orthonormal set $\{\phi_k\}_{k \geq 1}$ is an H-system if $\mathcal{F}(\phi_1, \cdots, \phi_n)$ is atomic with exactly n-atoms for $n \geq 1$. It is easy to see that any martingale $\{f_n, \mathcal{F}(\phi_1, \cdots, \phi_n)\}_{n \geq 1}$ can be written as $f_n = \Sigma_{k \geq 1} c_k \phi_k$ for suitable constants $\{c_k\}_{k \geq 1}$, and hence Gundy's result follows from Theorem 3.2.

Gundy's treatment was closely related to a classical differentiation theorem of Lusin (see [11, p. 217]) which states that for a Borel measurable function f: $[0, 1] \to (-\infty, \infty)$, there exists a continuous function F: $[0, 1] \to (-\infty, \infty)$ such that $F' = f$ a.e. (Lebesgue). Theorem 3.2 may be viewed as a martingale generalization of Lusin's theorem. Marcinkiewicz [9] used Lusin's theorem to prove the existence of certain "universal" functions F with the property that for every Borel measurable function f: $[0, 1] \to [-\infty, \infty]$, there exists a subsequence $\{b_n\}_{n \geq 1}$ of strictly positive numbers decreasing to 0 such that

$$\lim_n [F(x + b_n) - F(x)]/b_n = f \quad \text{a.e.}$$

We now prove a martingale version of this result.

Theorem 3.3. *Let $\{\mathcal{F}_n\}_{n \geq 1}$ be a d-sequence and assume that $\mathcal{F}_\infty$ is countably generated. There exists a martingale $\{f_n, \mathcal{F}_n\}_{n \geq 1}$ with the following universal convergence property: If f: $\Omega \to [-\infty, \infty]$ is $\mathcal{F}_\infty$-measurable, then there exists a subsequence $\{n_k\}_{k \geq 1}$ of the positive integers such that $\lim_k f_{n_k} = f$ a.e.*

Proof. There exists a countable set of real valued, $\mathcal{F}_\infty$-measurable functions $\{b_n\}_{n \geq 1}$ such that every f as described above can be written as the a.e. limit of a suitable subsequence $\{b_{n_k}\}_{k \geq 1}$. Choose $\{\delta_k\}_{k \geq 1}$ such that $\delta_k > 0$ and $\Sigma_{k \geq 1} \delta_k < \infty$.

Define $\{m_k, \{f_n^k, \mathcal{F}_n\}_{n\geq 1}\}_{k\geq 0}$ by induction as follows: Let $m_0 = 0$, $f_n^0 = 0$ for every $n \geq 1$. Assuming that $\{m_i, \{f_n^i, \mathcal{F}_n\}_{n\geq 1}\}_{1\leq i\leq k}$ have been defined, we choose m_{k+1} such that

$$P\left\{\left|h_k - \sum_{i=1}^{k} f_n^i\right| \geq \delta_k\right\} < \delta_k$$

if $n \geq m_k$. Let $\{f_n^{k+1}, \mathcal{F}_n\}_{n\geq 1}$ be a martingale such that $f_n^{k+1} = 0$ if $1 \leq n \leq m_{k+1}$, and $\lim_n f_n^{k+1} = h_{k+1} - h_k$ a.e. Such a martingale exists by Theorem 3.2. Now let $f_n = \sum_{k\geq 1} f_n^k$. It follows that $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ is a well-defined martingale and

$$P\{|f_{m_{k+1}} - h_k| \geq \delta_k\} < \delta_k. \tag{3.7}$$

If f is the a.e. limit of $\{h_{n_k}\}_{k\geq 1}$, then (3.7) and the Borel-Cantelli lemma imply that the subsequence of $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ corresponding to $\{m_{n_k+1}\}_{k\geq 1}$ converges to f a.e., and the proof is complete.

We will call a martingale with the property described in Theorem 3.4 a universal martingale. By taking subsequences of universal martingales we may construct various examples of strangely behaved martingales. For example, Luis Baez-Duarte [8] gave an example of a martingale which converges in measure but not a.e. To obtain a large class of such examples it suffices to take (in the proof of Theorem 3.3) $\{h_{n_k}\}_{k\geq 1}$ converging in measure but not a.e. A similar comment applies to convergence in distribution (see [4]).

