MINIMAL SEQUENCES IN SEMIGROUPS

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ABSTRACT. In this paper we generalize a result of Tamura on \( \mathcal{S} \)-indecomposable semigroups. Based on this, the concept of a minimal sequence between two points, and from a point to another, is introduced. The relationship between two minimal sequences between the same points is studied. The rank of a semigroup \( S \) is defined to be the supremum of the lengths of the minimal sequences between points in \( S \). The semirank of a semigroup \( S \) is defined to be the supremum of the lengths of the minimal sequences from a point to another in \( S \). Rank and semirank are further studied.

Introduction. Semilattice decompositions of semigroups were first defined and studied by Clifford [1]. Since then several people have worked on this topic, notably Tamura [5]–[9]. The author's work on the subject can be found in [3], [4]. In this paper, we start by generalizing a result of Tamura [8] (or [9]) on \( \mathcal{S} \)-indecomposable semigroups. Based on this, the concept of a minimal sequence between two points, and from a point to another, is introduced. The relationship between two minimal sequences between the same points is studied. The rank of a semigroup is defined to be the supremum of the lengths of the minimal sequences between points in the semigroup. The semirank of a semigroup is defined to be the supremum of the lengths of the minimal sequences from a point to another in the semigroup. Rank and semirank are further studied. To understand this paper, the reader need only be aware of the first few chapters of Clifford and Preston [2] and Tamura's decomposition theorem. (See any of [5], [6], [8] or [9]. It was rediscovered by Petrich [10].)

1 Preliminaries. Throughout, \( S \) will denote a semigroup and \( Z^+ \) the set of positive integers. A congruence \( \sigma \) on \( S \) is called a semilattice congruence if \( S/\sigma \) is a semilattice. \( S \times S \) is the universal congruence on \( S \). \( S \) is \( \mathcal{S} \)-indecomposable if \( S \times S \) is the only semilattice congruence on \( S \).

Definition. Let \( a, b \in S \). Then

(1) \( a \mathrel{\mid} b \) if and only if \( b \in S^i a S^i \), \( \mid \) is transitive and reflexive.

(2) \( \rightarrow \) is defined as \( a \rightarrow b \) iff \( a \mid b^i \) for some \( i \in Z^+ \); let \( \rightarrow^0 \) denote \( \rightarrow \), i.e., \( \rightarrow^0 = \rightarrow \).

(3) \( a \rightarrow^{*+1} b \) iff there exists \( x \in S \) such that \( a \rightarrow^n x \rightarrow b \).

(4) \( a \rightarrow^\infty b \) iff \( a \rightarrow^n b \) for some \( n \in Z^+ \).

(5) \( \equiv \) is defined as \( a \equiv b \) iff \( a \rightarrow \rightarrow a \); let \( \equiv^0 \) denote \( \equiv \), i.e., \( \equiv^0 = \equiv \).

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\textbf{Theorem [Tamura].} Let $S$ be a semigroup. Then $\rightarrow^\infty \cap (\rightarrow^\infty)^{-1}$ is the finest semilattice congruence on $S$ and each component is $\mathcal{S}$-indecomposable.

\textbf{Corollary [Tamura].} Let $S$ be an $\mathcal{S}$-indecomposable semigroup. Then $\rightarrow^\infty$ is the universal congruence on $S$.

We generalize these results to:

\textbf{Theorem 1.1.} Let $S$ be a semigroup. Then $\rightarrow^\infty$ is the finest semilattice congruence on $S$. $\rightarrow^\infty$ is also the equivalence relation generated by the relations $ab = aba = ba$, for all $a, b \in S^1$ and $ab \in S$.

\textbf{Corollary 1.2.} Let $S$ be an $\mathcal{S}$-indecomposable semigroup. Then $\rightarrow^\infty$ is the universal congruence on $S$.

It is easy to deduce Tamura’s result from ours.

To prove Theorem 1.1, we need the following

\textbf{Lemma 1.3.} Let $\sigma$ be an equivalence relation on a semigroup $S$ satisfying $xy \sigma x'y'x'y'$ for all $x, y \in S^1$. Then for all $a, b, c, d \in S^1$ (with the convention $1 \sigma 1$),

1. $abc \sigma ab'c$ for all $i \in \mathbb{Z}^+$,
2. $abcd \sigma acbd$,
3. $a \rightarrow^\infty b$ implies $xay \sigma xby$ for all $x, y \in S^1$.

In particular $\rightarrow$ $\subset \sigma$.

\textbf{Proof.} (1) $abc \sigma ab'c$ for all $a, b, c, d \in S^1$,
2. Using (1), for any $A, B, C \in S^1$,

\[
A BC \sigma A(BC)(BC) \sigma (ABCBC)(ABCBC)
\]

$= (AB)(CB)(BC)^2 \sigma (AB)(CB)(BC)BC$

$= (AB)(CB)(BC)^2 \sigma (AB)(CB)(BC)BC$

$= A(BC)^2(ACBCAB) \sigma A(BC)(ACBCAB)$

$= (ABCACB)(ACBC)(ACBC)(ACBC)$

$= (ABCACBA)(ACB)(ACB)(ACB)$

$= (ABCACBA)(ACB)(ACB)(ACB)$

$= (ABCACBA)(ACB)(ACB)(ACB)$

$= (ABC)(ACB)(ACB)^2 \sigma (ABC)(ACB)(ACB)$.

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In short $ABC \sigma (ABC)(ACB)(ABC)$. Interchanging $B$ and $C$, we have $ACB \sigma (ACB)(ABC)(ACB)$. But $(ACB)(ABC)(ACB) \sigma (ACB)(ABC)(ACB)$. So $ABC \sigma ACB$.

Thus $abcd \sigma d(abc) = (da)bc \sigma (da)cb = d(abc) \sigma (acb)d$.

(3) First suppose $a \rightarrow b$. So we solve $sat = b', s'bt' = a'$. Then using (1), (2), we have $xaby \sigma xab'y = xasaty \sigma xsaaty \sigma xsaty = xb'y \sigma xby$. Similarly, $xbay \sigma xay$. So by (2), $xay \sigma xby$. Now assume $a \rightarrow^* b$, $n \geq 1$. So $a \rightarrow a_1 \rightarrow \cdots \rightarrow a_n \rightarrow b$. By the above $xay \sigma xa_iy$, $xa_iy \sigma xa_{i+1}y$ $(i = 1, \ldots, n - 1)$, $xa_ny \sigma xby$. Thus, $xay \sigma xby$.

Thus $a \rightarrow^* b$ implies $xay \sigma xby$ for all $x, y \in S^1$.

