A MULTIPLIER THEOREM FOR FOURIER TRANSFORMS

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ABSTRACT. A function \( f \) analytic in the upper half-plane \( \Pi^+ \) is said to be of class \( E_p(\Pi^+) \) \( (0 < p < \infty) \) if there exists a constant \( C \) such that \( \int_{-\infty}^{\infty} |f(x + iy)|^p \, dx \leq C < \infty \) for all \( y > 0 \). These classes are an extension of the \( H_p \) spaces of the unit disc \( U \). For \( f \) belonging to \( E_p(\Pi^+) \) \( (0 < p \leq 2) \), there exists a Fourier transform \( \hat{f} \) with the property that \( \hat{f}(z) = (2\pi)^{-1} \int_{\mathbb{R}} f(e^{i\theta}) e^{-iz\theta} \, d\theta \). This makes it possible to give a definition for the multiplication of \( E_p(\Pi^+) \) \( (0 < p < 2) \) into \( L_q(0, \infty) \) that is analogous to the multiplication of \( H_p(U) \) into \( L_q \). In this paper, we consider the case \( 0 < p < 1 \) and \( p < q \) and derive a necessary and sufficient condition for multiplying \( E_p(\Pi^+) \) into \( L_q(0, \infty) \).

1. Introduction. A function \( f \) analytic in the unit disc \( U \) is said to be of class \( H_p(U) \) if there exists a constant \( C \) such that \( \int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta \leq C < \infty \) for all \( r < 1 \). For these classes there exists a rich and varied theory which is described in Duren's book [2]. Among the concepts studied is that of multipliers from \( H_p(U) \) to \( L_q \).

Definition 1. A sequence \( \{\lambda_n\} \) is said to multiply \( H_p(U) \) into \( L_q \) \( (0 < q < \infty) \), if for each \( f(z) = \sum a_n z^n \) belonging to \( H_p(U) \), \( \sum |a_n|^p |\lambda_n|^q < \infty \).

Duren and Shields have shown that a necessary and sufficient condition for \( \{\lambda_n\} \) to multiply \( H_p(U) \) \( (0 < p < 1) \) into \( L_q \) \( (p \leq q < \infty) \) is that

\[
\sum_{n=1}^{N} n^{\pi/p} |\lambda_n|^q = O(N^q) \quad [2], [3].
\]

It is our aim in this paper to consider classes of functions analytic in the upper half-plane \( \Pi^+ \), which are analogous to the classes \( H_p(U) \), and to prove a result similar to that of Duren and Shields.

2. The main result.

Definition 2. A function \( f \) analytic in \( \Pi^+ \) is said to be of class \( E_p(\Pi^+) \) \( (0 < p < \infty) \) if there exists a constant \( C \) such that

\[
M_p(y,f) = \left( \int_{-\infty}^{\infty} |f(x + iy)|^p \, dx \right)^{1/p} \leq C < \infty
\]

for all \( 0 < y < \infty \).

The expression \( M_p(y,f) \) is called a \( p \)th mean of \( f \). Also the expression \( M_\infty(y,f) = \sup_{-\infty < x < \infty} |f(x + iy)| \) is a \( p \)th mean of \( f \) and, if \( M_\infty(y,f) \) is bounded, \( f \) is said to belong to \( E_\infty(\Pi^+) \).

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Definition 3. If $f$ belongs to $E_p(\Pi^+) \ (0 < p < 1)$, then the Fourier transform of $f$ is

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x + iy)e^{-i(x+iy)t}dx \quad [6].$$

A proof of the fact that $\hat{f}$ exists and is independent of $y$ is given in §5. In addition, the facts $\hat{f}(t)$ is continuous, $\hat{f}(t) = 0$ for $t \leq 0$, and

$$f(z) = (2\pi)^{-1} \int_{t=0}^{t=\infty} \hat{f}(i)e^{it}dt,$$

are proved there.

