EQUIVARIANT METHOD FOR PERIODIC MAPS

BY

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ABSTRACT. The notion of coherency with submanifolds for a Morse function on a manifold is introduced and discussed in a general way. A Morse inequality for a given periodic transformation which compares the invariants called $q$th Euler numbers on fixed point set and the invariants called $q$th Lefschetz numbers of the transformations is thus obtained. This gives a fixed point theorem in terms of $q$th Lefschetz number for arbitrary $q$.

Let $f$ be a periodic transformation of a closed $m$-dimensional manifold $M$ with fixed point set $N$. We develop in this note an equivariant approach using Morse theory. We introduce in §2 the notion of coherency with a submanifold $S$ of $M$ for a Morse function and show that such $S$-coherent Morse functions are dense in $C^\infty(M)$. Furthermore, in this approximation $f$-invariance will be preserved (§3). The coherency with the fixed point set $N$ of $f$ makes it possible to compare the difference of $q$th Euler number of $N$ and $q$th Lefschetz number of $f$. More precisely, let $\beta_q(N)$ and $\lambda_q(f)$ be respectively the $q$th Betti numbers of $N$ and the trace of $f^*$ on the $q$th homology group $H_q(M)$ with real coefficients. Let $B_q(N)$ and $\Lambda_q(f)$ be their alternative sums respectively, i.e.,

$$\beta_q(N) = \beta_q(N) - \beta_{q-1}(N) + \cdots + (-1)^q \beta_q(N),$$

$$\Lambda_q(f) = \lambda_q(f) - \lambda_{q-1}(f) + \cdots + (-1)^q \lambda_q(f),$$

where $0 \leq q \leq m$. We establish in §5 an inequality for arbitrary $q$ that $|\beta_q(N) - \lambda_q(f)|$ is no greater than the $q$th Morse difference of an arbitrary $f$-invariant $N$-coherent Morse function. We obtain as corollaries a fixed point theorem in terms of arbitrary $\Lambda_q$ (when $q = m$, this is the Lefschetz fixed point theorem) and a more geometric proof of the fact that $\beta_q(N) = \Lambda_q(f)$, i.e., the Euler number of $N$ is equal to the Lefschetz number of $f$.

The Lemma 1 (§1) which states that a smooth function can be approximated by a Morse function with prescribed "boundary value" is essential to the construction of the approximations.

1. A Morse extension. For a real-valued smooth function $F$ on $M$, let $C(F)$ denote the set of all critical points of $F$. $F$ is called a Morse function if for any $p \in C(F)$, the determinant of the Hessian at $p$ does not vanish.

We assume without loss of generality that $M$ is a riemannian manifold with a metric $g$. Let $g_{ij}$ be the metric tensor of $g$ with respect to a local coordinate $(x^i)$.
and let $g^{ij}$ be the inverse of $g_{ij}$ as matrices. Using the metric $g$, the differential $dF(x)$ of $F$ at $x$ has a natural way to be identified with a tangent vector at $x$ which is called the gradient $\nabla F(x)$ at $x$. Locally we have $\nabla F(x) = g^{ij}(\partial F/\partial x^i)(\partial/\partial x^j)$.

We define $\|dF(x)\|$ by

$$\|dF(x)\|^2 = g(\nabla F, \nabla F) \quad \text{at} \quad x$$

and define $\|F\|_{b, \Omega}$ and $\|F\|_{l, \Omega}$ of $F$ on an open set $\Omega$ in $M$ by

$$\|F\|_{b, \Omega} = \sup\{|F(x)|; x \in \Omega\},$$

$$\|F\|_{l, \Omega} = \sup\{|F(x)| + \|dF(x)\|; x \in \Omega\}.$$

Let $\phi: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function with $0 \leq |\phi(r)| \leq 1$, $\phi(0) = 1$, $\phi''(0) < 0$ and $\phi(r) = 0$ for $|r| \geq 1$. We denote throughout the induced function of mollifier by $\phi_\epsilon$ for each positive number $\epsilon$, i.e. $\phi_\epsilon(r) = \phi(r/\epsilon)$.

