A MULTIPLIER THEOREM FOR FOURIER TRANSFORMS

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ABSTRACT. A function $f$ analytic in the upper half-plane $\Pi^+$ is said to be of class $E_p(\Pi^+)$ $(0 < p < \infty)$ if there exists a constant $C$ such that $\int_0^{\infty} |f(x + iy)|^p dx \leq C < \infty$ for all $y > 0$. These classes are an extension of the $H_p$ spaces of the unit disc $U$. For $f$ belonging to $E_p(\Pi^+)$ $(0 < p \leq 2)$, there exists a Fourier transform $\hat{f}$ with the property that $f(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{f}(t)e^{itz} dt$. This makes it possible to give a definition for the multiplication of $E_p(\Pi^+)$ $(0 < p \leq 2)$ into $L_q(0, \infty)$ that is analogous to the multiplication of $H_p(U)$ into $L_q$. In this paper, we consider the case $0 < p < 1$ and $p < q$ and derive a necessary and sufficient condition for multiplying $E_p(\Pi^+)$ into $L_q(0, \infty)$.

1. Introduction. A function $f$ analytic in the unit disc $U$ is said to be of class $H_p(U)$ if there exists a constant $C$ such that $\int_0^r \|f\|^p r^{p-1} dr \leq C < \infty$ for all $r < 1$. For these classes there exists a rich and varied theory which is described in Duren's book [2]. Among the concepts studied is that of multipliers from $H_p(U)$ to $L_q$.

Definition 1. A sequence $(\lambda_n)$ is said to multiply $H_p(U)$ into $L_q$ $(0 < q < \infty)$, if for each $f(z) = \sum a_n z^n$ belonging to $H_p(U)$, $\sum |a_n|^q|\lambda_n|^q < \infty$.

Duren and Shields have shown that a necessary and sufficient condition for $(\lambda_n)$ to multiply $H_p(U)$ $(0 < p < 1)$ into $L_q$ $(p < q < \infty)$ is that

$$\sum_{n=1}^{\infty} n^{q/p}/|\lambda_n|^q = O(N^q)$$

[2], [3].

It is our aim in this paper to consider classes of functions analytic in the upper half-plane $\Pi^+$, which are analogous to the classes $H_p(U)$, and to prove a result similar to that of Duren and Shields.

2. The main result.

Definition 2. A function $f$ analytic in $\Pi^+$ is said to be of class $E_p(\Pi^+)$ $(0 < p < \infty)$ if there exists a constant $C$ such that

$$M_p(y, f) = \left\{ \int_{-\infty}^{\infty} |f(x + iy)|^p dx \right\}^{1/p} \leq C < \infty$$

for all $0 < y < \infty$.

The expression $M_p(y, f)$ is called a $p$th mean of $f$. Also the expression $M_N(y, f) = \sup_{-\infty < x < \infty} |f(x + iy)|$ is a $p$th mean of $f$ and, if $M_N(y, f)$ is bounded, $f$ is said to belong to $E_p(\Pi^+)$. 

Received by the editors July 18, 1972 and, in revised form, December 14, 1972.

AMS (MOS) subject classifications (1970). Primary 30A78; Secondary 42A68.

Key words and phrases. $H_p$ spaces, multipliers.

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Definition 3. If \( f \) belongs to \( E_p(\Pi^+) \) \((0 < p < 1)\), then the Fourier transform of \( f \) is
\[
\hat{f}(t) = \int_{-\infty}^{\infty} f(x + iy)e^{-it(x+iy)}dx \quad [6].
\]
A proof of the fact that \( \hat{f} \) exists and is independent of \( y \) is given in §5. In addition, the facts \( \hat{f}(t) \) is continuous, \( \hat{f}(t) = 0 \) for \( t \leq 0 \), and
\[
f(z) = (2\pi)^{-1} \int_{0}^{\infty} \hat{f}(i)e^{it}dt,
\]
are proved there.

Definition 4. Let \( \phi(t) \) be a function measurable on \((0, \infty)\). Then \( \phi(t) \) is said to multiply \( E_p(\Pi^+) \) \((0 < p < 1)\) into \( L_q(0, \infty) \) \((0 < q < \infty)\), if for each \( f(z) = \int_{0}^{\infty} f(i)e^{it}dt \) belonging to \( E_p(\Pi^+) \),
\[
\int_{0}^{\infty} |\phi(t)|^q |\hat{f}(t)|^q dt < \infty.
\]
We now state the main result.