Definition 4. Let $\phi(t)$ be a function measurable on $(0, \infty)$. Then $\phi(t)$ is said to multiply $E_p(\Pi^+) \ (0 < p < 1)$ into $L_q(0, \infty) \ (0 < q < \infty)$, if for each $f(z) = \sum_{n=0}^{\infty} f_nJ_n(t)e^{int}dt$ belonging to $E_p(\Pi^+)$,

$$\int_{t=0}^{t=\infty} |\phi(t)|^q |\hat{f}(t)|^q dt < \infty.$$

We now state the main result.

**Theorem A.** Let $\phi(t)$ be a function measurable on $(0, \infty)$. Then $\phi(t)$ multiplies $E_p(\Pi^+)$ into $L_q(0, \infty) \ (p \leq q)$ if and only if

$$(1) \quad \int_{t=0}^{t=\infty} \frac{t^\alpha}{t^\lambda} |\phi(t)|^q dt \leq KX^q,$$

where $K$ is a positive constant.

The proof of Theorem A requires the use of two other results.

**Theorem B.** If $0 < p < q \leq \infty$, $f$ belongs to $E_p(\Pi^+)$, $\alpha = 1/p - 1/q$, and $\lambda \geq p$, then $\int_{t=0}^{t=\infty} y^\lambda M_q^\alpha(y,f)dy < \infty$.

The second of these results needs some introduction. If $f$ belongs to $E_p(\Pi^+) \ (0 < p < \infty)$, then $\lim_{y \to 0} f(x + iy) = f(x)$ exists a.e. and

$$\rho(f,g) = \int_{0}^{\infty} |f(x) - g(x)|^p dx,$$

where $f$ and $g$ belong to $E_p(\Pi^+)$, is a translation invariant metric on $E_p(\Pi^+)$. Moreover, under this metric, $E_p(\Pi^+) \ (0 < p < \infty)$ is a complete topological vector space. In other words, $E_p(\Pi^+) \ (0 < p < \infty)$ is an $F$-space [1], [2], [5]. Finally, we say that an operator $A$ from $E_p(\Pi^+)$ into $L_q(0, \infty)$ is bounded if there exists a constant $K$ such that $\|A(f)\|_q < K\|f\|_p$, where $\|f\|_p = \{\int_{0}^{\infty} |f(x)|^p dx\}^{1/p}$.

**Theorem C.** Let $\phi(t)$ be a function measurable on $(0, \infty)$. If $\phi(t)$ multiplies $E_p(\Pi^+) \ (0 < p < 1)$ into $L_q(0, \infty)$ then the operator $A(f)(t) = \phi(t)\hat{f}(t)$ is bounded.
We defer, for now, the proofs of Theorem B and Theorem C in order to give an immediate proof of Theorem A.

**Proof of Theorem A.** We begin by showing that (1) is necessary. So let us consider the function

\[ F(z) = F_p(z) = (2\pi)^{-1} \int_0^\infty t^{1/p} e^{-zt} e^{-t^q} dt. \]

Since the Laplace transform of \( t^{u-1} (u > 0) \) is \( \Gamma(u)/s^u \), where \( s \) is a complex number with \( \text{Re} \ s > 0 \), we see that setting \( u = 1 + 1/p \) and \( s = -iz + \rho \) gives

\[ F(z) = \Gamma(1 + 1/p)/(\rho - iz)^{1+1/p}. \]

From this it follows that \( F(z) \) belongs to \( E_q(\Pi^+) \) and \( \|F\|_p = M/p \). But by Theorem C there exists a constant \( K \) such that

\[ \|\Lambda(f)\|_q \leq K\|F\|_p, \]

so \( \|\hat{F}(i)\phi(i)\|_q \leq KM/p. \) Thus, our next step is to find \( \hat{F}(i) \). However, \( F(x + iy) = F_p(x) \) is in \( L_q(-\infty, \infty) \) and is the Fourier transform of

\[ g(t) = (2\pi)^{-1} t^{1/p} e^{-t^p} e^{-t^q} \quad \text{if } t > 0, \]

\[ = 0 \quad \text{if } t < 0, \]