**Proof of Theorem 1.1.** Consider the following.

(※) \[ xy = xyx = yx, \quad \text{for all } x, y \in S^1, xy \in S. \]

Let $a, b \in S^1$. Then $aba \parallel (ab)^2$, $ab \parallel ba$. So $aba \rightarrow ab$. Now $ab \parallel (ba)^2$, $ba \parallel (ab)^2$. Thus $ab \rightarrow ba$. So $ab \rightarrow ba$. Thus $-\rightarrow^*$ is an equivalence relation satisfying (※). By Lemma 1.3, we conclude that $-\rightarrow^*$ is the smallest equivalence relation satisfying (※). In the same lemma, replacing $\sigma$ by $-\rightarrow^*$, we have $-\rightarrow^*$ is a semilattice congruence. Since any semilattice congruence satisfies (※), we have that $-\rightarrow^*$ is the finest semilattice congruence on $S$.

Corollary 1.2 is now immediate. We will need the following lemmas later.

**Lemma 1.4.** Let $S$ be a semilattice of semigroups $S_a (a \in \Omega)$, $\delta$ the corresponding semilattice congruence.

(1) Let $a \in \Omega$, with $a, b \in S_a$. If $a \rightarrow b$ in $S$, then $a \rightarrow b$ in $S_a$.

(2) Let $a \in S_a$, $b \in S_b$, $a \rightarrow b$. Then $a \parallel b$.

(3) Let $a, b \in S$ with $a \rightarrow b$. Then for some $a \in \Omega$, $a, b \in S_a$ and $a \rightarrow b$ in $S_a$.

**Proof.** (1) For some $x, y \in S^1$, $xay = b'$. So $b'xayb' = b''$. Then $b'x = xay \delta xay = b' \delta b$. So $b'x \in S_a$. Similarly $yb' \in S_a$. So $a \parallel b'' \in S_a$, whence $a \rightarrow b$ in $S_a$.

(2) $\alpha \rightarrow \beta$ in $\Omega$. Since $\Omega$ is a semilattice we deduce $\alpha \parallel \beta$ in $\Omega$ and then that $\alpha \geq \beta$.

(3) Let $a \in S_a$, $b \in S_b$. By (2), $a \geq \beta$, $\beta \geq a$ and so $a = \beta$. By (1), $a \rightarrow b$ in $S_a$.

**Lemma 1.5.** Let $S$ be a semigroup and $a, b, c \in S$.

(1) Let $i \in Z^+$. Then $a \rightarrow b^i$ implies $a \rightarrow b$.

(2) $a \parallel b \rightarrow c$ implies $a \rightarrow c$.

(3) Let $i, j \in Z^+$. Then $a^i \rightarrow b^j$ implies $a \rightarrow b$.

**Lemma 1.6.** (1) Let $S$ be a semigroup with an ideal $I$ and $a, b \in S$. Suppose $b$ is not nilpotent in $S/I$, and $a \rightarrow b$ in $S/I$. Then $a \rightarrow b$ in $S$.

(2) Let $S$ be a semigroup with zero, and suppose $a \in S$. Then $0 \rightarrow a$ if and only if $a$ is nilpotent.

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Proof. (1) We can solve \( xay = b' \) in the semigroup \( S/I \). Since \( b \) is not nilpotent in \( S/I, b' \in S \setminus I \). So \( x, a, xa, y, xay \in S \setminus I \). Thus \( xay = b' \) in \( S \).

(2) If \( 0 \to a, \) then \( 0 \mid a' \) for some \( i \in Z^+ \). Hence \( a' = 0 \). Conversely if \( a' = 0 \) for some \( i \in Z^+ \), then \( 0 \mid 0 = a' \) whence \( 0 \to a \).


Definition. Let \( S \) be a semigroup, \( a, b \in S \).

(1) By a sequence between \( a \) and \( b \), we mean a (possibly empty) finite sequence \( \langle x_i \rangle_{i=1}^n \) in \( S \) such that \( a = x_1, x_1 = x_{i+1} (i = 1, \ldots, n-1), x_n = b \). We call \( n \) the length of \( \langle x_i \rangle \). By \( n = 0 \), or \( \langle x_i \rangle_{i=1}^n \) empty, we mean \( a = b \). We say \( \langle x_i \rangle \) is minimal if it is nonempty and there is no sequence of smaller length (including the empty sequence) between \( a \) and \( b \).

(2) By a sequence from \( a \) to \( b \), we mean a (possibly empty) finite sequence \( \langle x_i \rangle_{i=1}^n \), such that \( a \to x_1, x_1 \to x_{i+1} (i = 1, \ldots, n-1), x_n \to b \). Again \( n \) is the length of the sequence, and by \( n = 0 \) (or \( \langle x_i \rangle \) empty) we mean \( a \to b \). \( \langle x_i \rangle \) is minimal if it is nonempty and there is no sequence of smaller length (including the empty sequence) from \( a \) to \( b \).

Lemma 2.1. Let \( S \) be a semigroup with \( a, b \in S \).

(1) Let \( \langle x_i \rangle_{i=1}^n, \langle y_i \rangle_{i=1}^m \) be two sequences between \( a \) and \( b \) of the same length. If \( \langle x_i \rangle \) is minimal, then so is \( \langle y_i \rangle \).

(2) Let \( S \) be \( \ast \)-indecomposable. Then either \( a \to b \) or there is a minimal sequence between \( a \) and \( b \).

(3) Let \( S \) be \( \ast \)-indecomposable. Then either \( a \to b \) or there is a minimal sequence from \( a \) to \( b \).

Proof. (1) Obvious.

(2) and (3) are trivial using Corollary 1.2.

Lemma 2.2. Let \( S \) be a semilattice of semigroups \( S_a (a \in \Omega) \).

(1) Let \( a, b \in S \), with a sequence \( \langle x_i \rangle_{i=1}^n \) between \( a \) and \( b \). Then \( a, b \) and all the \( x_i \)'s lie in some \( S_a \). Moreover \( \langle x_i \rangle \) is a sequence between \( a \) and \( b \) in \( S_a \). The minimal sequences between \( a \) and \( b \) in \( S \) are exactly those in \( S_a \).

(2) Let \( a \in \Omega, a, b \in S_a \). Let \( \langle x_i \rangle_{i=1}^n \) be a sequence from \( a \) to \( b \) in \( S \). Then all the \( x_i \)'s lie in \( S_a \) and \( \langle x_i \rangle \) is a sequence from \( a \) to \( b \) in \( S_a \). The minimal sequences from \( a \) to \( b \) in \( S \) are exactly those in \( S_a \).

(3) Let \( a \in S_a, b \in S_b \). Suppose there exists a sequence from \( a \) to \( b \) in \( S \). Then \( a \geq \beta \).