**Theorem A.** Let \( \phi(t) \) be a function measurable on \((0, \infty)\). Then \( \phi(t) \) multiplies \( E_p(\Pi^+) \) into \( L_q(0, \infty) \) \((p < q)\) if and only if
\[
\int_{0}^{\infty} t^{q/p} |\phi(t)|^q dt \leq K X^q,
\]
where \( K \) is a positive constant.

The proof of Theorem A requires the use of two other results.

**Theorem B.** If \( 0 < p < q < \infty \), \( f \) belongs to \( E_p(\Pi^+) \), \( \alpha = 1/p - 1/q \), and \( \lambda \geq p \), then \( \int_{0}^{\infty} y^{\lambda-1} M_q^*(y, f) dy < \infty \).

The second of these results needs some introduction. If \( f \) belongs to \( E_p(\Pi^+) \) \((0 < p < \infty)\), then \( \lim_{y \to 0} f(x + iy) = f(x) \) exists a.e. and
\[
\rho(f, g) = \int_{0}^{\infty} |f(x) - g(x)|^p dx,
\]
where \( f \) and \( g \) belong to \( E_p(\Pi^+) \), is a translation invariant metric on \( E_p(\Pi^+) \). Moreover, under this metric, \( E_p(\Pi^+) \) \((0 < p < \infty)\) is a complete topological vector space. In other words, \( E_p(\Pi^+) \) \((0 < p < \infty)\) is an \( F \)-space \([1],[2],[5]\). Finally, we say that an operator \( \Lambda \) from \( E_p(\Pi^+) \) into \( L_q(0, \infty) \) is bounded if there exists a constant \( K \) such that \( \|\Lambda(f)\|_q < K\|f\|_p \), where \( \|f\|_p = \{\int_{0}^{\infty} |f(x)|^p dx\}^{1/p} \).

**Theorem C.** Let \( \phi(t) \) be a function measurable on \((0, \infty)\). If \( \phi(t) \) multiplies \( E_p(\Pi^+) \) \((0 < p < 1)\) into \( L_q(0, \infty) \) then the operator \( \Lambda(f)(t) = \phi(t)\hat{f}(t) \) is bounded.
We defer, for now, the proofs of Theorem B and Theorem C in order to give an immediate proof of Theorem A.

**Proof of Theorem A.** We begin by showing that (1) is necessary. So let us consider the function

\[ F(z) = F_p(z) = (2\pi)^{-1} \int_{0}^{\infty} t^{1/p} e^{-\pi t} e^{iz} \, dt. \]

Since the Laplace transform of \( t^{u-1} (u > 0) \) is \( \Gamma(u)/s^u \), where \( s \) is a complex number with \( \Re s > 0 \), we see that setting \( u = 1 + 1/p \) and \( s = -iz + \rho \) gives

\[ F(z) = \Gamma(1 + 1/p)(\rho - iz)^{1+1/p}. \]

From this it follows that \( F(z) \) belongs to \( E_p(\Pi^+) \) and \( \|F\|_p = M/p \). But by Theorem C there exists a constant \( K \) such that

\[ \| \Lambda(f) \|_q \leq K \| F \|_p, \]

so \( \| \hat{F}(i) \phi(i) \|_q \leq KM/p \). Thus, our next step is to find \( \hat{F}(i) \). However, \( F(x + iy) = F_p(x) \) is in \( L_1(-\infty, \infty) \) and is the Fourier transform of

\[ g(t) = (2\pi)^{-1} t^{1/p} e^{-\pi t} e^{-\pi t} \quad \text{if} \ t \geq 0, \]

\[ = 0 \quad \text{if} \ t < 0, \]

which also belongs to \( L_1(-\infty, \infty) \). Hence \( \hat{F}(i)e^{-\pi t} = \hat{F}_p(i) \) or \( \hat{F}(i) = t^{1/p} e^{-\pi t} \) if \( t \geq 0 \) and zero if \( t < 0 \). Consequently,

\[ \int_{0}^{\infty} t^{q/p} |\phi(t)|^q e^{-\pi t} \, dt \leq K^q M^q/\rho^q \]

and this implies that

\[ \int_{0}^{X} e^{-\pi t} t^{q/p} \phi(t)^q \, dt \leq K^q M^q/\rho^q \]

for \( X > 0 \). So taking \( \rho = 1/X \), we find

\[ \int_{0}^{X} t^{q/p} \phi(t)^q \, dt \leq K M^q e^{Xq}. \]

