

## A MATRIX REPRESENTATION FOR ASSOCIATIVE ALGEBRAS. II

BY

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**ABSTRACT.** The results of part I of this paper are applied to show that if  $F$  is a free algebra over the field  $K$  and  $W$  is a subset of  $F$  which is algebraically independent modulo the commutator ideal  $[F, F]$ , then  $W$  again generates a free algebra. On the way a similar theorem is proved for algebras that are free in the variety of  $K$ -algebras whose commutator ideal is nilpotent of class  $n$ .

It is also shown that if  $L$  is a Lie algebra with universal enveloping algebra  $F$ , and  $U, V$  are ideals of  $L$ , then  $FUF \cdot FVF \cap L = [U \cap V, U \cap V]$ . This is used to extend the representation theorem of part I to free Lie algebras.

**I. Introduction.** In this paper we give some further applications of the matrix representation theorem proved in part I [7]. We first prove a general result communicated to us by George Bergman: If  $R$  is a ring and  $I$  an ideal of  $R$  which is free as a right  $R$ -module, and  $x \in R$ , then  $x$  is a zero divisor in  $R/I$  if and only if it is a zero divisor in  $R/I^n$ . Next, let  $[F, F]$  be the commutator ideal of the free algebra  $F$ . The algebra  $R = F/[F, F]^n$  is a free algebra in the variety (in the sense of universal algebra) of algebras whose commutator ideal is nilpotent of class at most  $n$ . We show that a subset  $Z$  of  $R$  freely generates a (relatively) free subalgebra if, and only if,  $Z$  is algebraically independent modulo  $[R, R]$ . Since  $\bigcap_{i=1}^{\infty} [F, F]^i = 0$ , this allows us to show that a subset of  $F$  which is algebraically independent modulo  $[F, F]$  generates a free subalgebra of  $F$ . In another direction, if  $U, V$  are ideals of a Lie algebra  $L$ , and  $U_F, V_F$  are the corresponding ideals in the universal associative enveloping algebra  $F$ , we show that  $U_F V_F \cap L = [U \cap V, U \cap V]$ . If  $L$  is a free Lie algebra, the matrix representation [4] for  $F$  with kernel  $U_F V_F$  then gives a representation for  $L$  with kernel  $[U \cap V, U \cap V]$ .

The author would like to express his appreciation to George Bergman who read two versions of this paper and suggested a number of improvements. In particular, the result of the next section, in this generality, is due to him.

**II. Regular elements.** The results of this section are due to G. Bergman.

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**Lemma 1.** *Let  $R$  be a ring,  $f: M \rightarrow N$  a homomorphism of right  $R$ -modules,  $f': M' \rightarrow N'$  its restriction to an appropriate pair of submodules, and  $J$  a left ideal of  $R$ . Suppose that the induced maps  $M/M' \rightarrow N/N'$  and  $M/MJ \rightarrow N/NJ$  are injective, and that  $N$  and  $J$  are flat as (right, resp. left)  $R$ -modules. Then the induced map  $M/M'J \rightarrow N/N'J$  is also injective*

**Proof.** Injectivity of  $M/M' \rightarrow N/N'$  and flatness of  $J$  give the injectivity of the top arrow in the commutative diagram:

$$\begin{array}{ccc} (M/M') \otimes J & \longrightarrow & (N/N') \otimes J \\ \downarrow & & \downarrow \\ MJ/M'J & \longrightarrow & NJ/N'J. \end{array}$$

Here to construct the left-hand descending arrow, we tensor  $M' \rightarrow M \rightarrow M/M' \rightarrow 0$  with  $J$  and conclude that  $(M/M') \otimes J \cong (M \otimes J)/\text{Im}(M' \otimes J)$ , then use the natural map  $M \otimes J \rightarrow MJ$  to induce a map  $M \otimes J/\text{Im}(M' \otimes J) \rightarrow MJ/M'J$ . The right-hand map is similarly obtained, but because  $N$  is flat, the map  $N \otimes J \rightarrow NJ$  is an isomorphism, hence so is  $N \otimes J/\text{Im}(N' \otimes J) \rightarrow NJ/N'J$ , and we see that the right-hand descending map is an isomorphism. As the top arrow is 1-1, so is the diagonal composite map, hence so is the left-hand descending map. As the latter is onto, it too is an isomorphism; hence the bottom map is injective.