There exists a constant $a > 1$ such that

$$|\phi_\epsilon'(r)| < a/\epsilon.$$  

It is well known ([4] or [3]) that any given real-valued smooth function on a compact manifold $M$ can be approximated by a Morse function in the norm $\|\cdot\|_{1, M}$. The following lemma establishes this approximation theorem even when the “boundary value” of the desired Morse function has been given.

**Lemma 1.** Let $\Omega$ and $D$ be open sets of a smooth manifold $M$ such that $\Omega$ has a compact closure $\overline{\Omega}$ with smooth boundary $\partial \Omega$ and $D \subset \Omega$. Let $F$ be a Morse function defined on $M - D$. Then $F | M - \Omega$ can be extended to a Morse function $\tilde{F}: M \to \mathbb{R}$. Moreover if a smooth function $G$ on $M$ with $\|F - G\|_{b, M - D} < \epsilon$, is given, then the above Morse extension can be made so that $\|\tilde{F} - G\|_{b, M} < 2\epsilon$.

**Proof.** Choose a metric $g$ for $M$. For a point $x$ inside $\Omega$, we denote by $r(x)$ the distance with respect to $g$ from $x$ to $\partial \Omega$. Let $\Upsilon$ be the set $\{x \in \Omega \mid r(x) > r\}$. Since $C(F)$ is discrete and $\Omega$ is compact, there exist positive numbers $\eta$, $R$ and $\delta$ such that

$$\delta < \min\{1/2(1 + a), \sqrt{\epsilon/\eta}\} \quad \text{and} \quad \|dF(x)\| > \eta$$

for all $x$ in the strip $\overline{\Omega}_{R-\delta} - \Omega_{R+2\delta}$ contained in $\Omega - D$.

Define $H: M \to \mathbb{R}$ by patching together $F$ and $G$ in $\Omega_{R+\delta} - \Omega_{R+2\delta}$ as follows:

$$H(x) = F(x), \quad x \in M - \Omega_{R+\delta},$$

$$= G(x) + \phi_\delta(R + \delta - r(x))(F(x) - G(x)), \quad x \in \Omega_{R+\delta} - \Omega_{R+2\delta},$$

$$= G(x), \quad x \in \Omega_{R+2\delta}. $$

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It follows that \(\|H - G\|_{h,M} < \varepsilon\).

Let \(E\) be a Morse function on \(\Omega_{R-\delta}\) approximating \(H|\Omega_{R-\delta}\) such that

\[
\|E - H\|_{h,\Omega_{R-\delta}} < \delta^2 \eta < \varepsilon.
\]

Finally we define \(F\) on \(M\) by patching together \(E\) and \(F\) in the strip \(\Omega_{R-\delta} - \Omega_R\) as above. In order to see that \(F\) is a Morse function on \(M\), it suffices to show that \(F\) has no critical point in \(\Omega_{R-\delta} - \Omega_R\). In fact, for \(x\) in \(\Omega_{R-\delta} - \Omega_R\), we have

\[
H(x) = F(x) \quad \text{and} \quad \|dF(x)\| > \|dE(x) - dH(x)\| - \|d\varphi_p(R - r(x))\| \cdot \|E(x) - H(x)\| > n - 82v - (a/\delta)82 > 7, \quad (1 - 5(1 + a)) > \eta/2 > 0,
\]

since we have the estimates (1), (2) and (4). The approximation of \(F\) to \(G\) follows evidently from the construction.

2. Coherency with submanifold. Let \(S\) be a closed embedding submanifold of \(M\). In this section we define \(S\)-coherent Morse functions and show an approximation theorem of smooth functions by \(S\)-coherent Morse functions.