To prove that (1) is sufficient, we begin by considering the integral

\[ \int_{0}^{\infty} t^{q/p} |\phi(t)|^q e^{-\pi t} \, dt \quad (y > 0). \]

Letting \( S(t) = \int_{0}^{t} \tau^{q/p} |\phi(\tau)|^q \, d\tau \) and integrating by parts we find that when we use the estimate \( S(t) \leq K t^q \), the integral is less than or equal to \( K \int_{0}^{\infty} t^{q} e^{-\pi t} \, dt = KT(q + 1)/\rho^q \). Hence

\[ y^q \int_{0}^{\infty} t^{q/p} |\phi(t)|^q e^{-\pi t} \, dt \leq C < \infty \]
for $y > 0$. Next we note that for $\gamma = q(1/p - 1)$, Theorem B implies that for $f$ belonging to $E_p(\Pi^+)$ ($0 < p < 1$),

$$\int_0^\infty y^{\gamma-1}M_q^\gamma(y,f)\,dy < \infty.$$  
Thus for each $f$ belonging to $E_p(\Pi^+)$

$$\int_0^\infty y^{\gamma-1}M_q^\gamma(y,f)\left[\int_0^\infty t^{q/p}|\phi(t)|^q e^{-\alpha t}\,dt\right]\,dy < \infty,$$

or using Fubini’s theorem

$$\int_0^\infty \int_0^\infty t^{q/p}|\phi(t)|^q y^{\gamma-1}M_q^\gamma(y,f)e^{-\alpha t}\,dt\,dy < \infty.$$  
But from the definition of the Fourier transform for $f$, we have $|\hat{f}(t)|e^{-\alpha t} \leq M_q(y,f)$. Thus

$$\int_0^\infty |\phi(t)|^q |\hat{f}(t)|^{q+1} t^{q/p} y^{\gamma-1}e^{-(q+1)\alpha t}\,dt\,dy < \infty,$$

or

$$\frac{\Gamma(q/p)}{(q + 1)^{q/p}} \int_0^\infty |\phi(t)|^q |\hat{f}(t)|^q \,dt < \infty.$$  

Theorem A has the following interesting corollary.

**Corollary.** If $f$ belongs to $E_p(\Pi^+)$ ($0 < p < 1$), then $\int_0^\infty |\hat{f}(t)|^p t^{p-2}dt < \infty$.

This is an extension of the following results.

**Theorem (Hardy-Littlewood-Titchmarsh).** If $f$ belongs to $E_p(\Pi^+)$ ($1 < p < 2$), then $\int_0^\infty |\hat{f}(t)|^p t^{p-2}dt < \infty$ [8].

**Theorem (Hille-Tamarkin).** If $f$ belongs to $E_1(\Pi^+)$, then $\int_0^\infty |\hat{f}(t)|/t\,dt < \infty$ [4].

3. The proof of Theorem B. This proof is a consequence of several other theorems.

**Theorem 1.** Let $u(z)$ be a nonnegative subharmonic function defined on $\Pi^+$ and suppose

$$\int_{-\infty}^\infty u(x + iy)\,dx \leq C/y^\alpha \quad (y > 0),$$

where $\alpha \geq 0$. Then there exists a constant $K = K(\alpha)$ such that $u(x_0 + iy_0) \leq KC/y_0^{\alpha+1}$ for each point $z_0 = x_0 + iy_0$ ($y_0 > 0$).