which also belongs to \( L_q(-\infty, \infty) \). Hence \( \hat{F}(i) e^{-t^q} = \hat{F}_p(i) = 2\pi g(i) \) or \( \hat{F}(i) = t^{1/p} e^{-t^q} \) if \( t \geq 0 \) and zero if \( t < 0 \) [7]. Consequently,

\[ \int_0^\infty t^{q/p} |\phi(t)|^q e^{-t^q} dt \leq K^q M^q/p^q \]

and this implies that

\[ \int_0^X t^{q/p} |\phi(t)|^q dt \leq K^q M^q/p^q \]

for \( X > 0 \). So taking \( \rho = 1/X \), we find

\[ \int_0^X t^{q/p} |\phi(t)|^q dt \leq K M^q e^{qX^q}. \]

To prove that (1) is sufficient, we begin by considering the integral

\[ \int_0^\infty t^{q/p} |\phi(t)|^q e^{-t^q} dt \quad (y > 0). \]

Letting \( S(t) = \int_0^t t^{q/p} |\phi(t)|^q dt \) and integrating by parts we find that when we use the estimate \( S(t) \leq Kt^q, \) the integral is less than or equal to \( Ky \int_0^\infty t^q e^{-t^q} dt = KT(q + 1)/y^q. \) Hence

\[ y^q \int_0^\infty t^{q/p} |\phi(t)|^q e^{-t^q} dt \leq C < \infty \]
for \( y > 0 \). Next we note that for \( \gamma = q(1/p - 1) \), Theorem B implies that for \( f \) belonging to \( E_p(\Pi^+) \) \((0 < p < 1)\),

\[
\int_0^\infty y^{-1} M_\gamma^q(y,f) \, dy < \infty.
\]

Thus for each \( f \) belonging to \( E_p(\Pi^+) \)

\[
\int_0^\infty y^{-1} M_\gamma^q(y,f) \left[ y^q \int_0^\infty t^{q/p} |\varphi(t)|^q e^{-yt} \, dt \right] \, dy < \infty,
\]

or using Fubini's theorem

\[
\int_0^\infty \int_0^\infty t^{q/p} |\varphi(t)|^q y^{q-1} M_\gamma^q(y,f) e^{-yt} \, dt \, dy < \infty.
\]

But from the definition of the Fourier transform for \( f \), we have \(|\hat{f}(t)| e^{-yt} \leq M_\gamma(y,f)\). Thus

\[
\int_0^\infty |\varphi(t)|^q |\hat{f}(t)|^{q/p} t^{q/p} \int_0^\infty y^{q-1} e^{-(q+1)yt} \, dy \, dt < \infty,
\]

or

\[
\frac{\Gamma(q/p)}{(q+1)^{q/p}} \int_0^\infty |\varphi(t)|^q |\hat{f}(t)|^q \, dt < \infty. \quad \Box
\]

Theorem A has the following interesting corollary.

**Corollary.** If \( f \) belongs to \( E_p(\Pi^+) \) \((0 < p < 1)\), then \( \int_0^\infty |\hat{f}(t)|^{p} \, dt < \infty \).

This is an extension of the following results.

**Theorem (Hardy-Littlewood-Titchmarsh).** If \( f \) belongs to \( E_p(\Pi^+) \) \((1 < p < 2)\), then \( \int_0^\infty |\hat{f}(t)|^{2} \, dt < \infty \) \([8]\).  

**Theorem (Hille-Tamarkin).** If \( f \) belongs to \( E_1(\Pi^+) \), then \( \int_0^\infty |\hat{f}(t)| \, dt < \infty \) \([4]\).

3. **The proof of Theorem B.** This proof is a consequence of several other theorems.

**Theorem 1.** Let  \( u(z) \) be a nonnegative subharmonic function defined on  \( \Pi^+ \) and suppose

\[
\int_{-\infty}^{\infty} u(x + iy) \, dx \leq C/y^\alpha \quad (y > 0),
\]

where \( \alpha \geq 0 \). Then there exists a constant \( K = K(\alpha) \) such that \( u(x_0 + iy_0) \leq KC/y_0^{\alpha+1} \) for each point \( z_0 = x_0 + iy_0 \) \((y_0 > 0)\).