Definition 1. A Morse function \(F\) on \(M\) is called \(S\)-coherent if for each \(p\) in \(C(F|S)\), there is a coordinate neighborhood \((U,(x_i))\) with origin at \(p\), \(U \cap S = \{x_{s+1} = \cdots = x_m = 0\}\), and

\[
F(x_1 \cdots x_m) = F(0) - x_1^2 - \cdots - x_s^2 + \cdots + x_m^2
\]

where \(s\) is the dimension of \(S\) at \(p\) with \(s \geq \lambda\).

Such a \((U,(x_i))\) is called an \(S\)-coherent coordinate neighborhood of \(p\) for \(F\). Evidently, if \(F\) is an \(S\)-coherent Morse function on \(M\), then \(F|S\) is a Morse function on \(S\) with \(C(F|S) \subset C(F)\) and at each \(p\) of \(C(F|S)\), the index of \(F|S\) is equal to the index of \(F\).

For the convenience of later use, we fix the following notation:

Definition 2. Given a smooth function \(\psi\) defined on a closed embedding submanifold \(S\) of \(M\), we denote by \(\psi^*\) an extension of \(\psi\) on a tubular neighborhood \(T_\rho\) of \(S\) with radius \(\rho\) defined as follows. Let \(\rho\) be so small that for any \(x\) in \(T_\rho\), there is a unique geodesic joining \(x\) to a point \(x'\) of \(S\) and having the length equal to the distance \(r(x)\) from \(x\) to \(S\). Let

\[
\psi^*(x) = \psi(x') \cdot (2 - \varphi_p(r(x)))
\]

where \(\varphi_p\) is the mollifier relative to \(\rho\) (see §1).

If \(\psi\) is a Morse function, so is \(\psi^*\). In fact,

\[
C(\psi) = C(\psi^*) \quad \text{and} \quad \varphi''(0) < 0.
\]
Note that at $p \in C(\psi)$, the index of $\psi$ equals the index of $\psi^*$. 

**Theorem 1.** Given a closed submanifold $S$ of $M$, any smooth function $G$ on $M$ can be approximated uniformly by an $S$-coherent Morse function $F$.

**Proof.** Let $g$ be a Morse function on $S$ approximating $G|_S$. By Lemma 1, the $g^*$ on a tubular neighborhood of $S$ can be extended to a Morse function $F$ on $M$. $F$ is evidently $S$-coherent. If the tubular neighborhood of $S$ is sufficiently small, $F$ can be made to approximate $G$. Q.E.D.

3. Review of isometric actions. In general, for a compact riemannian manifold $(M, g)$, let $\text{ISO}(M, g)$ denote the full isometry group. Let $G$ be a closed subgroup of $\text{ISO}(M, g)$ and $p$ a point in $M$. By the isotropy group $G^p$, we mean the subgroup of isometries which leave $p$ fixed. The orbit $G(p)$ of $G$ at $p$ is the set $\{\gamma(p); \gamma \in G\}$.

Each orbit is a closed submanifold embedded in $M$. An orbit $G(p)$ is called principal if

1. for any $q \in M$, $\dim G^p \leq \dim G^q$, and
2. the number of components of $G^p$ is no greater than the number of components of $G^q$ whenever $\dim G^p = \dim G^q$.

We quote the following well-known result.

**Lemma 2** [5]. Let $G$ be a closed subgroup of $\text{ISO}(M, g)$ of a complete riemannian manifold $(M, g)$. Then the union of all the principal orbits of $G$ is open and dense in $M$.

We return to our given periodic map $f$ of $M$ with order $v$. Without loss of generality, we may assume that $f$ is an isometry of $(M, g)$ with some metric. In fact we can modify an arbitrarily given metric $\bar{g}$ by taking the mean of the induced metrics $(f^k)_* g$ for $k = 1, 2, \ldots, v$.