**Proof.** The case $\alpha = 0$ was proved by Krylov [5]. So assume $\alpha > 0$. Then setting $y_1 = y_0/2$ and $u_{y_1}(z) = u(x + i(y + y_1))$, we find

$$\int_{-\infty}^\infty u_{y_1}(x + iy)\,dx \leq C/y_1^\alpha \quad (y > 0).$$
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Hence, by the case $\alpha = 0$, we have $u_{y_1}(x_0 + iy_2) \leq KC/y_{y_2}$ ($y_2 > 0$), and putting $y_1 = y_2 = y_0/2$,

$$u(x_0 + iy_0) \leq 2^{n+1}KC/y_0^{n+1}. \quad \square$$

**Theorem 2.** Suppose $f(z)$ is analytic in $\Pi^+$ and

$$M_p(y,f) \leq C/y^\beta \quad (0 < p < \infty, \beta \geq 0).$$

Then there exists a constant $K = K(\beta,p,q)$ such that

$$M_q(y,f) \leq KC/y^{\beta+1/p-1/q} \quad (p < q \leq \infty).$$

**Proof.** It suffices to consider the case $q = \infty$. For suppose (2) has been proven for $q = \infty$ and $K \geq 1$ (which we may assume without loss of generality). Then

$$M_q(y,f) = \left\{ \int_0^\infty |f(x + iy)|^p |f(x + iy)|^{q-p} dx \right\}^{1/q} \leq [M_\infty(y,f)]^{\frac{q-p}{q}}[M_p(y,f)]^{p/q} \leq K^{q-p/q}C/y^\lambda,$$

where $\lambda = \beta + 1/p - 1/q$. Now to derive the theorem for $q = \infty$, let $u(z)$ be the nonnegative subharmonic function $|f(z)|^p$ and $\alpha = \beta p$. Then Theorem 1 implies

$$|f(x_0 + iy_0)|^p \leq KC/y_0^{\beta+1},$$

which is equivalent to (2). $\square$

**Theorem 3.** Suppose $f$ belongs to $E_p(\Pi^+)$. Then for $1 < p < \infty$, $-1 < b$, and $1 < a < \infty$,

$$\int_0^\infty y^b M_p^a(y,f) dy \leq C \int_0^\infty y^{a+b} M_p^a(y,f) dy,$$

where $C = C(a,b)$ is independent of $f$.

**Proof.** We begin by assuming that $f$ is analytic in the closed upper half-plane. Then integrating by parts we find

$$\int_0^\infty y^b M_p^a(y,f) dy = \frac{y_0^{b+1}}{b+1} M_p^a(y_0,f) - \frac{1}{b+1} \int_0^\infty y^{b+1} \frac{\partial}{\partial y} [M_p^a(y,f)] dy.$$

Thus our next step is to estimate $[\partial/\partial y] M_p^a(y,f)$. But

$$(\partial/\partial y) M_p^a(y,f) = (a/p) M_p^{a-p}(y,f) (\partial/\partial y) M_p^a(y,f),$$
so we need to estimate \(|(\partial/\partial y)M_p^\alpha(y,f)|\).

However,

\[
\left| \frac{\partial}{\partial y} |f(x + iy)|^p \right| = p |f(x + iy)|^{p-1} \left| \frac{\partial}{\partial y} |f(x + iy)| \right|
\]

and

\[
||f(x + iy_1) - f(x + iy_2)|| \leq \frac{|f(x + iy_1) - f(x + iy_2)|}{|y_1 - y_2|}
\]

implies

\[
|\frac{\partial}{\partial y} f(x + iy)|| \leq |f'(x + iy)|
\]

so

\[
\left| \frac{\partial}{\partial y} |f(x + iy)|^p \right| \leq p |f(x + iy)|^{p-1} |f'(x + iy)|.
\]

Thus Hölder’s inequality implies

\[
|(\partial/\partial y)M_p^\alpha(y,f)| \leq pM_p^{p-1}(y,f)M_p(y,f')
\]

and this implies

\[
|(\partial/\partial y)M_p^\alpha(y,f)| \leq aM_p^{p-1}(y,f)M_p(y,f').
\]