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a family of σ-fields on a probability space $(\Omega, \mathcal{F}, P)$. We will say that $\{\mathcal{F}_t\}_{t\geq 0}$ is a d-family if $\{\mathcal{F}_{t_n}\}_{n\geq 1}$ is a d-sequence for some (and hence every) increasing sequence $\{t_n\}_{n\geq 1}$ with $\lim_n t_n = \infty$. It is easy to adapt the results of this section to the continuous case and we obtain the following result:

Theorem 3.4. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a d-family. Let $f: \Omega \to (-\infty, \infty)$ be $\mathcal{F}_\infty$-measurable. There exists a martingale $\{f_t, \mathcal{F}_t\}_{t\geq 0}$ and an increasing function M: $[0, \infty) \to [0, \infty)$ such that*

(i) $|f_t| \leq M_t$ *a.e. for every* $t \geq 0$.

(ii) $\lim_{t\in Q} f_t = f$ *a.e., where Q is any countably dense subset of* $[0, \infty)$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \Theta_t, P^x)$ be a Hunt process with a locally compact and countably generated state space (see [2]). If there are no absorbing points in the state space, then $\{\mathcal{F}_t\}_{t\geq 0}$ is easily shown to be a d-family with respect to

each measure P^{μ}. It follows from Theorem 3.4 that if a measure P^{μ} is given and $f: \Omega \to (-\infty, \infty)$ is $\mathcal{F}_{\infty}$-measurable, then there is a separable martingale $\{f_t, \mathcal{F}_t\}_{t\geq 0}$ such that $\lim_t f_t = f$ a.e. For example, if we consider 1-dimensional Brownian motion with initial distribution μ, and if $\{f_t, \mathcal{F}_t\}_{t\geq 0}$ is chosen to be square integrable, then by a theorem of Kunita and Watanabe [6]

$$f_t = \int_0^t \psi_s \, dX_s$$

for some nonanticipating functional $\{\psi_s\}_{s\geq 0}$. Roughly speaking, this means that Brownian motion can be controlled to converge to any real valued measurable function f.

4. Representation of extended real valued functions. If $f: \Omega \to [-\infty, \infty]$ is $\mathcal{F}_{\infty}$-measurable and $\{\mathcal{F}_n\}_{n\geq 1}$ is a d-sequence of σ-fields, then it does not necessarily follow that there is a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ which converges to f a.e. Chow [1] proved that if $\{\mathcal{F}_n\}_{n\geq 1}$ is a regular sequence of atomic σ-fields, then no martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ can converge to ∞ (or $-\infty$) on a set of strictly positive P-measure.

Definition 4.1. For $n \geq 1$, let $S_n(\delta)$ be the δ-splitting set of $\mathcal{F}_n$ with respect to $\mathcal{F}_{n+1}$. Let $\{\delta_k\}_{k\geq 1}$ be a sequence of strictly positive numbers decreasing to 0. The regular part Ω_r of Ω with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ is defined as the complement of the set

$$\bigcap_{k\geq 1} \limsup_n S_n(\delta_k). \tag{4.1}$$

The sets $\limsup_n S_n(\delta)$ decrease as δ decreases and hence the intersection in (4.1) is independent of the particular sequence $\{\delta_k\}_{k\geq 1}$. If $\{\mathcal{F}_n\}_{n\geq 1}$ is a sequence of atomic σ-fields, then $\{\mathcal{F}_n\}_{n\geq 1}$ is regular in Chow's sense if and only if there is a δ_k such that $S_n(\delta_k) = \varnothing$ for all $n \geq 1$. We sharpen Chow's convergence result as follows:

Theorem 4.2. *If $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ is a martingale, then $\lim_n f_n$ exists and is finite a.e. on the set*

$$\Omega_r \cap \left(\left\{\sup_n f_n < \infty\right\} \cup \left\{\inf_n f_n > -\infty\right\}\right). \tag{4.2}$$