**Proof.** The case \( \alpha = 0 \) was proved by Krylov \([5]\). So assume \( \alpha > 0 \). Then setting \( y_1 = y_0/2 \) and \( u_{y_1}(z) = u(x + i(y + y_1)) \), we find

\[
\int_{-\infty}^{\infty} u_{y_1}(x + iy) \, dx \leq C/y_1^\alpha \quad (y > 0).
\]
Hence, by the case $\alpha = 0$, we have $u_{y_1}(x_0 + iy_2) \leq KC/y_1^\beta y_2$ ($y_2 > 0$), and putting $y_1 = y_2 = y_0/2$,

$$u(x_0 + iy_0) \leq 2^{a+1} KC/y_0^{q+1}. \quad \square$$

**Theorem 2.** Suppose $f(z)$ is analytic in $\Pi^+$ and

(1) $M_p(y, f) \leq C/y^\beta$ \hspace{1em} ($0 < p < \infty, \beta \geq 0$).

Then there exists a constant $K = K(\beta, p, q)$ such that

(2) $M_q(y, f) \leq KC/y^{\beta + \frac{1}{p} - \frac{1}{q}}$ \hspace{1em} ($p < q \leq \infty$).

**Proof.** It suffices to consider the case $q = \infty$. For suppose (2) has been proven for $q = \infty$ and $K \geq 1$ (which we may assume without loss of generality). Then

$$M_q(y, f) = \left\{ \int_0^\infty |f(x + iy)|^p |f(x + iy)|^{q-p} \, dx \right\}^{1/q}$$

$$\leq \left[ M_\infty(y, f) \right]^{q-p/q} \left[ M_p(y, f) \right]^{p/q}$$

$$\leq K^{q-p/q} C/y^\beta,$$

where $\lambda = \beta + 1/p - 1/q$. Now to derive the theorem for $q = \infty$, let $u(z)$ be the nonnegative subharmonic function $|f(z)|^p$ and $a = \beta p$. Then Theorem 1 implies

$$|f(x_0 + iy_0)|^p \leq KC/y_0^{q+1},$$

which is equivalent to (2). \hspace{1em} \square

**Theorem 3.** Suppose $f$ belongs to $E_p(\Pi^+)$. Then for $1 < p < \infty$, $-1 < b$, and $1 < a < \infty$,

(3) $\int_0^\infty y^b M_p^a(y, f) \, dy \leq C \int_0^\infty y^{a+b} M_p^a(y, f') \, dy,$

where $C = C(a, b)$ is independent of $f$.

**Proof.** We begin by assuming that $f$ is analytic in the closed upper half-plane. Then integrating by parts we find

$$\int_0^\infty y^b M_p^a(y, f) \, dy = \frac{y_0^{b+1}}{b + 1} M_p^a(y_0, f)$$

$$- \frac{1}{b + 1} \int_0^\infty y^b M_p^a(y_0, f) \, dy.$$

Thus our next step is to estimate $[(\partial/\partial y) M_p^a(y, f)]$. But

$$(\partial/\partial y) M_p^a(y, f) = (a/p) M_p^a(y, f) (\partial/\partial y) M_p^a(y, f),$$
so we need to estimate $|(\partial/\partial y)M_p^*(y,f)|$.

