RECAPTURING $H^2$-FUNCTIONS ON A POLYDISC

BY

D. J. PATIL

ABSTRACT. Let $U^2$ be the unit polydisc and $T^2$ its distinguished boundary. If $E \subset T^2$ is a set of positive measure and the restriction to $E$ of a function $f$ in $H^2(U^2)$ is given then an algorithm to recapture $f$ is developed.

Introduction. Let $U$ be the open unit disc in the complex plane and $T$ its boundary. Let $f$ be holomorphic in the unit polydisc $U^2 = U \times U$; then $f(z) = \sum c(n)z^n$, $z = (z_1, z_2) \in U^2$, $n = (n_1, n_2) \in \mathbb{Z}^2_+$. The function $f$ is in $H^2$ if and only if $\sum |c(n)|^2 < \infty$ [2, p. 50]. The functions $f$ in $H^2$ can be identified with the boundary value functions on the distinguished boundary $T^2 = T \times T$ of $U^2$ and these boundary value functions are precisely those $f \in L^2(T^2)$ whose Fourier coefficients $\hat{f}(n_1, n_2)$ are zero if either $n_1 < 0$ or $n_2 < 0$. It is known that if a nonzero $f$ is in $H^2$ then $\log|f|$ is in $L^1(T^2)$ and hence if $f = 0$ on a subset $E$ of $T^2$ of positive measure then $f$ is the zero function. It follows that the restriction to such a set $E$ of a function $f$ in $H^2$ determines $f$ uniquely. Following the methods in [1], we give a constructive algorithm to recapture the function $f$ from its values on $E$. The construction is in two steps. From the knowledge of $f$ on $E$, the first step obtains $f$ on $F \times T$ where $F$ is some subset of $T$ of positive measure. The second step is the 'conjugate' of the first and starting with $f$ on $F \times T$ we recover $f$ on the whole of $T \times T$.

The arrangement of the paper is as follows. After defining the notations, we prove some lemmas leading to Theorem 1 which gives the first step in recapturing $f$. We then discuss how the second step is a corollary of the first. This is followed by Theorem 2 which makes the algorithm more explicit.

Notations. In the following $H^2$ will stand for $H^2(T^2)$. By $L^2$ and $L^\infty$ will be meant $L^2(T^2)$ and $L^\infty(T^2)$ respectively. The subspace $L^2_+$ ($L^\infty_+$) will consist of those $f$ in $L^2$ ($L^\infty$) whose Fourier coefficients vanish in the lower half plane, thus, $L^2_+ = \{ f \in L^2: \hat{f}(n_1, n_2) = 0 \text{ for all } n_2 < 0 \}$. The orthogonal projection of $L^2$ onto $L^2_+$ will be denoted by $P$. The Toeplitz operator $T$ corresponding to $\phi \in L^\infty$ is defined by $T\phi = P(\phi)$, $f \in L^2_+$. If $f \in L^2_+$ we may consider $f$ as an $H^2$-function on $T$ with values in $L^2(T)$: $f(\theta_1, \theta_2) = \sum_n \hat{f}(\theta_1, \theta_2) e^{in\theta_2}$. The natural...
extension of this \( f \) to \( \overline{U} \) will also be denoted by \( f \); then for \( z_2 \in U \), \( f(z_2) = \sum_{n \geq 0} f_n(\theta_1)z_2^n \). Such functions \( f \) are holomorphic \( L^2(T) \)-valued functions on \( U \) and the relation \( f(r_2e^{i\theta_2}) \to f(\theta_1, \theta_2) \) as \( r_2 \to 1 \) holds for a.e. pointwise convergence and also for mean convergence [3, p. 186]. Thus the functions \( f(\theta_1, \theta_2) \) and \( f(z_2) \) above determine each other. The (normalized) Haar measures on \( T \) and \( T^2 \) will be denoted by \( \mu_1 \) and \( \mu_2 \) respectively, and when there is no risk of confusion \( d\mu_1(\theta_1) \) will be shortened to \( d\theta_1 \) etc. For \( z \in U \), \( \theta \in T \), let \( K(z, \theta) = (e^{i\theta} + z)/(e^{i\theta} - z) \) and \( P = \text{Re} \ K \).