Combining this with the injectivity of  $M/MJ \rightarrow N/NJ$ , we get the desired injectivity of  $M/M'J \rightarrow N/N'J$ .  $\square$

**Lemma 2.** *Let  $R$  be a ring (with 1),  $M$  a generator in the category of left  $R$ -modules, and  $J$  a two-sided ideal of  $R$ . If  $x \in R$  acts as a left-zero-divisor on  $R/J$ , then it acts as a zero-divisor on  $M/MJ$ .*

**Proof.** Since  $M$  is a generator, some direct sum  $N = M \oplus \dots \oplus M$  has a direct summand isomorphic to  $R$ . Hence  $N/JN$  has a summand isomorphic to  $R/J$ . Hence  $x$  acts as a zero divisor on  $N/JN = M/JM \oplus \dots \oplus M/JM$ . Hence  $x$  acts as a zero divisor on  $M/JM$ .  $\square$

**Theorem 1.** *Let  $R$  be a ring,  $x$  an element of  $R$ , and  $I$  a two-sided ideal of  $R$ . Then:*

(i) *If  $I$  is flat as a left  $R$ -module, and  $x$  is not a left zero-divisor on  $R/I$ , then  $x$  is not a left zero-divisor on  $R/I^n$  for any  $n$ .*

(ii) *If  $I$  is a generator of the category of left  $R$ -modules, and  $x$  is a left zero-divisor on  $R/I$ , then  $x$  is a left zero-divisor on  $R/I^n$  for all  $n$ .*

(iii) *If  $R$  is a free associative algebra over a field (and  $I$  arbitrary), then for all  $n > 0$ ,  $x$  is a zero-divisor in  $R/I$  if and only if it is a zero-divisor in  $R/I^n$ .*

**Proof.** (i) and (ii) are proved by induction on  $n$ . For (i), apply Lemma 1, with  $M = N = R$ ,  $f =$  left multiplication by  $x$ ,  $M' = N' = I^{n-1}$ , and  $J = I$ . For (ii),

apply Lemma 2 with  $M = I$ ,  $J = I^{n-1}$ . We get (iii) from (i) and (ii) and the symmetric statements on the right, and the result that in a free associative algebra  $R$ , any right or left ideal  $I$  is free as a (right or left)  $R$ -module (Cohn [4]).  $\square$

III. Free subalgebras. Recall from [7] that if  $F = K\langle X \rangle$  is a free  $K$ -algebra,  $r \rightarrow r'$ ,  $r \rightarrow r''$   $K$ -homomorphisms of  $F$  into  $K$ -algebras  $R'$ ,  $R''$  with kernels  $U$ ,  $V$  respectively and  $T = T(F; R', R'')$  a free  $(R', R'')$ -bimodule on generators  $\delta(x)$  ( $x \in X$ ), then the map  $x \rightarrow (\begin{smallmatrix} x \\ \delta(x) \end{smallmatrix} \begin{smallmatrix} 0 \\ x' \end{smallmatrix})$  extends to a homomorphism  $\mu: F \rightarrow (\begin{smallmatrix} R'' & 0 \\ T & R' \end{smallmatrix})$  with kernel  $UV$ . The induced map  $\delta: F \rightarrow T$  is a derivation. If we consider  $T$  as a right  $R'^{\text{opp}} \otimes R''$  bimodule, then we define, for each  $x$ , partial derivatives  $\partial/\partial x: F \rightarrow R'^{\text{opp}} \otimes R''$  by the formula  $\delta(f) = \sum_{x \in X} \delta(x)(\partial f/\partial x)$ . It is easily verified that if  $m \in F$  is a monomial  $m = x_1 x_2 \cdots x_n$  ( $x_i \in X$ ), then

$$(1) \quad \frac{\partial m}{\partial x} = \sum_{x_i=x} (x'_1 \cdots x'_{i-1})' \otimes (x_{i+1} \cdots x_n)''.$$