However,

$$\left| \frac{\partial}{\partial y} |f(x+iy)|^p \right| = p |f(x+iy)|^{p-1} \left| \frac{\partial}{\partial y} |f(x+iy)| \right|$$

and

$$\left| |f(x+iy_1)| - |f(x+iy_2)| \right| \leq \left| \frac{f(x+iy_1) - f(x+iy_2)}{y_1 - y_2} \right|$$

implies

$$|\frac{\partial}{\partial y} |f(x+iy)|| \leq |f'(x+iy)|,$$

so

$$\left| \frac{\partial}{\partial y} |f(x+iy)|^p \right| \leq p |f(x+iy)|^{p-1} |f'(x+iy)|.$$

Thus Hölder's inequality implies

$$|\frac{\partial}{\partial y} M_p^*(y,f)| \leq p M_p^{p-1}(y,f) M_p(y,f')$$

and this implies

$$|\frac{\partial}{\partial y} M_p^*(y,f)| \leq a M_p^{p-1}(y,f) M_p(y,f').$$

But now we have

$$\left| \int_0^a y^{b+1} \frac{\partial}{\partial y} (M_p^*(y,f)) \, dy \right| \leq a \int_0^a y^{b+1} M_p^{p-1}(y,f) M_p(y,f') \, dy$$

$$\leq a \left\{ \int_0^a y^b M_p^*(y,f) \, dy \right\}^{1/a} \left\{ \int_0^a y^{a+b} M_p^*(y,f') \, dy \right\}^{1/a},$$

where we have used Hölder's inequality again. Hence

$$\left\{ \int_0^a y^b M_p^*(y,f) \, dy \right\}^{1/a}$$

$$\leq \left( \frac{y^{b+1}}{b+1} \right)^{1/a} M_p^*(y_0,f) + \frac{a}{b+1} \left\{ \int_0^a y^{a+b} M_p^*(y,f') \, dy \right\}^{1/a},$$

where we have used the estimate

$$\int_0^a y^b M_p^*(y,f) \, dy \geq \frac{y_0^{b+1} M_p^*(y_0,f)}{b+1}. \quad (4)$$
which follows from the fact that the means $M_p(y, f)$ are nonincreasing functions of $y$ [5].

From (4), it is clear that in order to complete the proof for this case, we need only show that $y^{b+1}M_p^b(y_0, f)$ tends to zero as $y_0$ tends to infinity. But using Theorem 2, it is easy to see that $f(x + iy_0) = -i \int_0^{y_0} f'(x + iy) \, dy$ and applying Minkowski's inequality, we find

$$M_p(y_0, f) \leq \int_0^{y_0} M_p(y, f') \, dy.$$  

So suppose $r > 1$. Then

$$M_p^a(y_0, f) \leq [C(y_0)]^a \left[ \frac{1}{r - 1} \int_0^{y_0} y^r M_p(y, f') \frac{d(-1/\sqrt{y-1})}{C(y_0)} \right]^a,$$

where $C(y_0) = \int_0^{y_0} d(-1/\sqrt{y-1}) = 1/y_0^{b+1}$, and Jensen's inequality gives

$$M_p^a(y_0, f) \leq [C(y_0)]^{a-1} \frac{1}{(r - 1)^{a-1}} \int_0^{y_0} y^{ar-1} M_p^a(y, f') \, dy.$$  

Hence setting $r = (a + b)/(a - 1)$, we have

$$y^{b+1}M_p^b(y_0, f) \leq \frac{1}{((b + 1)/(a - 1))^{a-1}} \int_0^{y_0} y^{a+b} M_p^a(y, f') \, dy,$$

from which it follows that $y^{b+1}M_p^b(y_0, f)$ tends to zero as $y_0$ tends to infinity.

Finally we remove the restriction that $f$ is analytic in the closed upper half-plane. Since $f(z) = f(z + iy)$ is analytic in the closed upper half-plane, the theorem holds for $f(z)$. Thus the result for $f(z)$ follows from letting $y$ tend to zero and applying the monotone convergence theorem. \qed

These three theorems have prepared the way for a proof of Theorem B.