Lemma 1. Let \( E \subset T^2 \), \( m_2(E) > 0 \). For \( \lambda > 0 \) and \( (z_1, z_2) \in U^2 \) define

\[
t_\lambda(z_1, z_2) = \exp \left\{ \frac{\lambda}{2} \log(1 + \lambda) \int_E P(z_1, \theta_1)K(z_2, \theta_2) \, d\theta_1 \, d\theta_2 \right\}.
\]

Then we have the following:

(a) For all \( (z_1, z_2) \in U^2 \), \( 1 \leq |t_\lambda(z_1, z_2)| \leq \sqrt{1 + \lambda} \).

(b) There exists a set \( \Lambda \subset T \) with \( m_1(\Lambda) = 1 \) such that for every \( \theta_1 \in \Lambda \) and for every \( z_2 \in U \), the limit

\[
t_\lambda(\theta_1, z_2) = \lim_{r_2 \to 1} t_\lambda(r_1e^{i\theta_1}, z_2)
\]

exists and for each \( \theta_1 \in \Lambda \), the function \( t_\lambda(\theta_1, \cdot) \) is holomorphic in \( U \).

(c) For each \( \theta_1 \in \Lambda \), the limit

\[
t_\lambda(\theta_1, \theta_2) = \lim_{r_2 \to 1} t_\lambda(\theta_1, r_2e^{i\theta_2})
\]

exists for almost all \( \theta_2 \) in \( T \), and hence \( t_\lambda(\theta_1, \theta_2) \) is defined almost everywhere in \( T^2 \) and is in \( L^\infty \).

(d) For almost all \( (\theta_1, \theta_2) \) in \( T^2 \),

\[
|t_\lambda(\theta_1, \theta_2)|^2 = 1 + \lambda \chi_E(\theta_1, \theta_2).
\]

Proof. (a) holds since \( |t_\lambda(z_1, z_2)| \) is the exponential of the Poisson integral of \( \frac{\lambda}{2} \log(1 + \lambda) \chi_E \).

(b) For \( z_2 \in U \) and \( \theta_1 \in T \), letting

\[
\mathcal{B}(\theta_1, z_2) = \int_T \chi_E(\theta_1, \theta_2)K(z_2, \theta_2) \, d\theta_2,
\]

we see that \( |\mathcal{B}(\theta_1, z_2)| \leq 2/(1 - |z_2|) \), for all \( \theta_1 \in T \) and hence by Fatou's theorem the integral

\[
\int_E P(r_1e^{i\alpha_1}, \theta_1)K(z_2, \theta_2) \, d\theta_1 \, d\theta_2 = \int_T P(r_1e^{i\alpha_1}, \theta_1)\mathcal{B}(\theta_1, z_2) \, d\theta_1
\]

converges to \( \mathcal{B}(\alpha_1, z_2) \) a.e. \( (\alpha_1) \) as \( r_1 \to 1 \). Therefore
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(1) \[ t^*_\lambda(r_1 e^{i\theta_1}, z_2) \rightarrow \exp \{ \frac{1}{2} \log (1 + \lambda) \bar{\beta}(\theta_1, z_2) \} \]
a.e. $(\theta_1)$. The set of measure 1 where this convergence takes place depends on $z_2$. To see that there is a set $A$ of measure 1 such that for all $\theta_1 \in A$ and for all $z_2 \in U$, (1) holds it suffices to observe that for a fixed $\alpha_1$, the family \[ \{ F_{r_1} : 0 < r_1 < 1 \} \]
is equicontinuous on compact subsets of $U$. A standard argument such as the one in the proof of (5.16) in [5, p. 327] now proves the existence of the set $A$.