Let $\Gamma$ be the subgroup generated by $f$ in $\text{ISO}(M, g)$. $\Gamma$ is finite and cyclic with order $v$. By the order of an orbit of $\Gamma$, we mean the cardinal number of the orbit. For the integer $k$ such that there exists an orbit $\Gamma$ with order $k$, let $M_k$ be the union of the orbits of order $l$ where $l$ is a divisor of $k$. Thus we have a lattice consisting of these $M_k$'s with inclusion as the partial ordering. The lower bound of the lattice is evidently the fixed point set $N = M_1$.

We now consider some geometries about $N$ and more generally about $M_k$'s.

**Lemma 3.** The fixed point set $N$ of an isometry $f$ is a closed totally geodesic submanifold embedded in $M$ [2]. If the isometry $f$ is periodic, then each $M_k$, defined in the above, is a closed totally geodesic submanifold embedded in $M$ as well as in each $M_j$ with $j$ being a multiple of $k$.

**Proof.** For the first statement, one can refer to [2]. An elementary proof with clearer geometric insight can be obtained by using the following two facts as the basis of induction to construct, in an obvious way, local coordinates of $N$ for proving that $N$ is a submanifold of $M$. 

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(1) For two points $p$ and $q$ of $N$ which are sufficiently close to each other, the unique geodesic connecting $p$ and $q$ is contained in $N$.

(2) Let $\gamma_1$ and $\gamma_2$ be two geodesics of $M$ which are contained in $N$ and intersect with each other at a point $p$ of $N$. Then the parallel transportation of $\gamma_1$ along $\gamma_2$ generates a 1-parametered family of geodesics whose union is entirely contained in $N$.

For the second statement of the lemma, we need only to notice that $M_k$ is exactly the fixed point set of $f^k$ acting on $M$ as well as on $M_j$ with $j$ being a multiple of $k$. This completes the proof.

For any two $M_k$ and $M_j$, the intersection $M_k \cap M_j$ is evidently the $M_{(k,j)}$ where $(k, j)$ is the greatest common divisor of $k$ and $j$. On the other hand, $M = M_1$. In fact, for each $M_j$ and each $x$ in $M_j$, choose a convex neighborhood $U$ of $x$ such that for any $y$ in $U$, the geodesic joining $y$ to $x$ in $U$ is the only curve joining $y$ to $\Gamma(x)$ and having the length equal to the distance from $y$ to $\Gamma(x)$. It follows that $\Gamma^p \subset \Gamma^k$ and therefore the order of $\Gamma(x)$ is a divisor of that of $\Gamma(y)$. By Lemma 2, we see that the order of $\Gamma(x)$ is a divisor of $\nu$.

4. The approximation.

**Theorem 2.** Given a periodic transformation $f$ of $M$ with fixed point set $N$, an $f$-invariant smooth function $G: M \to \mathbb{R}$ can be uniformly approximated by an $f$-invariant $N$-coherent Morse function $F$.

**Proof.** We construct $F$ inductively in the following steps.

Step 1. Let $h_t$ be a Morse function on $N$ approximating $G | N$ uniformly. Recalling the Definition 2, we extend $h_t$ to $h_t^*$ on a tubular neighborhood $T_{2p}$ of $N$.

Step 2. For each prime number $p$ which is a divisor of $\nu$, we shall extend $h_t^* | T_p \cap M_p$ to an $f$-invariant Morse function $h_p^*: M_p \to \mathbb{R}$ which approximates $G | M_p$.

For a general $k$ with $1 \leq k \leq \nu$, let $U_k$ denote the union of all orbits of order $k$. By Lemma 2, $U_k$ is open and dense in $M_k$. Now $h_t^* | T_p \cap U_p$ induces a Morse function

$$\tilde{h}_t^*: (T_p \cap U_p) / \Gamma \to \mathbb{R}$$

where the quotient by $\Gamma$ means the orbit space of $T_p \cap U_p$ under $\Gamma$. By Lemma 1, $\tilde{h}_t^*$ can be extended to a Morse function

$$\tilde{h}_p: U_p / \Gamma \to \mathbb{R}$$

approximating $G / \Gamma$ restricted on $U_p / \Gamma$. This $\tilde{h}_p$ induces an $f$-invariant $N$-coherent Morse extension $h_p: M_p \to \mathbb{R}$ of $h_t^* | T_p \cap M_p$. $h_p$ evidently still approximates $G | M_p$.