Proof. (1) That \( x_i \)'s, \( a, b \) lie in some \( S_a \) and that \( \langle x_i \rangle \) is a sequence between \( a \) and \( b \) within \( S_a \) follow from Lemma 1.4. So a minimal sequence between \( a \) and \( b \) in \( S \) is a sequence between \( a \) and \( b \) in \( S_a \) and obviously minimal in \( S_a \). Let \( \langle y_i \rangle \) be a minimal sequence between \( a \) and \( b \) in \( S_a \). Let \( \langle z_i \rangle \) be a sequence between \( a \) and \( b \) in \( S \). By the above, \( \langle z_i \rangle \) is a sequence between \( a \) and \( b \) in \( S_a \). So \( \langle z_i \rangle \) has length at least that of \( \langle y_i \rangle \). So \( \langle y_i \rangle \) is minimal in \( S \).
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(2) If \( \langle x_i \rangle \) is empty, \( a \rightarrow b \) in \( S \) and so in \( S_a \), by Lemma 1.4. Otherwise let \( x_i \in S_{a_i}, i = 1, \ldots, n \). Then \( a \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \rightarrow b \). By Lemma 1.4, \( a \geq a_1 \geq \cdots \geq a_n \geq a \). Consequently, \( a = a_1 = \cdots = a_n \). Now \( \langle x_i \rangle \) is a sequence from \( a \) to \( b \) within \( S_a \), by Lemma 1.4. The rest follows as in (1).

(3) If the sequence is empty, \( a \rightarrow b \) and so by Lemma 1.4, \( a \geq \beta \). Otherwise, \( a \rightarrow x_1 \cdots x_n \rightarrow b, x_i \in S_{a_i} \). By Lemma 1.4, \( \alpha \geq a_1 \geq \cdots \geq a_n \geq \beta \). So \( a \geq \beta \).

Definition. (1) A semigroup \( S \) is a \( \Gamma \)-semigroup iff for any \( a, b \in S \), either \( a \rightarrow b \) or \( b \rightarrow a \). Clearly any semigroup \( S \) with \( J \)-classes linearly ordered (equivalently the ideals are linearly ordered or still equivalently for any \( a, b \in S \), \( a \downarrow b \) or \( b \downarrow a \)) is a \( \Gamma \)-semigroup. Such an example is the full transformation semigroup. The null semigroup with more than one element is a \( \Gamma \)-semigroup, but its \( J \)-classes are not linearly ordered.

(2) \( S \) is a \( \Gamma^* \)-semigroup iff \( S \) is a semilattice of \( \Gamma \)-semigroups.

Lemma 2.3. Let \( S \) be a semigroup. Then the following are equivalent.

(1) \( S \) is a \( \Gamma^* \)-semigroup.
(2) \( S \) is a semilattice of \( \Gamma^* \)-semigroups.
(3) The \( S \)-indecomposable components of \( S \) are \( \Gamma \)-semigroups.

Proof. (1) \( \Rightarrow \) (2). Obvious.

(2) \( \Rightarrow \) (3). Let \( S \) be a semilattice of \( \Gamma^* \)-semigroups \( S_a (a \in \Omega) \). Let \( T \) be an \( S \)-indecomposable component of \( S \). Then \( T \subseteq S_a \) for some \( a \in \Omega \). \( S_a \) is a semilattice of \( \Gamma \)-semigroups \( U_\beta \). So \( T \subseteq U_\beta \) for some \( \beta \). Let \( a, b \in T \). Then \( a, b \in U_\beta \). So \( a \rightarrow b \) or \( b \rightarrow a \) in \( U_\beta \) and hence in \( S \). By Lemma 1.4, \( a \rightarrow b \) or \( b \rightarrow a \) in \( T \). Consequently \( T \) is a \( \Gamma \)-semigroup.

(3) \( \Rightarrow \) (1). Obvious.

Definition. Let \( a, b \in S \). Then \( a \triangleleft b \) iff \( a^i \rightarrow b \) for all \( i \in \mathbb{Z}^+ \).

Lemma 2.4. Let \( S \) be a semigroup with \( a, b, c \in S \).

(1) If \( a \rightarrow b \triangleleft c \), then \( a \rightarrow c \).
(2) If \( S \) is a \( \Gamma \)-semigroup, then either \( a \triangleleft b \) or \( b \triangleleft a \).

Proof. (1) \( a \downarrow b^i \rightarrow c \) for some \( i \in \mathbb{Z}^+ \). So \( a \rightarrow c \).

(2) Suppose \( a \rightleftharpoons b \). Then \( a^i \rightleftharpoons b \) for some \( i \in \mathbb{Z}^+ \). So for any \( k \in \mathbb{Z}^+ \), \( a^i \rightarrow b^k \). Hence \( b^k \rightarrow a^i \) and so \( b^k \rightarrow a \). Since \( k \) is arbitrary, \( b \rightleftharpoons a \).

Theorem 2.5. Let \( S \) be a \( \Gamma^* \)-semigroup with \( a, b \in S \). Let \( \langle x_i \rangle_{i=1}^n, \langle y_i \rangle_{i=1}^n \) be two minimal sequences between \( a \) and \( b \). Then \( x_i - y_i \) for \( i = 1, \ldots, n \). We can further conclude (if \( n > 1 \)) that for \( i = 1, \ldots, n - 1 \), either \( x_i - y_{i+1} \) or \( y_i - x_{i+1} \).

Proof. \( S \) is a semilattice of \( \Gamma \)-semigroups \( S_a (a \in \Omega) \). Using Lemma 2.2, we deduce that if the theorem is true for each \( S_a \), it is true for \( S \). So we can assume that \( S \) is a \( \Gamma \)-semigroup. We use Lemma 2.4 without further remark.
First we prove the theorem for $n = 1$. We have

$$\begin{array}{cc}
\bullet & x_1 \\
\circ & a \\
\circ & b \\
\bullet & y_1 \\
\end{array}$$

Now $a \sim b$ or $b \sim a$. By symmetry, we assume $a \sim b$. Since $a \not\sim b$, we conclude that $b \not\sim a$. Now either $x_1 \sim a$ or $a \sim x_1$. If $x_1 \sim a$, we have (since $b \rightarrow x_1$) that $b \not\rightarrow a$, a contradiction. So, $a \sim x_1$. Since $y_1 \rightarrow a$, we have $y_1 \rightarrow x_1$. Similarly $x_1 \rightarrow y_1$. Thus $x_1 \sim y_1$.