(c) For each $\theta_1 \in A$, $t^*_\lambda(\theta_1, \cdot)$ is a bounded holomorphic function on $U$ and hence by Fatou’s theorem $t^*_\lambda(\theta_1, z_2)$ has a limit—say $t(\theta_1, \theta_2)$—as $r_2 \rightarrow 1$, a.e. $(\theta_2)$. Since $t^*_\lambda(\theta_1, \theta_2)$ is a repeated limit of continuous functions, its domain $\Delta$ is a measurable set in $T^2$ and since almost every $\theta_1$-section of $\Delta$ has measure 1, we must have that $m_2(\Delta) = 1$.

(d) We have for almost every $(\theta_1, \theta_2) \in T^2$,

\[ |t^*_\lambda(\theta_1, \theta_2)|^2 = \lim_{r_2 \rightarrow 1} \exp \left\{ \log (1 + \lambda) \int_{T^2} \beta(\theta_1, \alpha_2) P(r_2 e^{i\theta_2}, \alpha_2) d\alpha_2 \right\} = \exp \{ \log (1 + \lambda) \beta(\theta_1, \theta_2) \} = 1 + \lambda \beta(\theta_1, \theta_2). \]

Lemma 2. Let $E \subset T^2$, $m_2(E) > 0$. Let $S$ be the Toeplitz operator on $L^2_+$ corresponding to the characteristic function $\chi_E$ of the set $E$. If $t^*_\lambda$ is as in Lemma 1 and $s^*_\lambda = 1/t^*_\lambda$, then $(1 + \lambda S) = T_{s^*_\lambda} T_{t^*_\lambda}$.

Proof. We observe that $s^*_\lambda \in L^\infty_+$. The rest of the proof is similar to those of Lemmas 1, 2 in [1].

Lemma 3. Let $s(\theta_1, \theta_2) = \sum_{n \geq 0} s_n(\theta_1) e^{in\theta_2}$ be in $L^\infty_+$ and let $s(\theta_1, z_2) = \sum_{n \geq 0} s_n(\theta_1) z_2^n$, $\theta_1 \in T$, $z_2 \in U$. Define for $(\theta_1, \theta_2) \in T^2$ and $z \in U$,

\[ e_z^{(j)}(\theta_1, \theta_2) = (1 - e^{i\theta_j})^{-1}, \quad j = 1, 2. \]

Then for $z_1, z_2 \in U$, $e_z^{(1)} e_z^{(2)}$ is in $H^2$ (and so in $L^2$) and

\[ T_z (e_z^{(1)} e_z^{(2)})(\theta_1, \theta_2) = \bar{s}(\theta_1, z_2) (e_z^{(1)} e_z^{(2)})(\theta_1, \theta_2). \]

Proof. The equality follows by checking that the inner products of both sides with $\exp(i m_1 \theta_1 + i m_2 \theta_2)$, are the same for every integer $m_1$ and every nonnegative integer $m_2$. 

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Lemma 4. Let $E \subset T^2$ with $m_2(E) > 0$. Let for each $\theta_1 \in T$, $E(\theta_1) = \{\theta_2 \in T : (\theta_1, \theta_2) \in E\}$ and let, for each $\delta > 0$, $F(\delta) = \{\theta_1 \in T : m_1(E(\theta_1)) > \delta\}$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that $m_1(F(\delta_2)) > \delta_1$.

Proof. Observe that as $n \to \infty$,
\[
\int_{F(1/n)} m_1(E(\theta_1)) d\theta_1 \uparrow \int_T m_1(E(\theta_1)) d\theta_1 = m_2(E),
\]
and hence, if $0 < \delta_1 < m_2(E)$, there exists $N$ such that
\[
\delta_1 < \int_{F(1/N)} m_1(E(\theta_1)) d\theta_1 \leq m_1(F(1/N)).
\]
Now take $\delta_2 = 1/N$.