But now we have

\[
\left| \int_0^1 y_{b+1} \frac{\partial}{\partial y} (M_p^\alpha(y,f)) dy \right| \leq a \int_0^1 y_{b+1} M_p^{p-1}(y,f) M_p(y,f') dy
\]

\[
\leq a \left\{ \int_0^1 y^b M_p^\alpha(y,f) dy \right\}^{1-\alpha} \left\{ \int_0^1 y^{ab+b} M_p^\alpha(y,f') dy \right\}^\alpha,
\]

where we have used Hölder’s inequality again. Hence

\[
\left\{ \int_0^1 y^b M_p^\alpha(y,f) dy \right\}^{\alpha}
\]

\[
\leq \left( \frac{y_{b+1}}{b + 1} \right)^{\alpha} M_p(y_0,f) + \frac{a}{b + 1} \left\{ \int_0^1 y^{ab+b} M_p^\alpha(y,f') dy \right\}^{\alpha}.
\]

where we have used the estimate

\[
\int_0^1 y^b M_p^\alpha(y,f) dy \geq \frac{y_{b+1}}{b + 1} M_p^\alpha(y_0,f).
\]
which follows from the fact that the means $M_p(y, f)$ are nonincreasing functions of $y$ [5].

From (4), it is clear that in order to complete the proof for this case, we need only show that $y_0^{b+1} M_p^a(y_0, f)$ tends to zero as $y_0$ tends to infinity. But using Theorem 2, it is easy to see that $f(x + iy_0) = -i \int_0^\infty f'(x + iy) \, dy$ and applying Minkowski's inequality, we find

$$M_p(y_0, f) \leq \int_0^\infty M_p(y, f') \, dy.$$  

So suppose $r > 1$. Then

$$M_p(y_0, f) \leq [C(y_0)]^a \left[ \frac{1}{r - 1} \int_0^\infty y^r M_p(y, f') \frac{d(-1/y^{r-1})}{C(y_0)} \right]^a,$$

where $C(y_0) = \int_0^\infty d(-1/y^{r-1}) = 1/y_0^{-1}$, and Jensen's inequality gives

$$M_p^a(y_0, f) \leq [C(y_0)]^{a-1} \frac{1}{(r - 1)^{a-1}} \int_0^\infty y^{a-r} M_p^a(y, f') \, dy.$$  

Hence setting $r = (a + b)/(a - 1)$, we have

$$y_0^{b+1} M_p^a(y_0, f) \leq \frac{1}{((b + 1)/(a - 1))^{a-1}} \int_0^\infty y^{a+b} M_p^a(y, f') \, dy,$$

from which it follows that $y_0^{b+1} M_p^a(y_0, f)$ tends to zero as $y_0$ tends to infinity.

Finally we remove the restriction that $f$ is analytic in the closed upper half-plane. Since $f(z) = f(z + iy)$ is analytic in the closed upper half-plane, the theorem holds for $f(z)$. Thus the result for $f(z)$ follows from letting $y$ tend to zero and applying the monotone convergence theorem. □

These three theorems have prepared the way for a proof of Theorem B.

**Proof of Theorem B.** We first reduce the theorem to the case $\lambda = p = 2$. By Theorem 2

$$M_p^\lambda(y, f) \leq K^{\lambda-p} M_p^f(y, f)/y^{a(\lambda-p)},$$

so

$$\int_0^\infty y^{a \lambda - 1} M_p^\lambda(y, f) \, dy \leq K^{\lambda-p} \int_0^\infty y^{\alpha p - 1} M_p^\alpha(y, f) \, dy.$$

Hence we can assume $\lambda = p$. Next assume the theorem is true for $\lambda = p = 2$ and $f(z) \neq 0$ in $\Pi^+$ and belongs to $E_2(\Pi^+)$. Then $g(z) = |f(z)|^{p/2}$ belongs to $E_2(\Pi^+)$ and

$$\int_0^\infty y^{-p/q} M_p^\mu(y, f) \, dy = \int_0^\infty y^{-2/\mu} M_2^2(y, g) \, dy < \infty,$$
where $s = 2q/p > 2$. In case $f(z)$ has zeros in $\Pi^+$, it is possible to write it as a sum of two nonzero functions in $E_p(\Pi^+)$ [2] and still show that it suffices to take $p = 2$.