**Proof of Theorem B.** We first reduce the theorem to the case $\lambda = p = 2$. By Theorem 2

$$M_\lambda^\lambda(y, f) \leq K^{\lambda-p} M_\lambda^\lambda(y, f)/y^{\lambda(a-\lambda)p},$$

so

$$\int_0^{y_0} y^{a-1} M_\lambda^\lambda(y, f) \, dy \leq K^{\lambda-p} \int_0^{y_0} y^{a-1} M_\lambda^\lambda(y, f) \, dy.$$  

Hence we can assume $\lambda = p$. Next assume the theorem is true for $\lambda = p = 2$ and $f(x) \neq 0$ in $\Pi^+$ and belongs to $E_2(\Pi^+)$. Then $g(z) = |f(z)|^{p/2}$ belongs to $E_2(\Pi^+)$ and

$$\int_0^{y_0} y^{-p/2} M_\lambda^p(y, f) \, dy = \int_0^{y_0} y^{-2/2} M_\lambda^2(y, g) \, dy < \infty,$$
where \( s = \frac{2q}{p} > 2 \). In case \( f(z) \) has zeros in \( \Pi^+ \), it is possible to write it as a sum of two nonzero functions in \( E_\rho(\Pi^+) \) [2] and still show that it suffices to take \( p = 2 \).

So let \( f \in E_\rho(\Pi^+) \). Then using the Paley-Wiener theorem [7], we can write

\[
f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t)e^{zt} dt,
\]

where \( \hat{f}(t) \) is the Fourier transform of the boundary function \( f(x) \) of \( f(z) \). Also

\[
f'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t\hat{f}(t)e^{zt} dt.
\]

Next we assume \( 2 < q < \infty \). Then by Theorem 3

\[
\int_{-\infty}^{\infty} y^{-2/q} M_q^2(y,f) \, dy \leq C \int_{-\infty}^{\infty} y^{-2-2/q} M_2^2(y,f') \, dy,
\]

and by Theorem 2 \( M_q(y,f') \leq K y^{q-1/2} M_2(y/2,f') \), so

\[
\int_{-\infty}^{\infty} y^{-2/q} M_q^2(y,f) \, dy \leq CK \int_{0}^{\infty} y M_2^2(y/2,f') \, dy.
\]

Finally, by Plancherel's theorem [7], we find

\[
\int_{-\infty}^{\infty} y^{-2/q} M_q^2(y,f) \, dy \leq \frac{CK}{2\pi} \int_{0}^{\infty} y \int_{0}^{\infty} |\hat{f}(t)|^2 t^2 e^{-yt} dt \, dy
\]

\[
= \frac{CK}{2\pi} \int_{0}^{\infty} |\hat{f}(t)|^2 t^2 \int_{0}^{\infty} ye^{-yt} \, dy \, dt
\]

\[
= \frac{CK}{2\pi} \int_{0}^{\infty} |\hat{f}(t)|^2 \, dt
\]

\[
= CK \int_{0}^{\infty} |f(x)|^2 \, dx < \infty.
\]

If \( q = \infty \), then the estimate

\[
M_\infty^2(y,f) \leq K M_2^2(y/2,f)/y^{2/r}
\]

for some \( r > 2 \) can be used to derive the desired results. \( \square \)

4. The proof of Theorem C. Since \( E_\rho(\Pi^+) \) is an \( F \)-space under the metric \( \rho(f,g) = \int_{-\infty}^{\infty} |f(x) - g(x)|^p \, dx \), we can use the closed graph theorem. Thus we need to show that \( \Lambda \) is a closed operator. So let \( \{f_n\} \) be a sequence which converges in \( E_\rho(\Pi^+) \) to \( f \) and also suppose \( \Lambda(f_n)(t) = \phi(t)f_n(t) \) converges to \( g(t) \) in \( L_q(0,\infty) \). Then we need to show that \( \Lambda(f)(t) = g(t) \) a.e.