We now proceed by induction on $n$. We have,

$$\begin{array}{cc}
\bullet & x_1 \\
\circ & x_2 \\
\circ & \cdots \\
\bullet & x_n \\
\cdots & \\
\circ & y_n \\
\bullet & b \\
\end{array}$$

Now either $a \sim b$ or $b \sim a$. By symmetry, we assume $a \sim b$. Since $a \not\sim b$, $b \not\sim a$. Again, either $x_n \sim a$ or $a \sim x_n$. If $x_n \sim a$, we obtain (since $b \rightarrow x_n$), $b \rightarrow a$, a contradiction. So $a \sim x_n$. Now assume $a \sim x_{n+1}$. So $a \sim x_n$. Now assume $a \sim x_{n+1}$. For otherwise, $a = x_{j+1}$ and so $(x_{j+1}, x_{j+1})$ is a sequence between $a$ and $b$, contradicting the minimality of $(x_{j+1}, x_{j+1})$. Now either $x_j \sim a$ or $a \sim x_j$. If $x_j \sim a$, then since $x_{j+1} \rightarrow x_j$ we have $x_{j+1} \rightarrow a$, a contradiction. So $a \sim x_j$. Thus $a \sim x_i$ for all $i = 1, \ldots, n$. Similarly $a \sim y_i$ for all $i = 1, \ldots, n$. In particular $a \sim x_1$, $a \sim y_1$. Since $x_1 \rightarrow a$, we have $x_1 \rightarrow y_1$. Similarly, $y_1 \rightarrow x_1$. Thus $x_1 \sim y_1$. We further have $y_1 \rightarrow a \sim x_2$, and so $y_1 \rightarrow x_2$. Similarly, $x_1 \rightarrow y_2$. Now $x_1 \sim y_1$ or $y_1 \sim x_i$. We assume $x_1 \sim y_1$, the other case being taken care of similarly. So $x_2 \rightarrow x_1 \sim y_1$ and hence $x_2 \rightarrow y_1$. Since we already established $y_1 \rightarrow x_2$, we have $x_2 \sim y_1$. Thus we obtain:

$$\begin{array}{cc}
\bullet & x_1 \\
\circ & x_2 \\
\circ & \cdots \\
\bullet & x_n \\
\cdots & \\
\circ & y_n \\
\bullet & y_1 \\
\cdots & \\
\circ & y_2 \\
\bullet & b \\
\end{array}$$

In the figure on the right, the sequence $(y_i)_{i=2}^n$ is a minimal sequence between $y_1$ and $b$. This is because a sequence between $y_1$ and $b$ of length less than $n - 1$ would produce a sequence between $a$ and $b$ of length less than $n$, contradicting the minimality of $(y_i)_{i=2}^n$. By Lemma 2.1, $(y_i)_{i=2}^n$ is a minimal sequence between $y_1$ and $b$. By our induction hypothesis, we have $x_i \sim y_i$ for $i \geq 2$. Also if $n > 2$, $x_i \sim y_{i+1}$, or $y_i \sim x_{i+1}$, $i = 2, \ldots, n - 1$. Since we already know $x_1 \sim y_1$, $y_1 \sim x_2$, the theorem is proved.

We will see later that Theorem 2.5 is not true for arbitrary semigroups, even for $n = 1$.

**Problem 2.6.** In Theorem 2.5 can we conclude that $x_i \sim y_{i+1}$ and $y_i \sim x_{i+1}$, $i = 1, \ldots, n - 1$?
Problem 2.7. Call a sequence \( <x_i> \) between \( a \) and \( b \) indecomposable if \( <x_i> \) is nonempty and no proper subsequence of \( <x_i> \) is a sequence between \( a \) and \( b \). Clearly a minimal sequence is indecomposable but not conversely. An indecomposable sequence of length 1 is minimal. Is Theorem 2.5 true for indecomposable sequences of the same length? For \( n \leq 2 \), the proof goes through.

Lemma 2.8. Let \( S \) be a \( \Gamma \)-semigroup and \( a, b \in S \). Let \( <x_i>_{i=1}^n \) be a minimal sequence from \( a \) to \( b \). Then \( <x_i>_{i=1}^n \) is a minimal sequence between \( a \) and \( b \).

Proof. We have \( a \rightarrow x_1 \cdots x_n \rightarrow b, n \geq 1 \). Set \( x_0 = a, x_{n+1} = b \). For \( n \geq i \geq 1, x_i \rightarrow x_{i+1} \) implies \( x_{i-1} \rightarrow x_{i+1} \) (since \( x_{i-1} \rightarrow x_i \)) contradicting the minimality of \( <x_i>_{i=1}^n \). So \( x_{i+1} \rightarrow x_i \). Thus \( x_i \rightarrow x_{i+1}, n \geq i \geq 1 \). Now \( a \leftrightarrow x_2 \) by minimality of \( <x_i>_{i=1}^n \). So \( x_2 \rightarrow a \). Since \( x_1 \rightarrow x_2 \), we have \( x_1 \rightarrow a \). So \( a \rightarrow x_1 \). Consequently, \( <x_i>_{i=1}^n \) is a sequence between \( a \) and \( b \). Since any sequence between \( a \) and \( b \) is a sequence from \( a \) to \( b \), we have that \( <x_i> \) is a minimal sequence between \( a \) and \( b \).

Corollary 2.9. Let \( S \) be a \( \Gamma \)-semigroup with \( a, b, c \in S \). Let \( <x_i>_{i=1}^n \) and \( <y_i>_{i=1}^n \) be minimal sequences of the same length from \( b \) to \( a \) and \( c \) to \( a \) respectively. Then \( x_i \rightarrow y_i \) for \( i = 1, \ldots, n \). For \( n > 1 \), we can further conclude that for each \( i = 1, \ldots, n - 1 \) either \( x_i \rightarrow y_{i+1} \) or \( y_i \rightarrow x_{i+1} \).

Proof. Now either \( x_1 \rightarrow y_1 \) or \( y_1 \rightarrow x_1 \). By symmetry, we assume \( x_1 \rightarrow y_1 \). Since \( b \rightarrow x_1 \), we have \( b \rightarrow y_1 \). Thus \( <y_i>_{i=1}^n \) is a sequence from \( b \) to \( a \). Since \( <x_i>_{i=1}^n \) is minimal, we obtain that \( <y_i>_{i=1}^n \) is also a minimal sequence from \( b \) to \( a \). By Lemma 2.8, \( <x_i> \) and \( <y_i> \) are minimal sequences between \( b \) and \( a \). By Theorem 2.5, we are done.

Problem 2.10. Is Corollary 2.9 true for \( \Gamma^* \)-semigroups?

3. Rank and semirank.

Definition. Let \( S \) be a semigroup.

1. The rank \( p_1(S) \) of a semigroup \( S \) is zero if there is no minimal sequence between any two points. Otherwise \( p_1(S) \) is the supremum of the lengths of the minimal sequences between points in \( S \).