Let now  $A = R' = R'' = F/[F, F]$ , with  $[F, F]$  the commutator ideal of  $F$ . Since  $[F, F]^2$  is the kernel of  $\mu$ ,  $\mu$  induces an injection  $\bar{\mu}: \bar{F} = F/[F, F]^2 \rightarrow (\begin{smallmatrix} A & 0 \\ T & A \end{smallmatrix})$  and  $\partial/\partial x$  induces a partial derivative  $\bar{\partial}/\partial x: \bar{F} \rightarrow A \otimes A$ . Each  $\bar{\partial}/\partial x$ , when restricted to  $[\bar{F}, \bar{F}]$  is an  $A$ -bimodule homomorphism. An element  $w \in [\bar{F}, \bar{F}]$  is zero if and only if for all  $x$ ,  $\bar{\partial} w/\partial x = 0$ . Let  $\nu$  be the natural homomorphism  $\nu: A \otimes_K A \rightarrow A \otimes_A A = A$ . For  $m_1, m_2 \in \bar{F}$ , then

$$\frac{\bar{\partial}}{\partial x} (m_1 m_2 - m_2 m_1) = \frac{\bar{\partial} m_1}{\partial x} m'_2 - m'_2 \frac{\bar{\partial} m_1}{\partial x} + \left( m'_1 \frac{\bar{\partial} m_2}{\partial x} - \frac{\bar{\partial} m_2}{\partial x} m'_1 \right).$$

Since  $\nu$  acts by removing tensor signs, the composed map  $(\bar{\partial}/\partial x)\nu$  is zero at  $m_1 m_2 - m_2 m_1$ . Since  $(\bar{\partial}/\partial x)\nu$  is an  $A$ -bimodule homomorphism on  $[\bar{F}, \bar{F}]$  it follows that  $(\bar{\partial}/\partial x)\nu$  is zero on  $[\bar{F}, \bar{F}]$ , and thus can be considered as a derivation  $d/dx': A = \bar{F}/[\bar{F}, \bar{F}] \rightarrow A$ . It is immediate from (1) that if we write  $x'$  for the coset  $x + [F, F]$ , then,  $d/dx'$  is indeed the usual partial derivative with respect to  $x'$  in the polynomial ring  $A = K[X']$ .

**Lemma 3.** Let  $A = R' = R_1 = \bar{F}/[F, F]$  and  $R'' = R_d = F/[F, F]^d$ . Let  $w_1, \dots, w_n$  be elements of  $F = K\langle x_1, \dots, x_n \rangle = K\langle X \rangle$  such that  $w'_1, \dots, w'_n$  are algebraically independent in  $R_1 = K[X']$ . Then the Jacobian matrix  $\Gamma_d = (\partial w_i/\partial x_j)$  is regular over  $R_1 \otimes R_d$ .

**Proof.** We first treat the case  $d = 1$ .

Let  $\nu_*: (A \otimes_K A)_* \rightarrow A_*$  be the map induced by  $\nu$  on  $n \times n$  matrices. Then, by the foregoing discussion,  $\Gamma_1 \nu_* = (dw'_i/dx'_j)$ , the Jacobian matrix of the elements  $w'_i$  of  $A$ . The independence of  $w'_1, \dots, w'_n$  insures that  $\Gamma_1 \nu_*$  is regular in  $A_n$ . Thus  $\Gamma_1 \nu_*$  has a nonzero determinant. Since  $\nu$  is a ring homomorphism,

( $\det \Gamma_1$ ) $\nu = \det (\Gamma_1\nu_*)$ . So  $\Gamma_1$  also has a nonzero determinant, and, hence, is regular.

Let now  $\pi^d$  be the natural map  $\pi^d: R_1 \otimes_K F \rightarrow R_1 \otimes_K R_d$ , and let  $\pi_*^d$  be the map induced on  $n \times n$  matrices. Then the kernel of  $\pi^d$  is  $R_1 \otimes [F, F]^d$ , so that  $\text{Ker } \pi_*^d$  is  $(R_1 \otimes [F, F]^d)_* = ((R_1 \otimes_K [F, F])_*^d)$ . Now, since  $F$  is a free algebra,  $[F, F]$  is free as a left module and it follows that  $R_1 \otimes [F, F]$  is free as a left  $R_1 \otimes F$ -module (with basis  $1 \otimes (F\text{-basis for } [F, F])$ ). Thus, when we go to  $n \times n$  matrices,  $(R_1 \otimes [F, F])_*$  is free as a left  $(R_1 \otimes F)_*$  module. Now if  $\partial_0/\partial_0 x$  are the partial derivatives with values in  $R_1^{\text{opp}} \otimes F = R_1 \otimes F$ , then  $\Gamma_d = (\partial_0 w_i / \partial_0 x) \pi_*^d$ ,  $\Gamma_1 = (\partial_0 w_i / \partial_0 x) \pi_*^1$ . We may now apply Theorem 1 with  $R = (R_1 \otimes F)_*$  and  $I = (R_1 \otimes [F, F])_*$  to deduce from the first part of the proof that since  $\Gamma_1$  is left regular, so is  $\Gamma_d$ . The same argument works just as well qua right modules, so  $\Gamma_d$  is also right regular.  $\square$