So let $f \in E_2(\Pi^+)$. Then using the Paley-Wiener theorem [7], we can write

$$f(z) = \frac{1}{2\pi} \int_0^\infty \hat{f}(t)e^{itz}dt,$$

where $\hat{f}(t)$ is the Fourier transform of the boundary function $f(x)$ of $f(z)$. Also

$$f'(z) = \frac{1}{2\pi} \int_0^\infty t\hat{f}(t)e^{itz}dt.$$

Next we assume $2 < q < \infty$. Then by Theorem 3

$$\int_0^\infty y^{-2/q} M^2_q(y, f) dy \leq C \int_0^\infty y^{-2-2/q} M^2_q(y, f') dy,$$

and by Theorem 2 $M_q(y, f') \leq Ky^{1/q-1/2} M_q(y/2, f')$, so

$$\int_0^\infty y^{-2/q} M^2_q(y, f) dy \leq CK \int_0^\infty y M^2_q(y/2, f') dy.$$

Finally, by Plancherel's theorem [7], we find

$$\int_0^\infty y^{-2/q} M^2_q(y, f) dy \leq \frac{CK}{2\pi} \int_0^\infty y \int_0^\infty |\hat{f}(t)|^2 t^2 e^{-yt^2}dt dy$$

$$= \frac{CK}{2\pi} \int_0^\infty |\hat{f}(t)|^2 t^2 \int_0^\infty ye^{-yt} dy dt$$

$$= \frac{CK}{2\pi} \int_0^\infty |\hat{f}(t)|^2 dt$$

$$= CK \int_0^\infty |f(x)|^2 dx < \infty.$$

If $q = \infty$, then the estimate

$$M^2_q(y, f) \leq KM^2_q(y/2, f)/y^{2/r}$$

for some $r > 2$ can be used to derive the desired results. □

4. The proof of Theorem C. Since $E_p(\Pi^+)$ is an $F$-space under the metric $\rho(f, g) = \int_0^\infty |f(x) - g(x)|^p dx$, we can use the closed graph theorem. Thus we need to show that $\Lambda$ is a closed operator. So let $\{f_n\}$ be a sequence which converges in $E_\infty(\Pi^+)$ to $f$ and also suppose $\Lambda(f_n)(t) = \phi(t)f_n(t)$ converges to $g(t)$ in $L_q(0, \infty)$. Then we need to show that $\Lambda(f)(t) = g(t)$ a.e.

Considering the sequence $\{f_n\}$ and $f$ first, we find by Theorem 2 that

$$\left\{\int_{-\infty}^\infty |f_n(x + iy_0) - f(x + iy_0)|^2 dx\right\}^{1/2} \leq \frac{K\|f_n - f\|_p}{y_0^{1/p-1/2}}.$$
where \( y_0 > 0 \). Thus \( f_{0,y}(z) = f_{0}(z + iy_0) \) converges to \( f_{0}(z) = f(z + iy_0) \) in \( E_2(\Sigma^+) \). Moreover, it is easy to see that the Fourier transform of \( f_{0,y}(x) \) is \( \hat{f}_n(t)e^{-iy_0t} \), while the Fourier transform of \( f_{0}(x) \) is \( f(t)e^{-iy_0t} \). Consequently, Plancherel's theorem [7] implies that \( \hat{f}_n(t)e^{-iy_0t} \) converges to \( f(t)e^{-iy_0t} \) in \( L_2(0, \infty) \). Hence, there exists a subsequence \( \{\hat{f}_n(t)\} \cap \{\{f_n(t)\} \} \) converging to \( f(t) \) a.e. But the sequence \( \{A(\Lambda(f_n))\} \) also converges to \( g(t) \) in \( L_2(0, \infty) \). Therefore, there exists a subsequence of \( \{A(\Lambda(f_n))\} \), which we also denote by \( \{A(\Lambda(f_n))\} \), converging to \( g(t) \) a.e. Thus \( \{\phi(t)\hat{f}_n(t)\} \) converges to \( \phi(t)f(t) \) a.e. and also to \( g(t) \) a.e., which implies

\[
\phi(t)f(t) = g(t) \quad \text{a.e.} \quad \square
\]

5. Fourier transform. The Fourier transform defined in §2 certainly exists since Theorem 2 implies that \( f_{0}(x) = f(x + iy) \) belongs to \( L_1(-\infty, \infty) \). In fact, if \( C \) is a constant such that \( M_y(f) \leq C \) for \( y > 0 \), then there exists a constant \( K = K(0,p,1) \) such that

\[
\int_{-\infty}^{\infty} |f(x + iy)| \, dx \leq CK/y^{1/p-1}
\]

for \( y > 0 \).