Proof. Let $m \geq 1$ be an integer and $\lambda \in (-\infty, \infty)$ be given. Define a stopping time with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ by

$$T(m, \lambda) = \inf\{i \geq m: P\{f_{i+1} > \lambda | \mathcal{F}_i\} > 0\}.$$

It follows that $\{f_{T(m,\lambda)\wedge n}, \mathcal{F}_n\}_{n\geq 1}$ is a martingale with $f_{T(m,\lambda)\wedge n} \leq f_m \vee \lambda$ for $n \geq m$. Hence $\lim_n f_{T(m,\lambda)\wedge n}$ exists and is finite a.e. To prove our theorem, it suffices to show that for a.e. $\omega_0 \in \Omega_r \cap \{\sup_n f_n < +\infty\}$ there is a corresponding pair (m, λ) such that $T(m, \lambda)(\omega_0) = \infty$. We omit references to exceptional sets of P-measure 0 in the arguments which follow.

If $\omega_0 \in \Omega_r \cap \{\sup_n f_n < \infty\}$, then

(a) there is a number $\delta > 0$ and an integer $j \geq 1$ such that $\omega_0 \in S_n(\delta)'$ for $n \geq j$.

(b) there is a number $\lambda \in (-\infty, \infty)$ such that $\sup_n f_n(\omega_0) \leq \lambda$. It follows from (a) that

$$P\{f_{i+1} > \lambda | \mathcal{F}_i\}(\omega_0) > \delta \text{ or } = 0 \quad \text{for } i \geq j. \tag{4.3}$$

Now

$$0 \leq P\{f_{i+1} > \lambda | \mathcal{F}_i\}(\omega_0) \leq P\left\{\sup_n f_n > \lambda \middle| \mathcal{F}_i\right\}(\omega_0)$$

$$\to \begin{cases} 1 & \text{if } \sup_n f_n(\omega_0) > \lambda, \\ 0 & \text{if } \sup_n f_n(\omega_0) \leq \lambda, \end{cases} \tag{4.4}$$

as $i \to \infty$. It follows from (4.3) and (4.4) that there is an integer $m \geq j$ such that $P\{f_{i+1} > \lambda \mid F_i\}(\omega_0) = 0$ for $i \geq m$. Hence $T(m, \lambda)(\omega_0) = \infty$, and the proof is complete.

Theorem 4.3. *If $B \in \mathcal{F}_\infty$ and $B \subset \Omega_r'$, then there exists a martingale $\{f_n, \mathcal{F}_n\}_{n\geq 1}$ such that $\lim_n f_n$ exists and is finite a.e. on B', and $\lim_n f_n = \infty$ a.e. on B.*

Proof. Choose $\{\delta_k\}_{k\geq 1}$ such that $\delta_k > 0$ and $\Sigma_{k\geq 1} \delta_k < \infty$. Choose $\{m_k, B_k\}_{k\geq 1}$ such that $B_k \in \mathcal{F}_{m_k}$ and $P\{B \,\Delta\, B_k\} < \delta_k$. Define a sequence of stopping times $\{T_k\}_{k\geq 0}$ with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ by induction as follows:

$$T_0 = 1, \qquad T_{k+1} = \inf\{i > T_k \vee m_k : \omega \in S_i(\delta_{k+1})\},$$

where $S_i(\delta_k)$ is the δ_k-splitting set of $\mathcal{F}_i$ with respect to $\mathcal{F}_{i+1}$. Let $A_i(\delta_k) \in \mathcal{F}_{i+1}$ correspond to $S_i(\delta_k)$ as in Proposition 2.1. Note that $T_k < T_{k+1} < \infty$ on B for $k \geq 1$, and T_i is eventually equal to ∞ on B'.