Let now  $\mathcal{C}_d$  be the variety of all  $K$ -algebras whose commutator ideal is nilpotent of class  $d$ . (This is the class of all algebras which satisfy the polynomial identity  $\prod_{i=1}^d [x_i, y_i]z_i = 0$ .) Then  $R_d = F/[F, F]^d$  is a free algebra in this variety, and we say that a subset  $W$  of  $R_d$  freely generates  $R_d$  if every relation of  $W$  is a polynomial identity of  $R_d$ . We may ask which subalgebras of  $R_d$  are again  $\mathcal{C}_d$ -free.

**Theorem 2.** *Let  $R_d = F/[F, F]^d$ . A subset of  $R_d$   $\mathcal{C}_d$ -freely generates a  $\mathcal{C}_d$ -free subalgebra if and only if it is algebraically independent modulo  $[R_d, R_d]$ .*

**Proof.** We first consider the case where  $F$  is finitely generated,  $F = K\langle x_1, \dots, x_n \rangle$  and  $W$  consists of precisely  $n$  elements,  $w_1 + [F, F]^d, \dots, w_n + [F, F]^d$ , which are algebraically independent modulo  $[R_d, R_d]$ . From the discussion at the beginning of this section,  $R_d$  is embedded in the algebra

$$\begin{pmatrix} R_{d-1} & 0 \\ T & R_1 \end{pmatrix},$$

$T$  a free  $(R_1, R_{d-1})$ -bimodule with basis  $\delta(x_1), \dots, \delta(x_n)$ , via the map

$$x_i + [F, F]^d \rightarrow \begin{pmatrix} x_i + [F, F]^{d-1} & 0 \\ \delta(x_i) & x_i + [F, F] \end{pmatrix}.$$

If  $w_i + [F, F]^d$  corresponds to

$$\begin{pmatrix} w_i + [F, F]^{d-1} & 0 \\ t_i & w_i + [F, F] \end{pmatrix}$$

then, considering  $T$  as a right  $R_1 \otimes R_{d-1}$  module,  $(t_1, \dots, t_n) = (\delta(x_1), \dots, \delta(x_n))\Gamma_{d-1}$ . Since, by Lemma 3,  $\Gamma_{d-1}$  is regular,  $\{t_1, \dots, t_n\}$  is again a basis for a free  $R_1 \otimes R_{d-1}$  module. Now,  $w_1 + [F, F], \dots, w_n + [F, F]$  are algebraically independent, so the map  $x_i + [F, F] \rightarrow w_i + [F, F]$  extends to an injection  $R_1 \rightarrow R_1$  and we may assume by induction that  $x_i + [F, F]^{d-1} \rightarrow w_i + [F, F]^{d-1}$  also extends to an injection  $R_{d-1} \rightarrow R_{d-1}$ . The freedom of the  $t_i$ 's now insures that the map  $x_i + [F, F]^d \rightarrow w_i + [F, F]^d$  (which extends to an endomorphism of  $R_d$  by the freedom of  $R_d$ ) extends to an injection.

Now for the general case. It is clear that for a subset  $Z$  of  $R_d$  to freely generate a  $\mathbb{C}_d$ -free subalgebra it is sufficient that every finite subset of  $Z$  freely generates a free subalgebra. Consider  $\{z_1, \dots, z_n\}$ . Then, for some integer  $s$ ,  $\{z_1, \dots, z_n\}$  is in the subalgebra  $S$  of  $R_d$  generated by  $x_1 + [F, F]^d, \dots, x_s + [F, F]^d$ . Since  $z_1 + [R_d, R_d], \dots, z_n + [R_d, R_d]$  are algebraically independent, we can find  $z'_{n+1}, \dots, z'_s$  in  $S$  such that  $\{z_1, \dots, z_n, z'_{n+1}, \dots, z'_s\}$  is still algebraically independent modulo  $[S, S]$ . From the special case above we deduce that  $\{z_1, \dots, z_n, z'_{n+1}, \dots, z'_s\}$ , and hence  $\{z_1, \dots, z_n\}$ , freely generates a free subalgebra.