2. The semirank \( p_2(S) \) of \( S \) is zero if there is no minimal sequence from a point to another in \( S \). Otherwise \( p_2(S) \) is the supremum of the lengths of the minimal sequences from one point to another in \( S \).

The following is an easy consequence of Lemma 2.1.

Lemma 3.1. Let \( S \) be an \( \mathcal{S} \)-indecomposable semigroup, and \( a, b \in S \). Then there exists a sequence between \( a \) and \( b \) of length at most \( p_1(S) \). Also there exists a sequence from \( a \) to \( b \) of length at most \( p_2(S) \).

Lemma 3.2. Let \( S \) be the semilattice of \( \mathcal{S} \)-indecomposable semigroups \( S_a (\alpha \in \Omega) \). Then

\[
(1) \ p_1(S) = \sup_{\alpha \in \Omega} p_1(S_a),
\]

\[
(2) \ p_2(S) = \sup_{\alpha \in \Omega} p_2(S_a).
\]

(*) If \( p_1(S) = \infty \), the length is less than \( p_1(S) \). Similarly for \( p_2(S) \).
Proof. (1) Immediate from Lemma 2.2.

(2) By Lemma 2.2, we have \( \rho_1(S_\alpha) \leq \rho_2(S) \) for all \( \alpha \in \Omega \). So \( \sup_{\alpha \in \Omega} \rho_2(S_\alpha) \leq \rho_2(S) \). Now let \( a, b \in S, \langle x_i \rangle_{i=1}^n \) a minimal sequence from \( a \) to \( b \). We have to show that \( n \leq \sup_{\alpha \in \Omega} \rho_2(S_\alpha) \). Let \( a \in S, b \in S_\beta \). By Lemma 2.2, \( \gamma \geq \beta \). So \( ab \in S_\beta \). By Lemma 3.1, there exists a sequence \( \langle y_i \rangle_{i=1}^k \) from \( ab \) to \( b \), \( k \leq \rho_2(S_\beta) \). Since \( a | ab \), by Lemma 1.5, \( \langle y_i \rangle_{i=1}^k \) is a sequence from \( a \) to \( b \). By minimality of \( \langle y_i \rangle_{i=1}^k \) we have \( n \leq k \leq \rho_2(S_\beta) \) \( \leq \sup_{\alpha \in \Omega} \rho_2(S_\alpha) \). Thus \( \rho_2(S) \leq \sup_{\alpha \in \Omega} \rho_2(S_\alpha) \). Combined with the previous result, \( \rho_2(S) = \sup_{\alpha \in \Omega} \rho_2(S_\alpha) \).

A semigroup \( S \) is archimedean if and only if for all \( a, b \in S, a \rightharpoonup b \) (see [3], [7] and [8]).

Theorem 3.3. Let \( S \) be a semigroup.

(1) \( \rho_1(S) \) is the smallest \( n \leq \infty \) for which \( \rightharpoonup^n \) is transitive (i.e., \( \rightharpoonup^n = \rightharpoonup^\infty \) or equivalently \( \rightharpoonup^n = \rightharpoonup^{n+1} \)).

(2) \( \rho_2(S) \) is the smallest \( n \leq \infty \) for which \( \rightharpoonup^n \) is transitive (i.e., \( \rightharpoonup^n = \rightharpoonup^{n+1} \)).

(3) Let \( S \) be a semilattice of semigroups \( S_\alpha (\alpha \in \Omega) \). Then \( \rho_i(S) = \sup_{\alpha \in \Omega} \rho_i(S_\alpha), i = 1, 2 \).

(4) \( \rho_2(S) \leq \rho_1(S) \).

(5) \( \rho_i(S) = 0 \) if and only if \( \rho_2(S) = 0 \) if and only if \( S \) is a semilattice of archimedean semigroups.

(6) If \( S \) is a \( \Gamma^\ast \)-semigroup, \( \rho_1(S) = \rho_2(S) \).

(7) A finite semigroup has finite semirank and rank.

Proof. (1) and (2) are easy consequences of the definition.

(3) For each \( \alpha \in \Omega, S_\alpha \) is the semilattice of the \( \mathcal{S} \)-indecomposable components of \( S \), contained in \( S_\alpha \). So the \( \mathcal{S} \)-indecomposable components of \( S \) are just those of all the \( S_\alpha \)'s. Now the result follows from Lemma 3.2.

(4) By (3), we can assume \( S \) is \( \mathcal{S} \)-indecomposable. Let \( a, b \in S \) and \( \langle x_i \rangle_{i=1}^n \) a minimal sequence from \( a \) to \( b \). By Lemma 3.1, there exists a sequence \( \langle y_i \rangle_{i=1}^m \) between \( a \) and \( b \), such that \( m \leq \rho_1(S) \). But \( \langle y_i \rangle \) can be considered a sequence from \( a \) to \( b \). By the minimality of \( \langle x_i \rangle \), \( n \leq m \leq \rho_1(S) \). So \( \rho_2(S) \leq \rho_1(S) \).

(5) Again we can assume \( S \) is \( \mathcal{S} \)-indecomposable. Clearly if \( S \) is archimedean, it has no minimal sequences and so \( \rho_1(S) = \rho_2(S) = 0 \). If for \( i = 1 \) or 2, \( \rho_i(S) = 0 \), then by Lemma 3.1, \( S \) is archimedean.

(6) By Lemma 2.3 and by (3) and (4) above, we can assume that \( S \) is an \( \mathcal{S} \)-indecomposable \( \Gamma \)-semigroup and that \( \rho_2(S) \leq \rho_1(S) \). We have to show \( \rho_1(S) \leq \rho_2(S) \). Let \( a, b \in S \) and let \( \langle x_i \rangle_{i=1}^n \) be a minimal sequence between \( a \) and \( b \).

Then \( a \rightharpoonup b \). So either \( a \rightharpoonup b \) or \( b \rightharpoonup a \). By symmetry we assume \( a \rightharpoonup b \). By Lemma 2.1, there exists a minimal sequence \( \langle y_i \rangle_{i=1}^m \) from \( a \) to \( b \). So \( m \leq \rho_2(S) \).

By Lemma 2.8, \( \langle y_i \rangle_{i=1}^m \) is a minimal sequence between \( a \) and \( b \). Thus \( n = m \leq \rho_2(S) \). So \( \rho_1(S) \leq \rho_2(S) \), whence \( \rho_1(S) = \rho_2(S) \).

(7) Obvious.
Lemma 3.4. Let $S$ be an $\mathcal{S}$-indecomposable semigroup and $T$ a homomorphic image of $S$. Then $\rho_i(T) \leq \rho_i(S)$, $i = 1, 2$.