For $k \geq 1$, let

$$A_{T_k} = \bigcup_{i\geq 1} A_i(\delta_k) \cap \{T_k = i\} \in \mathcal{F}_{T_k+1}, \qquad S_{T_k} = \bigcup_{i\geq 1} S_i(\delta_k) \cap \{T_k = i\} \in \mathcal{F}_{T_k},$$

and note that

$$S_{T_k} = \{T_k < \infty\}, \tag{4.5}$$

$$P\{A_{T_k}|\mathcal{F}_{T_k}\} = P\{A_i(\delta_k)|\mathcal{F}_i\} \quad \text{a.e. on } \{T_k = i\}. \tag{4.6}$$

Equations (4.5) and (4.6) imply that

$$P\{A_{T_k}|\mathcal{F}_{T_k}\} \le \delta_k \quad \text{a.e.}, \qquad \{0 < P\{A_{T_k}|\mathcal{F}_{T_k}\} \le \delta_k\} = S_{T_k}.$$

Define a sequence of functions $\{f_n\}_{n\ge1}$ by induction as follows:

$$f_1 = \cdots = f_{T_1} = 0,$$

$$f_{T_k+1} = \cdots = f_{T_{k+1}} = \begin{cases} f_{T_k+1} & \text{on } A'_{T_k} \cap B_k, \\ g_k & \text{on } A_{T_k} \cap B_k, \\ f_{T_k} & \text{on } B'_k, \end{cases} \tag{4.7}$$

where

$$g_k = (f_{T_k} P\{A_{T_k}|\mathcal{F}_{T_k}\} - P\{A'_{T_k}|\mathcal{F}_{T_k}\})P\{A_{T_k}|\mathcal{F}_{T_k}\}^{-1}$$

on $A_{T_k} \cap B_k$. Routine calculations show that f_n is P-integrable for every $n \ge 1$, and that $\{f_n, \mathcal{F}_n\}_{n\ge1}$ is a martingale.

Now $P\{A_{T_k}\} = E\{P\{A_{T_k}|\mathcal{F}_{T_k}\}\} \le \delta_k$, and hence $P\{\limsup_k A_{T_k}\} = 0$. In addition, $P\{\limsup_k (B \,\Delta\, B_k)\} = 0$. Hence, if $\omega_0 \in B$, then ω_0 is eventually in $A'_{T_k} \cap B_k$ and $\lim_n f_n(\omega_0) = \infty$. On the other hand if $\omega_0 \in B'$, then ω_0 is eventually in B'_k and $f_n(\omega_0)$ is eventually constant. The proof is complete.

If, in (4.7) we replace $f_{T_k} + 1$ by $f_{T_k} + (-1)^k$, then the martingale $\{f_n, \mathcal{F}_n\}_{n\ge1}$ will oscillate boundedly on B and converge on B'. Thus, either of the two types of behavior ruled out on Ω_r by Theorem 4.2 can occur on Ω'_r. The following theorem is the main result of this paper.

Theorem 4.4. *Let $\{\mathcal{F}_n\}_{n\ge1}$ be a d-sequence. Let $f: \Omega \to [-\infty, \infty]$ be $\mathcal{F}_\infty$-measurable. There exists a martingale $\{f_n, \mathcal{F}_n\}_{n\ge1}$ such that $\lim_n f_n = f$ a.e. if and only if $|f| < \infty$ a.e. on Ω_r.*

Proof. The function f must be finite a.e. on Ω_r by Theorem 4.2. For the converse, we may assume that $f \ge 0$. If $B = \{f = \infty\}$, then Theorem 4.3 shows that there is a martingale $\{b_n, \mathcal{F}_n\}_{n\ge1}$ such that $\lim_n b_n$ exists and is finite a.e. on B' and $\lim_n b_n = \infty$ a.e. on B. Let $b = f - \lim_n b_n$ on B' and $b = 0$ on B.

According to Theorem 3.2, there exists a martingale $\{g_n, \mathcal{F}_n\}_{n\geq 1}$ such that $\lim_n g = b$ a.e. The proof is completed by taking $f_n = g_n + b_n$.

In order to find H-systems of orthonormal functions which can be used to represent every extended real valued, Borel measurable function f on $[0, 1]$, Gundy [5] introduced the concept of an H^*-system. We will not repeat the definition of an H^*-system here, but remark that if $\{\phi_k\}_{k\geq 1}$ is an H^*-system, then $P\{\Omega_r\} = 0$ and Gundy's representation theorem follows from Theorem 4.4.

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