Step 3. If $\nu \neq p$, we extend $h_p$ to an $f$-invariant Morse function $H_p$ defined on a tubular neighborhood $T_p(M_p)$ of $M_p$ by considering $h_p^*: T_{kp}(M_p) \to \mathbb{R}$, and then patching $h_p^*$ and $\tilde{h}_p$ together near $N$ as follows.
\[
H_p(x) = h^\dagger(x), \quad x \in T_\eta \cap T_p(M_p),
\]
\[
= h^\dagger(x) + \varphi_\eta(r(x) - \eta)(h^\gamma(x) - h^\dagger(x)), \quad x \in (T_\eta - T_\gamma) \cap T_p(M_p),
\]
\[
= h^\gamma(x), \quad x \in T_p(M) - T_\eta,
\]
where \( \eta = \rho/3 \) and \( r(x) \) denotes the distance from \( x \) to \( N \).

By taking \( \rho_\eta \) sufficiently small, \( h^\dagger \) and \( h^\gamma \) as well as their derivatives will differ from each other only by a small amount in the patching strip. This guarantees that no critical point of \( H_p \) will appear in the strip. Clearly \( H_p \) approximates \( G \). \( H_p \) is also \( f \)-invariant, since \( h^\dagger \) and \( h^\gamma \) are \( f \)-invariant and \( \varphi_\eta \) is symmetric with respect to \( 0 \).

**Step 4.** For \( M_k \), we assume according to the induction hypothesis that for each divisor \( l \) of \( k \), \( H_l \) has been constructed. By the Lemma 1, we extend the function

\[
\bigcup H_l \mid M_k \cap \left( \bigcup T_h(M_l) \right)
\]

to an \( f \)-invariant \( N \)-coherent Morse function \( h_k : M_k \to \mathbb{R} \) in the way similar to that described in Step 2. \( h_k \) approximates \( G \) again. If \( k < r \), we construct again \( h^\dagger \) and patch together \( h^\dagger_k \) and \( h^\gamma \), for all divisors \( l \) of \( k \), as in Step 2 to obtain \( H_k \). If \( k = r \), we take \( F = h_r \). This completes the construction of \( F \).

**Remark.** Such \( F \) is indeed \( M_r \)-coherent for all \( l \).

5. **The Inequality and its applications.** In general, for \( Y \subset X \subset M \), let

\[
\beta_q(X, Y) = \text{the Betti number of the pair } (X, Y),
\]
\[
\lambda_q(X, Y) = \text{the trace of } f_\alpha \text{ on } H_q(X, Y),
\]
and let

\[
B_q(X, Y) = \beta_q(X, Y) - \beta_{q-1}(X, Y) + \cdots + (-1)^q \beta_0(X, Y),
\]
\[
\Lambda_q(X, Y) = \lambda_q(X, Y) - \lambda_{q-1}(X, Y) + \cdots + (-1)^q \lambda_0(X, Y).
\]

We fix an \( f \)-invariant \( N \)-coherent Morse function \( F \) chosen arbitrarily. For a real number \( a \), let \( M^a \) be the set \( \{ x \in M \mid F(x) \leq a \} \).

Let all the critical values \( c_\alpha \)'s of \( F \) be ordered such that \( c_1 > c_2 > \cdots > c_k \). Let \( p_1^\alpha, \ldots, p_l^\alpha \) be all the critical points of \( F \) with critical value \( c_\alpha \) and of indices \( v_1^\alpha, \ldots, v_l^\alpha \) respectively, where \( p_1^\alpha, \ldots, p_l^\alpha \) are precisely the ones contained in \( N \). (\( l \) and \( k \) depend on \( \alpha \). The superscript \( \alpha \) will be omitted everywhere when no confusion can occur.)