Proof. Let $a, b \in S$, $\varphi: S \to T$ an onto homomorphism. Let there be a minimal sequence $\langle y_i \rangle_{i=1}^{m}$ in $T$ between $\varphi(a)$ and $\varphi(b)$. So $\varphi(a) \neq \varphi(b)$ and so $a \neq b$. By Lemma 2.1, there exists a minimal sequence $\langle x_i \rangle_{i=1}^{n}$ between $a$ and $b$. Thus $m \leq \rho_1(S)$, $\langle \varphi(x_i) \rangle_{i=1}^{m}$ is a sequence between $\varphi(a)$ and $\varphi(b)$. By minimality of $\langle y_i \rangle_{i=1}^{m}$, we have $n \leq m \leq \rho_1(S)$. Thus $\rho_1(T) \leq \rho_1(S)$. A similar argument shows that $\rho_2(T) \leq \rho_2(S)$.

Problem 3.5. Is Lemma 3.4 true for arbitrary semigroups?

Theorem 3.6. Let $S$ be an $\mathcal{S}$-indecomposable semigroup with an ideal $I$. Then

$$\rho_2(S/I) \leq \rho_2(S) \leq \rho_2(I) + \rho_2(S/I).$$

Proof. That $\rho_2(S/I) \leq \rho_2(S)$ follows from Lemma 3.4. By [5] (or [10]), both $I$ and $S/I$ are $\mathcal{S}$-indecomposable. Let $a, b \in S$. We have to show the existence of a sequence from $a$ to $b$ of length at most $\rho_2(I) + \rho_2(S/I)$.

Case 1. $a \in S, b \in I$. Then $ab \in I$ and $S/I$ is nilpotent in $S/I$. Then $y^k \in I$ for some $k \in \mathbb{Z}^+$. Then by Lemma 3.1, there exists a sequence $\langle y_i \rangle_{i=1}^{m}$ from $a$ to $b$ in $I$ such that $m \leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle_{i=1}^{m}$ is a sequence from $a$ to $b$.

Case 2. $a \in I, b \in S/I$. Then $b^k \in I$ for some $k \in \mathbb{Z}^+$. Then by Lemma 3.1, there exists a sequence $\langle y_i \rangle_{i=1}^{m}$ from $a$ to $b^k$, $m \leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle_{i=1}^{m}$ is a sequence from $a$ to $b$.

Case 3. $a \in I, b \in S/I$. Then $b^k \in I$ for some $k \in \mathbb{Z}^+$. Then $b^k \in I$ for some $k \in \mathbb{Z}^+$. Then by Lemma 2.1, there exists a minimal sequence $\langle x_i \rangle_{i=1}^{n}$ from $0$ to $b$. So $n \leq \rho_2(S/I)$. So in $S/I$, $0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n \rightarrow b$. By Lemma 1.6, $0 \rightarrow y_j \rightarrow \cdots \rightarrow y_n \rightarrow b$. By Lemma 1.6, $\langle y_i \rangle_{i=1}^{m}$ is a sequence from $a$ to $b$.

Case 4. $a \in S/I, b \in S/I$. If $b$ is nilpotent in $S/I$, then $b^k \in I$ for some $k \in \mathbb{Z}^+$. By Case 1, there exists in $S$ a sequence $\langle y_i \rangle_{i=1}^{m}$ from $a$ to $b^k$ of length $\leq \rho_2(I)$. By Lemma 1.5, $\langle y_i \rangle_{i=1}^{m}$ is a sequence from $a$ to $b$ in $S$.

Thus we may assume $b$ is not nilpotent in $S/I$. So if $a \rightarrow b$ in $S/I$, then by Lemma 1.6, $a \rightarrow b$ in $S$ and we would be done. So we assume $a \leftrightarrow b$ in $S/I$. By Lemma 2.1, there exists a minimal sequence $\langle x_i \rangle_{i=1}^{n}$ in $S/I$ from $a$ to $b$. So $n \leq \rho_2(S/I)$. If none of the $x_i$'s is nilpotent, then $\langle x_i \rangle_{i=1}^{n}$ is a sequence from $a$ to $b$ in $S$, by Lemma 1.6. So let some $x_j$ be nilpotent in $S/I$. Then $a \rightarrow x_j \rightarrow \cdots \rightarrow x_n \rightarrow b$. By Lemma 1.6, $\langle y_i \rangle_{i=1}^{m}$ is a sequence from $a$ to $b$.

(?) The theorem is trivial if $\rho_2(I) = \infty$ or $\rho_2(S/I) = \infty$. So we assume $\rho_2(I) < \infty$ and $\rho_2(S/I) < \infty$. 

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\[ \cdots \rightarrow x_n \rightarrow b \text{ in } S/I. \] By minimality of \( \langle x_i \rangle_{i=1}^{n}, j = 1. \) Thus \( x_1 \) is nilpotent and \( x_j \) is not nilpotent for \( j > 1. \) Thus by Lemma 1.6, \( x_1 \rightarrow \cdots \rightarrow x_n \rightarrow b \) in \( S. \) From what we proved above there exists a sequence \( \langle y_i \rangle_{i=1}^{m} \) in \( S \) from \( a \) to \( x_1 \) such that \( m \leq \rho_2(I). \) Thus in \( S, \)

\[ a \rightarrow y_1 \rightarrow \cdots \rightarrow y_m \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \rightarrow b, \quad m + n \leq \rho_2(I) + \rho_2(S/I). \]

**Problem 3.7.** Is Theorem 3.6 true for arbitrary semigroups? In Theorem 3.6, can we replace \( \rho_2 \) by \( \rho_1? \)

Consider the following condition on semigroups:

\[ a \in S \text{ implies there exists a fixed } n = n(a) \in Z^+ \text{ such that for all } i \in Z^+, \ a^n \mid a^i. \]

Clearly any semigroup with a power of each element lying in a subgroup (in particular a periodic semigroup) satisfies (A).

**Lemma 3.8.** Let \( S \) be a semigroup satisfying (A). Let \( a, b \in S, \ k \in Z^+, \) such that \( b \rightarrow a^k. \) Then \( b \mid a^k. \)

**Proof.** For some \( i \in Z^+, b \mid a^i \mid a^{(i+2)m} \mid a^k. \) So \( b \mid a^{i+2m}. \)

**Theorem 3.9.** Let \( S \) be a semigroup satisfying (A). Suppose \( \rho_2(S) < 1. \) Then \( \rho_1(S) < 4. \)

**Proof.** Let \( \delta \) be the finest semilattice congruence on \( S \) and \( S_\alpha (\alpha \in S/\delta) \) the \( \mathcal{S} \)-indecomposable components of \( S. \) Let \( a \in S_\alpha. \) By (A), there exists \( n = n(a) \in Z^+ \) such that for all \( i \in Z^+, \ a^{(i+2)m} \mid a^n \) in \( S. \) So there exists \( x, y \in S^1 \) such that \( xa^n \mid a^y = a^n. \) But then \( xa^n = x(axa^{(i+2)m})^y \delta \) \( xa^{(i+2)m}y = a^n \delta. \) So \( xa^n \in S_\alpha. \) Similarly \( a^n \in S_\alpha. \) Thus \( a^n \mid a^i \) in \( S_\alpha. \) Consequently each \( S_\alpha \) satisfies (A).