Theorem 1. Let $E \subset T^2$, $m_2(E) > 0$. Choose $\delta_1, \delta_2$ as in Lemma 4 and denote $F(\delta_2)$ by $F$. Let $M$ be the operator defined by
\[
(Mf)(\theta_1, \theta_2) = \chi_F(\theta_1)(\theta_1, \theta_2), \quad f \in L^2, (\theta_1, \theta_2) \in T^2.
\]
Suppose that $S$ is as in Lemma 2. Then for every $f \in H^2$,
\[
\lim_{\lambda \to \infty} \lambda M(I + \lambda S)^{-1} Sf = Mf.
\]
Proof. Since $M(I - \lambda(I + \lambda S)^{-1} S) = M(I + \lambda S)^{-1}$, and the set $\{e_n^{(1)} e_n^{(2)} : z_1, z_2 \in U\}$ is fundamental in $H^2$, the theorem will be proved if the following are verified: (i) $\sup_\lambda \|M(I + \lambda S)^{-1}\| < \infty$, and (ii) for all $z_1, z_2 \in U$,
\[
M(I + \lambda S)^{-1} e_n^{(1)} e_n^{(2)} \to 0, \quad \lambda \to \infty.
\]
By Lemma 1(a), $|s_\lambda(\theta_1, \theta_2)| \leq 1$, a.e. and therefore $\|T_s^\lambda\| = \|T_{s_\lambda}\| \leq 1$. Also trivially, $\|M\| \leq 1$ and hence in view of Lemma 2, $\|M(I + \lambda S)^{-1}\| = \|MT_{s_\lambda} T_{s_\lambda}\| \leq 1$. This proves (i).

To check (ii), note that for $\theta_1 \in F$, using the notation in the proof of Lemma 1(b),
\[
\Re B(\theta_1, z_2) = \int_{E(\theta_1)} p(z_2, \theta_2) d\theta_2 \geq \frac{1 - |z_2|}{1 + |z_2|},
\]
and hence for almost all $\theta_1 \in F$,
\[
|s_\lambda(\theta_1, z_2)| \leq (1 + \lambda)^{-p} \quad z_2 \in U
\]
where $p = \delta_2 (1 - |z_2|)/2(1 + |z_2|) > 0$. Now using Lemma 3, we have
\[
M(I + \lambda S)^{-1}(e_n^{(1)} e_n^{(2)})(\theta_1, \theta_2) = \chi_F(\theta_1)s_\lambda(\theta_1, \theta_2) s_\lambda(\theta_1, z_2) e_n^{(1)} e_n^{(2)}(\theta_1, \theta_2),
\]
and hence
\[ \|M(I + \lambda S)^{-1}(e^{(1)}_z e^{(2)}_z)\|_2 \leq (1 + \lambda)^{-\theta} \|e^{(1)}_z e^{(2)}_z\|_2. \]

This last expression tends to zero as \( \lambda \to \infty \) and (ii) is verified.

**Discussion.** The limit relation \( \lambda M(I + \lambda S)^{-1} Sf \to Mf \) of Theorem 1 provides the first step in recapturing \( f \) from its values on \( E \). The knowledge of \( f \) on \( E \) yields \( Sf \) and \( (I + \lambda S)^{-1} \) is obtained in terms of \( s_\lambda \) which depends only on the set \( E \). Thus \( \lambda M(I + \lambda S)^{-1} Sf \) can be computed and the limit as \( \lambda \to \infty \) gives \( Mf \), i.e. values of \( f \) on \( F \times T \). The second step of going from \( F \times T \) to \( T \times T \) now follows easily. We proceed basically as in the first step, but we interchange the roles of the \( \theta_1 \)- and \( \theta_2 \)-coordinates and employ the appropriate substitute for \( L^2 \).

The original set \( E \) is now replaced by \( E' = F \times T \). Recalling the relationship that \( F \) bears to \( E \) (Lemma 4), we see that the set \( F' \) which corresponds in a similar way (but with \( \theta_1, \theta_2 \) interchanged) to \( E' \) can be chosen to be \( T \) itself! Thus a theorem such as Theorem 1 with suitable changes will lead us to \( f \) on \( T \times F' = T \times T \). This is the sought-for second step, and the algorithm for recapturing \( f \) is complete.