For the converse, suppose  $z_1, \dots, z_n \in R_d$  are not algebraically independent modulo  $[R_d, R_d]$ . Then there is a nonzero element  $p(x_1, \dots, x_n) \notin [F, F]$  such that  $p(z_1, \dots, z_n) \in [R_d, R_d]$ . Thus  $p(z_1, \dots, z_n)^d = 0$ . If  $z_1, \dots, z_n$  freely generate a  $\mathbb{C}_d$ -free subalgebra, then  $p(x_1, \dots, x_n)^d$  is an identity of  $R$ , a contradiction since all the identities of  $R_d$  are in the commutator ideal of  $F$ .  $\square$

Note the close resemblance between Theorem 2 and the following result of G. Baumslag [2, Theorem 2]:

*Let  $G$  be a group which is free in the variety  $\mathcal{S}_d$  of groups that are solvable of derived length at most  $d$ . A subset of  $G$   $\mathcal{S}_d$ -freely generates an  $\mathcal{S}_d$ -free group if and only if it is linearly independent module  $[G, G]$ , the commutator subgroup of  $G$ .*

In one direction, Theorem 2 can be extended to (absolutely) free algebras.

**Theorem 3.** *Let  $W$  be a subset of the free algebra  $F = K\langle X \rangle$  which is algebraically independent modulo  $[F, F]$ . Then  $W$  freely generates a free subalgebra of  $F$ .*

**Proof.** By extending  $W$  if necessary, we may assume that  $W$  is indexed by the elements of  $X$ . Consider the endomorphism  $\alpha$  of  $F$  defined by  $x \rightarrow w_x$ , and suppose  $0 \neq u \in \text{Ker } \alpha$ . Choose  $d$  such that  $u \notin [F, F]^d$ . (This can be done since  $\bigcap_{k=1}^\infty [F, F]^k = 0$ .) Then  $\alpha$  induces a map  $\alpha_d: F/[F, F]^d \rightarrow F/[F, F]^d$  which, by the previous theorem, is injective. However,  $(u + [F, F]^d)\alpha_d = u\alpha + [F, F]^d = 0$ , so that  $u \in [F, F]^d$ , a contradiction. Thus  $\text{Ker } \alpha = 0$  and we are done.  $\square$

**IV. Lie algebras.** In this section, we prove a representation theorem for Lie algebras.

Throughout what follows,  $L$  will be a Lie algebra over a field  $K$ , and  $A$  its universal enveloping algebra. We denote Lie multiplication by  $[ , ]$ , so that it agrees with the commutator operation in  $A$ . We must begin with some results on the relation between  $L$  and  $A$ .

If  $B$  is a  $K$ -basis for  $L$ , and we totally order  $B$ , then an *ascending monomial* will mean an expression  $x_1 x_2 \cdots x_n$  (representing an element of  $A$ ), with  $n \geq 0$ ,  $x_i \in B$ , and  $x_1 \leq x_2 \leq \cdots \leq x_n$ . By the Poincaré-Birkhoff-Witt theorem, the ascending monomials form a  $K$ -basis for  $A$ . There is an explicit reduction procedure for expressing any monomial in  $B$  as a  $K$ -linear combination of ascending monomials (cf. Jacobson [5, Chapter V], also Bergman [3, §4]). If  $y = y_1 y_2 \cdots y_m$  is a monomial and  $y_i > y_{i+1}$  for some  $i$ , we write:

$$(2) \quad \begin{aligned} y &= y_1 \cdots y_i y_{i+1} \cdots y_m \\ &= (y_1 \cdots y_{i+1} y_i \cdots y_m) + (y_1 \cdots [y_i, y_{i+1}] \cdots y_m), \end{aligned}$$

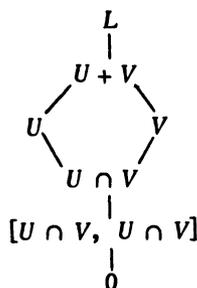
and then express  $[y_i, y_{i+1}] \in L$  as a  $K$ -linear combination of elements of  $B$ . Thus,  $y$  is a linear combination of monomials of smaller length,  $m - 1$ , and a monomial which is closer to being ascending. So this process, applied to the terms of any expression, will eventually terminate in a linear combination of ascending monomials, the "canonical form" for the element in question.