By Theorem 3.3, it suffices to prove the theorem for each \( S_\alpha. \) Consequently we may and do assume that \( S \) is an \( \mathcal{S} \)-indecomposable semigroup. Let \( a, b \in S. \) We have to show the existence of a sequence between \( a \) and \( b \) of length at most 4. We use Lemma 3.1 and Lemma 3.8 without further remark. Let \( n_1 = n(a), \)

\[ n_1 = n(b). \] There exists \( c \in S \) such that \( a^n \rightarrow c \rightarrow b^n. \) Set \( n_3 = n(c). \) So \( a^n \mid c^n. \) There exists \( d_1 \in S \) such that \( c^n \rightarrow d_1 \rightarrow a^n. \) Set \( m_1 = n(d_1). \) So \( d_1 \mid a^{m_1} \mid c^n \mid d_{1m}. \) So \( d_1 \mid c^n \) and \( a^{m_1} \mid d_{1m}. \) Thus \( c^n - d_1 - a^n. \) By Lemma 1.5, \( c - d_1 - a. \) Now since \( c \rightarrow b^n, \ c \mid b^n. \) There exists \( d_2 \in S \) such that \( b^{m_2} \rightarrow d_2 \rightarrow c. \) Let \( m_2 = n(d_2). \) Then \( c \mid b^{m_2} \mid d_{2m}. \) Thus \( c \mid d_{2m} \) and hence, \( b^{m_2} \rightarrow d_2 \rightarrow c. \) There exists \( d_3 \in S \) such that \( d_{2m} \rightarrow d_3 \rightarrow b^n. \) Let \( m_3 = n(d_3). \) So \( d_3 \mid b^{m_3} \mid d_{2m} \mid d_{3m}. \) Hence \( d_3 \mid d_{2m} \) and \( b^{m_3} \mid d_{3m}. \) Thus \( d_{2m} \rightarrow d_3 \rightarrow b^n. \) By Lemma 1.5, \( d_2 - d_3 - b. \) Hence \( c - d_3 - d_2 - b. \) Consequently \( a - d_1 - c - d_2 - d_3 - b, \) and the theorem is proved.

**Problem 3.10.** Can the bound on \( \rho_1 \) be improved in Theorem 3.9? Does a semigroup of finite semirank necessarily have finite rank?
4. Examples. It can be deduced from Corollary 1.2 or from Tamura’s corollary that a 0-simple semigroup is \( \omega \)-indecomposable if and only if it has a nonzero nilpotent element. Also notice that for an \( \omega \)-indecomposable semigroup \( S \) with zero, \( \rho_1(S) = 0 \) if and only if \( \rho_1(S) = 0 \) if and only if \( S \) is nil.

**Example 4.1.** Let \( S \) be a 0-simple semigroup with a nonzero nilpotent element \( b \). Then \( 0 \rightarrow b \). If \( x \) is a nonnilpotent element in \( S \), then \( 0 \rightarrow b \rightarrow x \). These are minimal sequences. So \( \rho_1(S) = \rho_2(S) = 1 \).

**Example 4.2.** Let \( S_1, S_2 \) be two 0-simple semigroups with \( b_1 \in S_1, b_2 \in S_2 \) being nonzero nilpotent elements of \( S_1 \) and \( S_2 \) respectively. Identify the zeros of \( S_1 \) and \( S_2 \). Let \( S = S_1 \cup S_2 \) and \( S_1 \cap S_2 = S_2 S_1 = \{0\} \). Let \( x \in S_1 \), \( y \in S_2 \) be nonnilpotent. Then \( x \rightarrow b_1 \rightarrow b_2 \rightarrow y \) is a minimal sequence between \( x \) and \( y \). So \( \rho_1(S) = 2 \). But \( x \rightarrow b_2 \rightarrow y, y \rightarrow b_1 \rightarrow x \), whence \( \rho_2(S) = 1 \). Thus the rank of a semigroup can be strictly larger than the semirank.

**Example 4.3.** Let \( S, S_1, S_2 \) be as in Example 4.2. This time choose \( b_1 \in S_1, b_2 \in S_2 \) such that \( b_1, b_2 \neq 0, b_1^2 = b_2^2 = 0 \). Let \( T = S \cup \{u\}, u \notin S \). Define \( u^2 = 0, xu = xb_1, ux = b_1 x, uy = b_2 y, yu = yb_2 \) where \( x \in S_1, y \in S_2 \). It can be seen that \( T \) is a semigroup with ideal \( S \). If \( s_1 \in S_1, s_2 \in S_2 \), then \( u \rightarrow b_1 \rightarrow s_1, u \rightarrow b_2 \rightarrow s_2, s_1 \rightarrow u^2 = 0, s_2 \rightarrow u^2 = 0 \). Hence \( s_1 \rightarrow u \rightarrow s_2 \). Thus \( \rho_1(T) = 1 \). But \( \rho_1(S) = 2 \) by Example 4.2. Thus the rank of an ideal of a semigroup can be greater than that of the semigroup. Can a similar thing happen with semirank? Can it happen for \( \Gamma \)-semigroups? Can the rank of a semigroup be less than that of an ideal by an arbitrary number?