The above algorithm recaptures the boundary values of \( f \) from its values on \( E \). It is sometimes convenient to have a formula which gives the values of \( f \) inside \( U \) directly. In the following theorem such a formula is obtained.

**Theorem 2.** Let the hypotheses and the notations be as in Theorem 1.

(a) If for \( \lambda > 0 \) and \( z_1, z_2 \in U \),
\[ f_\lambda(z_1, z_2) = \frac{\lambda}{(2\pi i)^2} \int_E \frac{f(w_1', w_2') s_\lambda(w_1', w_2') s_\lambda(w_1, z_2)x_{z_1}(w_1)}{(w_1 - z_1)(w_2 - z_2)} \, dw_1 dw_2, \]
then as \( \lambda \to \infty \), \( f_\lambda \) converges in \( H^2 \) to some \( \phi \) and a fortiori uniformly on compact subsets of \( U^2 \).

(b) If for \( \lambda > 0 \) and \( z_1 \in U \),
\[ b_\lambda(z_1) = \exp \left\{ -\frac{1}{2} \log (1 + \lambda) \int_P K(z_1, \theta_1) \, d\theta_1 \right\}, \]
and \( \phi \) is as in (a) above then for each \( (z_1, z_2) \in U^2 \),
\[ f(z_1, z_2) = \lim_{\lambda \to \infty} \lim_{r \to 1} \frac{\lambda}{2\pi i} b_\lambda(z_1) \int_{c_r} \frac{\phi(w_1, z_2) \overline{b_\lambda(w_1)}}{w_1 - z_1} \, dw_1, \]
where \( c_r \) is the circle \( |w_1| = r \) and \( |z_1| < r < 1 \).

**Proof.** (a) By Theorem 1, we have that as \( \lambda \to \infty, \lambda M(I + \lambda S)^{-1} Sf \to Mf \). Taking the inner product with \( e^{(1)}_z e^{(2)}_z, z_1, z_2 \in U \), we get
We will prove that the first member of (2) equals \( f_\lambda(z_1, z_2) \) and hence if the second member of (2) is denoted by \( \phi(z_1, z_2) \) the proof of (a) will be complete.

Since \( M \) is the multiplication by \( \chi_{E'}(\theta_1) \), a function of \( \theta_1 \) only, \( M \) commutes with \( S \) and therefore with \((I + \lambda S)^{-1}\). Thus

\[
(M(I + \lambda S)^{-1}Sf, e^{(1)}_{z_1}e^{(2)}_{z_2}) = (SMf, (I + \lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2} = (Mf, (I + \lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2}).
\]

Using Lemmas 2 and 3 to write the expressions for \((I + \lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2}\), we see that the left member of (2) is in fact \( f_\lambda(z_1, z_2) \).

(b) From the proof in (a), we see that \( \phi(z_1, z_2) = (Mf, e^{(1)}_{z_1}e^{(2)}_{z_2}) \) and hence

\[
\phi(z_1, z_2) = \frac{1}{(2\pi)^2} \int_{F \times T} \frac{f(w_1, w_2)}{(w_1 - z_1)(w_2 - z_2)} \, dw_1 \, dw_2.
\]

Let us now define \( L^2_\sim \) to be the subspace \( f / \in L^2_\sim: \hat{f}(m, n) = 0 \) for all \( m < 0 \) via \( \hat{P} \) the orthogonal projection of \( L^2_\sim \) onto \( L^2_\sim \), \( S \) the Toeplitz operator on \( L^2_\sim \) corresponding to the function \( \chi_{E'} \), where \( E' = F \times T \), i.e. \( \hat{S}f = \hat{P}(\chi_{E'}f), f \in L^2_\sim \), and \( \sigma_\lambda, b_\lambda \) to be

\[
\sigma_\lambda(z_1, z_2) = \exp \left\{ -\frac{1}{2} \log \left( 1 + \lambda \right) \int_{E'} K(\theta, z_1)P(\theta, z_2) \, d\theta_1 \, d\theta_2 \right\},
\]