For each \( p_j, 1 \leq j \leq k \), there is an \( N \)-coherent coordinate neighborhood \( (x_j) \) of \( p_j \). Let \( e_j \) be the \( r_j \)-cell \( \{ x_{j+1} = x_{j+2} = \cdots = x_m = 0 \} \). Consider numbers \( a_0, a_1, \ldots, a_n \) such that
When \(a_\alpha\) is chosen sufficiently close to \(c_\alpha\), we can have

1. \(e_j\)'s are disjoint and \(\partial e_j \subset M^{\alpha_\alpha}\);
2. \(\{(e_j, \partial e_j) \mid j = 1, \ldots, l\}\) and \(\{(e_j, \partial e_j) \mid j = 1, \ldots, l, \ldots, k\}\) are respectively the generators of the homology groups \(H(N^{\alpha_\alpha-1}, N^{\alpha_\alpha})\) and \(H(M^{\alpha_\alpha-1}, M^{\alpha_\alpha})\); and
3. for \(1 \leq j \leq l\), \(f\) is the identity map on \(e_j\) and for \(l < j \leq k\), \(f_\alpha(e_j, \partial e_j) = (e_i, \partial e_i)\) with \(i \neq j\), where \(f_\alpha\) is the induced map of \(f\) on \(H(M^{\alpha_\alpha-1}, M^{\alpha_\alpha})\).

It follows that for each \(q\) and \(\alpha\) both of \(\beta_q(N^{\alpha_\alpha-1}, N^{\alpha_\alpha})\) and \(\lambda_q(M^{\alpha_\alpha-1}, M^{\alpha_\alpha})\) are equal to the number of \(e_j\)'s with \(y_j = q\) and \(1 \leq j \leq l\). Hence we have

\[
\beta_q(N^{\alpha_\alpha-1}, N^{\alpha_\alpha}) = \lambda_q(M^{\alpha_\alpha-1}, M^{\alpha_\alpha}),
\]
\[
B_q(N^{\alpha_\alpha-1}, N^{\alpha_\alpha}) = \Lambda_q(M^{\alpha_\alpha-1}, M^{\alpha_\alpha}).
\]

From the exactness of

\[
0 \rightarrow \partial_\alpha(H_{q+1}(N, N^{\alpha_\alpha-1})) \rightarrow H_q(N^{\alpha_\alpha}, N^{\alpha_\alpha-1}) \rightarrow H_q(N, N^{\alpha_\alpha})
\]

we have

\[
B_q(N, N^{\alpha_\alpha}) = B_q(N^{\alpha_\alpha-1}, N^{\alpha_\alpha}) + B_q(N, N^{\alpha_\alpha-1}) - \varepsilon_{q,\alpha}
\]

where \(\varepsilon_{q,\alpha}\) is the rank of \(\partial_\alpha(H_{q+1}(N, N^{\alpha_\alpha-1}))\). Similarly, we have

\[
\Lambda_q(M, M^{\alpha_\alpha}) = \Lambda_q(M^{\alpha_\alpha-1}, M^{\alpha_\alpha}) + \Lambda_q(M, M^{\alpha_\alpha-1}) - \eta_{q,\alpha}
\]

where \(\eta_{q,\alpha}\) is the trace of \(f_\alpha\) on \(\partial_\alpha(H_{q+1}(M, M^{\alpha_\alpha-1}))\). By induction we have

\[
B_q(N) = \sum_{\alpha} B_q(N^{\alpha_\alpha}, N^{\alpha_\alpha-1}) - \sum_{\alpha} \varepsilon_{q,\alpha}
\]

and

\[
\Lambda_q(f) = \sum_{\alpha} \Lambda_q(M^{\alpha_\alpha}, M^{\alpha_\alpha-1}) - \sum_{\alpha} \eta_{q,\alpha}.
\]