Now suppose  $U$  is a Lie ideal in  $L$ . Let us first choose a  $K$ -basis  $B(U)$  for  $U$ , then extend this to a basis  $B$  of  $L$ . Suppose a monomial  $y$  in  $B$  has at least one factor  $y_j$  belonging to  $B(U)$ . Then we claim that a reduction (2) takes  $y$  to a linear combination of monomials which again have a factor in  $B(U)$ . Indeed, if  $j \neq i, i + 1$  in (2), this is trivial, while if  $j = i$  or  $i + 1$ , then  $[y_i, y_{i+1}]$  again lies in  $U$  because  $U$  is an ideal, so  $[y_i, y_{i+1}]$  is a linear combination of elements of  $B(U)$ ; the assertion is then clear.

Let  $U_A \subseteq A$  denote  $AUA$ , the ideal of  $A$  generated by  $U$ . The elements of  $U_A$  are by definition those that can be written as linear combinations of (arbitrary) monomials having a factor in  $B(U)$ . It follows from the above observations that the canonical form for an element of  $U_A$  will still be as a linear combination of such monomials. In particular, since  $L$  is the subspace of  $A$  spanned by monomials of length 1, we have  $U_A \cap L = U$ . This can also be seen directly from the Poincaré-Birkhoff-Witt theorem applied to  $L/U$ , which has universal enveloping algebra  $A/U_A$ . But the analogous result we shall now obtain for *two* ideals is not so trivial.

Let  $U$  and  $V$  be two Lie ideals of  $L$ . Let us construct the basis  $B$  for  $L$  as follows. First choose a basis  $B([U \cap V, U \cap V])$  for the ideal  $[U \cap V, U \cap V] \subseteq L$ ; extend this to a basis  $B(U \cap V)$  of the ideal  $U \cap V$ ; extend  $B(U \cap V)$  on the one hand to a basis  $B(U)$  of  $U$ , and on the other hand to a basis  $B(V)$  of  $V$ , and note

that  $B(U) \cup B(V)$  will form a basis of the ideal  $U + V$  (cf. diagram).



We extend  $B(U) \cup B(V)$  to a basis  $B = B(L)$  of  $L$ .

We choose any total ordering of  $B$  subject to the condition that all elements of  $B(U)$  not lying in  $B(U \cap V)$  should be  $<$  all elements of  $B(U \cap V)$ , which should in turn be  $<$  all other elements of  $B(V)$ .

Now put  $U_A = AUA$  and  $V_A = AVA$ . We wish to study the ideal  $U_A V_A = AUAVA$ . For this purpose, let  $S$  denote the set of all monomials on  $B$  which either (i) involve an element of  $B(U)$  to the left of (i.e., preceding, possibly with some other terms in between) an element of  $B(V)$ , or (ii) involve an element of  $B([U \cap V, U \cap V])$ . Clearly every member of  $S$ , and hence every linear combination of members of  $S$  represents an element of  $U_A V_A$ , and conversely,  $U_A V_A$  is spanned by the elements represented by monomials in  $S$ .

We now claim that any reduction of the form (2) will take a monomial  $y \in S$  to a linear combination of monomials in  $S$ . It is clear from earlier observations that if  $y$  satisfies condition (ii), so will all the terms in the expression to which it reduces. Suppose, on the other hand, that  $y$  has a term  $y_j \in B(U)$  to the left of a term  $y_k \in B(V)$  (i.e.,  $j < k$ ). If neither  $j$  nor  $k$  belongs to  $\{i, i + 1\}$  (in the notation of (2)), the result is trivial, and if only one of them does, it is straightforward, like our earlier result. So suppose, finally, that  $j = i, k = i + 1$ . In order for the reduction (2) to be applicable, the element  $y_j \in B(U)$  must be greater, under our ordering, than  $y_k \in B(V)$ . By the conditions on our ordering, this can only happen if  $y_j$  and  $y_k$  both lie in  $B(U \cap V)$ . In this case, when we apply (2) the term of length  $m$  resulting contains the sequence  $y_k y_j$ , and hence still has a member of  $B(U)$  (actually a member of  $B(U \cap V)$ ) to the left of a member of  $B(V)$  (ditto), while  $[y_j, y_k] \in [U \cap V, U \cap V]$ , so the monomials of length  $m-1$  all involve terms in  $B([U \cap V, U \cap V])$ .