\[
b_\lambda(z_1) = \exp \left\{ -\frac{1}{2} \log \left( 1 + \lambda \right) \int_{E} K(\theta, z_1) \, d\theta \right\},
\]

where \( \lambda > 0 \) and \( z_1, z_2 \in U \). Then for all \( z_1, z_2 \in U \), \( \sigma_\lambda(z_1, z_2) = b_\lambda(z_1) \) and as in Theorem 1 we will get, for every \( f \in H^2 \), \( \lambda \hat{M}(I + \lambda S)^{-1}f \rightarrow \hat{M}(\lambda \rightarrow \infty) \), where \( \hat{M} \) is the multiplication by \( \chi_{E'}(\theta_2) \), \( F' \) corresponding to \( E' \) according to Lemma 4 but with roles of \( \theta_1, \theta_2 \) reversed. However, since \( E' = F \times T \), we can take \( F' = T \) and so for \( f \in H^2 \), \( \lambda(I + \lambda S)^{-1}f \rightarrow f \). If, in this last relation, we take the inner product with \( e^{(1)}_{z_1}e^{(2)}_{z_2} \), then as \( \lambda \rightarrow \infty \),

\[
(\lambda(I + \lambda S)^{-1}f, e^{(1)}_{z_1}e^{(2)}_{z_2}) \rightarrow (f, e^{(1)}_{z_1}e^{(2)}_{z_2}) = (f, e^{(2)}_{z_1}e^{(1)}_{z_2}) = (f, \phi_\lambda(z_1, z_2)).
\]

Now noting that \( \hat{P}(\chi_{E'}f) = \phi_\lambda \), we see that

\[
((I + \lambda S)^{-1}f, e^{(1)}_{z_1}e^{(2)}_{z_2}) = (\hat{S}f, (I + \lambda S)^{-1}e^{(1)}_{z_1}e^{(2)}_{z_2}) = (\phi, b_\lambda e^{(1)}_{z_1}e^{(2)}_{z_2})b_\lambda(z_1).
\]
In the last step we used results similar to Lemmas 2 and 3 for \((I + \lambda S)^{-1}\). From (3) it now follows that as \(\lambda \to \infty\),
\[
\lambda\langle \phi, b_k e^{(1)} z_1 z_2 \rangle \to f(z_1, z_2).
\]
It is easy to see that \(\langle \phi, b_k e^{(1)} z_1 z_2 \rangle\) equals the inner product in \(H^2(U^1)\) of \(\phi(\cdot, z_2)\) with \(b_k e^{(1)} z_1 \). The proof is finished by observing that if \(u, v \in H^2(U^1)\) and \(u_e^{i\theta} = u(re^{i\theta})\) and \(v_r\) is similarly defined then the product \(\langle u, v \rangle_{e^{(1)}}\) is the limit as \(r \to 1\) of
\[
\frac{1}{2\pi i} \oint_{C_r} \frac{u(w)\overline{v}(w)}{w-z} \, dw.
\]

Remarks. (1) An alternative proof of a part of the one-variable theorem (Theorem I of [1] as regards the convergence on compact sets) has been suggested by Wainger (see Appendix B of [4]). This proof depends on the following statement which is true if \(n = 1\): To each nonnegative function \(\psi \in L^\infty(T^n)\) there is \(f \in H^\infty(U^n)\) such that \(|f| = \psi\) a.e. The statement is false for \(n > 1\) [2, p. 54ff]. Moreover for \(n > 1\) even when such a function \(f\) exists an explicit formula for \(f\) does not seem to be known. It would thus appear that a proof on the lines suggested by Wainger is not possible for \(n > 1\) and that recourse to the techniques such as the ones used in the present work is necessary.

(2) The Theorem 1 above can easily be generalized to functions in \(H^2(T^n)\) with \(n > 2\). The algorithm to recapture the function takes \(n\) steps and the generalization does not need any new ideas.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MILWAUKEE, MILWAUKEE, WISCONSIN 53201