**Example 4.4.** We are now going to construct \( \omega \)-indecomposable \( \Gamma \)-semigroups of every rank (and hence semirank). A group is an example of an \( \omega \)-indecomposable, \( \Gamma \)-semigroup of rank and semirank zero. Now let \( \langle S_i \rangle_{i \in \mathbb{Z}^+} \) be a sequence of 0-simple semigroups. Assume \( S_i \) has zero \( 0_i \), a nonzero nilpotent element \( b_i \), \( b_i^2 = 0 \), and a nonzero idempotent \( e_i \). (For instance \( S_i \) could be a completely 0-simple semigroup which is not a Clifford semigroup.) Now identify \( e_i \) and \( 0_i+1 \). Thus \( S_i \cap S_{i+1} = \{e_i\} = \{0_i+1\} \). Let \( S = \bigcup_{i \in \mathbb{Z}^+} S_i \). Set \( I_i = \bigcup_{j \leq i} S_j \). We define multiplication on \( S \) by defining multiplication on each \( I_i \). \( I_i = S_i \) is a semigroup. Assume multiplication has been defined on \( I_i \). Let \( x \in I_i \) and \( y \in S_{i+1} \). Then define \( xy = xe_i, yx = e_i x \). Then it can be seen that the multiplication is consistent with the previous (i.e., when \( x \) or \( y = e_i = 0_i+1 \)), and also that now \( I_{i+1} \) is a semigroup. Consequently, we obtain a semigroup \( S \). Let \( 0_i = 0 \). Then \( 0 \) is the zero of \( S \), and each \( I_i \) is an ideal of \( S \). Now let \( x_i \) be a nonnilpotent element of \( S_i \). For \( k \in \mathbb{Z}^+, b_k = e_k b_{k+1}^2 \) and so \( b_k = b_{k+1}^2 \). Thus \( 0 = b_0 \rightarrow b_2 \rightarrow \cdots \rightarrow b_i \rightarrow x_i \) is a minimal sequence between \( 0 \) and \( x_i \) of length \( i \) in both \( I_i \) and \( S \). It now follows easily that \( \rho_1(S) = \infty, \rho_1(I_i) = i \). Since \( S, I_i \) are \( \Gamma \)-semigroups, \( \rho_2(S) = \infty \) and \( \rho_2(I_i) = i \). \( S \) and each \( I_i \) are \( \omega \)-indecomposable since there is a sequence between any two points. It is also clear that if we choose \( S_i \)'s finite, we obtain finite semigroups of every finite semirank and rank.

**Example 4.5.** Now we are going to show that Theorem 2.5 need not be true for arbitrary (even finite) semigroups. Let \( S, S_1, S_2 \) be as in Example 4.2. Now we
further assume that there exists a nonzero idempotent $e_i$ in $S_i$ a nonzero nilpotent element $b_i \in S_i$ such that $b_i^2 = e_i b_i = b_i e_i = 0$, $i = 1, 2$. (For instance $S_i$ could be the Rees-matrix semigroup over the trivial group $\{1\}$ with the $3 \times 3$ identity sandwich matrix and

$$e_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } b_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Let $T = S \cup \{u_1, u_2, u_3, u_4\} , u_k \not\in S, k = 1, 2, 3, 4, u_i \neq u_j$ for $i \neq j$. We have

$$\begin{array}{cccc}
\hat{e}_1 & u_1 & u_2 & u_3 \\
\hat{b}_1 & u_4 & & \\
S_1 \setminus \{0\} & & & S_2 \setminus \{0\} \\
\end{array}$$

We complete the multiplication table as follows:

1. $u_i u_j = 0$ for $i \neq j$.
2. $u_1^2 = e_1$.
3. $u_2^2 = 0$.
4. $u_3^2 = e_2$.
5. $u_4^2 = 0$.

$$\begin{array}{cccc}
x \in S_1 & y \in S_2 \\
u_1 x = e_1 x & u_1 y = b_2 y \\
x u_1 = x e_1 & y u_1 = y b_2 \\
u_2 x = b_1 x & u_2 y = 0 \\
x u_2 = x b_1 & y u_2 = 0 \\
u_3 x = b_1 x & u_3 y = e_2 y \\
x u_3 = x b_1 & y u_3 = y e_2 \\
u_4 x = 0 & u_4 y = b_2 y \\
x u_4 = 0 & y u_4 = y b_2 \\
\end{array}$$

The multiplication intersects when $x = y = 0$ but then the values are identically equal to 0. It can be shown with some effort that $T$ is a semigroup with zero 0. Furthermore

(i) $u_1$ divides every element of $S$.
(ii) $u_2$ divides every element of $S_1$ but no element of $S_2 \setminus \{0\}$.
(iii) $u_3$ divides every element of $S$.
(iv) $u_4$ divides every element of $S_2$ but no element of $S_1 \setminus \{0\}$. Thus,
But,

\[ u_2 \mid e_1 = u_1^2, \quad u_4 \mid 0 = u_2^2, \quad \text{whence } u_1 - u_2. \]

\[ u_3 \mid e_1 = u_3^2, \quad u_1 \mid e_2 = u_3^2, \quad \text{whence } u_1 - u_3. \]

\[ u_4 \mid e_2 = u_4^2, \quad u_3 \mid 0 = u_4^2, \quad \text{whence } u_3 - u_4. \]

\[ u_4 \mid 0 = u_2^2, \quad u_2 \mid 0 = u_4^2, \quad \text{whence } u_2 - u_4. \]

Thus,

\[ u_4 \mid e_1 = u_1^2 \quad \text{and so } \quad u_1 \not\sim u_4. \]

\[ u_2 \mid e_2 = u_3^2 \quad \text{and so } \quad u_2 \not\sim u_3. \]

So Theorem 2.5 is not true for \( T \). Note that \( u_3 \mid 0 = u_2 \) and so \( u_3 \not\rightarrow u_2 \). A slightly more complicated example can be given where

\[ u_2 \not\rightarrow u_4, \quad u_3 \not\rightarrow u_4, \quad u_3 \not\rightarrow u_2. \]

**Example 4.6.** Let \( X \) be a finite set \( |X| > 2 \), \( T_X \) the full transformation semigroup on \( X \). Let \( \mathcal{V}_X = T_X \ast S_X \) where \( S_X \) is the group of permutations on \( X \). Then \( \mathcal{V}_X \) is a prime ideal in \( T_X \). \( \mathcal{V}_X \) is \( S \)-indecomposable. Moreover, there exists a fixed \( a_0 \in \mathcal{V}_X \) such that for all \( c \in \mathcal{V}_X, a_0 - c \). So \( \rho_1(\mathcal{V}_X) = \rho_2(\mathcal{V}_X) = \rho_1(T_X) = \rho_2(T_X) = 1 \). Since \( T_X \), \( \mathcal{V}_X \) are \( \Gamma \)-semigroups, Theorem 2.5 is true for these semigroups. As a side remark we mention that \( \mathcal{V}_X \) cannot even be decomposed into disjoint union of proper subsemigroups.

**Example 4.7.** Let \( X \) be an infinite set and \( T_X \) the full transformation semigroup on \( X \). Then there exists a fixed \( a_0 \in T_X \) such that for all \( c \in T_X, a_0 - c \). Furthermore \( T_X \) is a \( \Gamma \)-semigroup. So \( T_X \) is an \( S \)-indecomposable semigroup of rank and semirank 1. Furthermore Theorem 2.5 holds for \( T_X \). Can \( T_X \) be decomposed into a disjoint union of proper subsemigroups?

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