The well-known Morse inequality states that given an arbitrary Morse function on \(M\), we have

\[
B_q(M) \leq C_q \overset{\text{def}}{=} c_0 - c_{q-1} + \cdots + (-1)^q c_0
\]

where \(c_q\) denotes the number of critical points of the Morse function with index \(q\). The difference \(C_q - B_q(M)\) is given by

\[
\sum_{\alpha} \text{rank}[\partial_\alpha(H_{q+1}(M^{\alpha_\alpha}, M^{\alpha_\alpha-1}))]
\]
if we adopt the subdivision of $M$ according to the Morse function as we did in
the above.

**Definition 3.** We call the difference $C_q - B_q(M)$ the $q$th Morse difference.
We denote the $q$th Morse difference of $F$ by $\delta_q(F)$. However,

$$|\eta_{q,a} - e_{q,a}| \leq \text{rank}[\partial_* (H_{q+1}(M^{a_*}, M^{a_{q-1}}))]$$

Therefore we obtain

**Theorem 3.** Given a periodic transformation $f$ of a compact smooth $m$-dimensional
manifold $M$ with fixed point set $N$, we have the inequality

$$|\Lambda_q(f) - B_q(N)| \leq \delta_q(F)$$

for each $q = 0, \ldots, m$ and each $f$-invariant $N$-coherent Morse function $F$, where

$$\Lambda_q(f) = \sum_{r=0}^{q} (-1)^{q-r} \text{trace of } f^r \text{ on } H_r(M),$$

$$B_q(N) = \sum_{r=0}^{q} (-1)^{q-r} \text{rth Betti number of } N,$$

and $\delta_q(F)$ is the $q$th Morse difference of $F$.

As corollaries we obtain a fixed point set theorem.

**Theorem 4.** Given a periodic transformation $f$ of a compact smooth manifold $M$, if

$$|\Lambda_q(f)| > \delta_q(F) \text{ for some } q = 1, \ldots, m \text{ and some } f \text{-invariant Morse function } F \text{ on } M,$$

then $f$ has a fixed point.

**Proof.** Suppose $f$ is fixed point free. Then every Morse function is $N$-coherent.
Also $B_q(N) = 0$. These lead to a contradiction.

**Remark 2.** In particular when $q = m$, $\Lambda_m$ is the usual Lefschetz number and
$\delta_m(F) = 0$ for all $F$. Therefore this corollary is a generalization of the Lefschetz
fixed point theorem for a periodic map.

**Remark 3.** Such a fixed point theorem based on $\Lambda_q$ and $\delta_q(F)$ for arbitrary $q$
and $F$ gives the best possible estimation. In fact, let $T^2 = S^1 \times S^1 = \{e^{i\theta}, e^{i\varphi} \mid 0 \leq \theta, \varphi < 2\pi\}$ and consider $f: (e^{i\theta}, e^{i\varphi}) \to (e^{i\theta}, e^{-i\varphi})$ and $F(e^{i\theta} + e^{i\varphi}) = \cos \theta + \cos 2\varphi$. Then $F$ is an $f$-invariant Morse function with $\Lambda_1 = 1 = \delta_1(F)$ but $f$ has
no fixed point.

Since $\delta_m(F) = 0$, we obtain

**Corollary 1.** Given a periodic transformation $f$ on a compact smooth manifold $M^m$
with fixed point set $N$, we have the Lefschetz number of $f$ equal to the Euler number
of the fixed point set $N$ and therefore equal to the integral over $N$ of the restricted
"intrinsic curvature" in the sense of Chern [1].
This statement can be regarded as a generalization of the Gauss-Bonnet theorem. A stronger result for any isometry can be proven rather directly by Mayer-Vietoris sequence applying on a tubular neighborhood of \( N \). However, the above approach using the viewpoint of Morse theory may help one to have better geometric insight.

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