To see that \( \hat{f} \) is independent of \( y \), fix \( 0 < y_1 < y_2 < \infty \) and for each \( \alpha > 0 \) let \( \Gamma_\alpha \) be the rectangular contour with vertices \( \pm \alpha + iy_1 \) and \( \pm \alpha + iy_2 \). By Cauchy's theorem

\[
\int_{\Gamma_\alpha} f(z)e^{-iz\beta} \, dz = 0.
\]

Next let \( I = [y_1, y_2] \) and put

\[
\Phi(\beta) = i \int_{y_1}^{y_2} f(\beta + iu)e^{-i\beta u} \, du.
\]

Then \( |\Phi(\beta)| \leq e^{\alpha u} \int_{y_1}^{y_2} |f(\beta + iu)| \, du \). Now if we let

\[
\Psi(\beta) = \int_{y_1}^{y_2} |f(\beta + iu)| \, du,
\]

then Fubini's theorem and (1) imply

\[
\int_{-\infty}^{\infty} \Psi(\beta) \, d\beta = \int_{y_1}^{y_2} \int_{-\infty}^{\infty} |f(\beta + iy)| \, d\beta \, dy \leq \frac{CK}{y_2^{1/p-1}}(y_2 - y_1).
\]

Thus there exists a sequence \( \{\alpha_j\} \) such that \( \alpha_j \to \infty \) as \( j \to \infty \) and \( \Psi(\alpha_j) + \Psi(-\alpha_j) \to 0 \) as \( j \to \infty \). Hence we have

\[
\Phi(\alpha_j) \to 0 \quad \text{and} \quad \Phi(-\alpha_j) \to 0
\]

as \( j \to \infty \). Now combining (1), (2), and (3), we find

\[
\int_{-\infty}^{\infty} f(x + iy_1)e^{-i(x+iy_1)\beta} \, dx
\]

\[
= \int_{-\infty}^{\infty} f(x + iy_2)e^{-i(x+iy_2)\beta} \, dx,
\]

i.e., \( \hat{f} \) is independent of \( y \).
If we let \( f_y(z) = f(z + iy) \), then (4) becomes
\[
\hat{f}(t) = e^{\pi^2 t} \hat{f}_y(t) = e^{\pi^2 t} \hat{f}_y(t).
\]
Since \( \hat{f}_y \) is the Fourier transform of an \( L^1(-\infty, \infty) \) function, it is continuous and hence \( \hat{f} \) is continuous.

Using (1) again, we see that
\[
|\hat{f}(t)e^{-\gamma t}| = |\hat{f}_y(t)| \leq \|f_y\| \leq CKy_0^{\frac{1}{p-1}}
\]
for a fixed \( y_0 < y \). Thus if we fix \( t < 0 \) and let \( y \to \infty \), we find \( \hat{f}(t) = 0 \). Hence \( \hat{f}(t) \) is identically zero on \((0, \infty)\) and by continuity it is zero at \( t = 0 \). Also note \( \hat{f}_y(t) = 0 \) on \((-\infty, 0] \).

As we have noted, \( \hat{f}(t) = f_y(t)e^{\gamma t} \), so \( \hat{f}_y(t) = \hat{f}(t)e^{-\gamma t} = \hat{f}_y(t)e^{\gamma(y_0 - y)} \), and letting \( y_0 = y/2 \), we have
\[
\int_0^\infty |\hat{f}_y(t)| dt \leq \|f_y\| \int_0^\infty e^{\gamma(y_0 - y)} dt
\]
\[
\leq \frac{KC}{y_0^{\frac{1}{p-1}} y - y_0}
\]
\[
= \frac{2^{\frac{1}{p}} KC}{y^{\frac{1}{p}}}.
\]
Hence for \( y > 0 \), \( \hat{f}_y \) belongs to \( L^1(-\infty, \infty) \) and we can apply the inversion theorem [7], to find
\[
f(z) = f_y(x) = (2\pi)^{-1} \int_0^\infty \hat{f}_y(t)e^{itx} dt
\]
\[
= (2\pi)^{-1} \int_0^\infty \hat{f}(t)e^{-\gamma t} e^{itx} dt
\]
\[
= (2\pi)^{-1} \int_0^\infty \hat{f}(t)e^{itx} dt.
\]

REFERENCES


A MULTIPLIER THEOREM FOR FOURIER TRANSFORMS


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