It follows that  $U_A V_A$  consists of those elements whose canonical representation involves only monomials from  $S$ . Intersecting with  $L$ , spanned by the monomials of length 1, we get:

**Theorem 4.** *Let  $U, V$  be ideals of the Lie algebra  $L$ . Let  $A$  be the universal*

associative enveloping algebra of  $L$  and let  $U_A = AUA$ ,  $V_A = AVA$  be the ideals generated by  $U$  and  $V$  respectively in  $A$ . Then  $U_A V_A \cap L = [U \cap V, U \cap V]$ .  $\square$

Note that the improper ideal  $L$  of  $L$  induces the augmentation ideal  $J = L_A \subseteq A$ .

**Corollary.** *If  $U$  is an ideal of  $L$ , and  $J$  the augmentation ideal of  $A$ , then,  $[U, U] = U_A J \cap L = U_A^2 \cap L = J U_A \cap L$ .*  $\square$

If  $L$  is the free Lie algebra on the set  $X$ , then  $A = K\langle X \rangle = F$  is free associative. So using Theorem 1 of [7] we obtain:

**Theorem 5.** *Let  $L$  be a free Lie algebra on the set  $X$ ,  $L'$ ,  $L''$  Lie algebras,  $l \rightarrow l'$ ,  $l \rightarrow l''$  maps of  $L$  into  $L'$  and  $L''$  with kernels  $U, V$  respectively,  $R'$ ,  $R''$  the universal enveloping algebras of  $L', L''$  and  $T$  a free  $(R', R'')$ -module generated by  $\{\delta(x); x \in X\}$ . Then the kernel of the map*

$$L \rightarrow \begin{pmatrix} L'' & 0 \\ T & L' \end{pmatrix} \text{ defined by } x \rightarrow \begin{pmatrix} x'' & 0 \\ \delta(x) & x' \end{pmatrix}$$

is  $[U \cap V, U \cap V]$ .

**Proof.** By Theorem 1 of [7],

$$x \rightarrow \begin{pmatrix} x'' & 0 \\ \delta(x) & x' \end{pmatrix} \text{ induces a map } K\langle X \rangle \rightarrow \begin{pmatrix} R'' & 0 \\ T & R' \end{pmatrix}$$

whose kernel is  $U_F V_F$ . The result follows immediately.  $\square$

In particular with  $L'' = 0$ , we find from the corollary that the map

$$x \rightarrow \begin{pmatrix} 0 & 0 \\ \delta(x) & x' \end{pmatrix} \text{ defines an embedding } L/[U, U] \rightarrow \begin{pmatrix} 0 & 0 \\ T & L' \end{pmatrix}.$$

If  $U$  is an ideal of  $L$ , and  $R'$  is the universal enveloping algebra of  $L/U$ , then the adjoint representation of  $L$  induces an  $R'$ -module structure on  $U/[U, U]$ . If  $J$  is the augmentation ideal of the free algebra  $F = K\langle X \rangle$ , then the injection  $U \rightarrow J$  induces an  $R'$ -module homomorphism  $U/[U, U] \rightarrow J/U_F J$ . The corollary shows that this map is injective. (This was shown by Labute [6] if  $U$  is generated by a single element.)

If  $L$  is finitely generated and  $R'$  is left Noetherian, then  $J/U_F J$  is left Noetherian and hence so is  $U/[U, U]$ . This can be used to give an alternate proof of a recent theorem of Amayo and Stewart [1]:

*Let  $L$  be a finitely generated Lie algebra having an Abelian ideal  $U$  such that the universal enveloping algebra of  $L/U$  is left Noetherian. Then  $L$  satisfies the maximal condition for ideals.*

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