Considering the sequence \( \{f_n\} \) and \( f \) first, we find by Theorem 2 that

\[
\left\{ \int_{-\infty}^{\infty} |f_n(x + iy_0) - f(x + iy_0)|^2 \, dx \right\}^{1/2} \leq \frac{K\|f_n - f\|_p}{y_0^{1/p - 1/2}}.
\]
where $y_0 > 0$. Thus $f_{y_0,n}(z) = f_n(z + iy_0)$ converges to $f_{y_0}(z) = f(z + iy_0)$ in $E_2(\Pi^+)$. Moreover, it is easy to see that the Fourier transform of $f_{y_0}(x)$ is $\hat{f}_n(t)e^{-y_0't}$, while the Fourier transform of $f_{y_0}(x)$ is $f(t)e^{-y_0't}$. Consequently, Plancherel's theorem [7] implies that $\hat{f}_n(t)e^{-y_0't}$ converges to $\hat{f}(t)e^{-y_0't}$ in $L_2(0, \infty)$. Hence, there exists a subsequence $\{\hat{f}_n(t)\}$ of $\{\hat{f}(t)\}$ converging to $\hat{f}(t)$ a.e. But the sequence $\{\Lambda(f_n)\}$ also converges to $g(t)$ in $L_2(0, \infty)$. Therefore, there exists a subsequence of $\{\Lambda(f_n)\}$, which we also denote by $\{\Lambda(f_n)\}$, converging to $g(t)$ a.e. Thus $\{\phi(t)f_n(t)\}$ converges to $\phi(t)f(t)$ a.e. and also to $g(t)$ a.e., which implies

$$\phi(t)\hat{f}(t) = g(t) \quad \text{a.e.} \quad \square$$

5. Fourier transform. The Fourier transform defined in §2 certainly exists since Theorem 2 implies that $f_n(x) = f(x + iy)$ belongs to $L_1(-\infty, \infty)$. In fact, if $C$ is a constant such that $M_p(y,f) \leq C$ for $y > 0$, then there exists a constant $K = K(0,p,1)$ such that

$$\int_{-\infty}^{\infty} |f(x + iy)| \, dx \leq CK/y^{1/p-1}$$

for $y > 0$.

To see that $\hat{f}$ is independent of $y$, fix $0 < y_1 < y_2 < \infty$ and for each $\alpha > 0$ let $\Gamma_\alpha$ be the rectangular contour with vertices $\pm \alpha + iy_1$ and $\pm \alpha + iy_2$. By Cauchy's theorem

$$\int_{\Gamma_\alpha} f(z)e^{-iz\beta} \, dz = 0.$$  

Next let $I = [y_1, y_2]$ and put

$$\Phi(\beta) = i\int_I f(\beta + iu)e^{-iu\beta}e^{iu} \, du.$$  

Then $|\Phi(\beta)| \leq e^{\alpha I} \int_{y_1}^{y_2} |f(\beta + iu)| \, du$. Now if we let

$$\Psi(\beta) = \int_{y_1}^{y_2} |f(\beta + iu)| \, du,$$

then Fubini's theorem and (1) imply

$$\int_{-\infty}^{\infty} \Psi(\beta) d\beta = \int_{y_1}^{y_2} \int_{-\infty}^{\infty} |f(\beta + iy)| \, d\beta \, dy \leq \frac{CK}{y_1^{1/p-1}}(y_2 - y_1).$$

Thus there exists a sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ as $j \to \infty$ and $\Psi(\alpha_j) + \Psi(-\alpha_j) \to 0$ as $j \to \infty$. Hence we have

$$\Phi(\alpha_j) \to 0 \quad \text{and} \quad \Phi(-\alpha_j) \to 0$$

as $j \to \infty$. Now combining (1), (2), and (3), we find

$$\int_{-\infty}^{\infty} f(x + iy_1)e^{-i(x+iy_1)\beta} \, dx$$

$$= \int_{-\infty}^{\infty} f(x + iy_2)e^{-i(x+iy_2)\beta} \, dx,$$

i.e., $\hat{f}$ is independent of $y$.  

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If we let \( f_y(z) = f(z + iy) \), then (4) becomes

\[
\hat{f}(t) = e^{\pi i t} \hat{f}_y(t) = e^{\pi i t} f_y(t).
\]

Since \( \hat{f}_y \) is the Fourier transform of an \( L^1(-\infty, \infty) \) function, it is continuous and hence \( \hat{f} \) is continuous.

Using (1) again, we see that

\[
|\hat{f}(t)| e^{-\pi y} = |\hat{f}_y(t)| \leq \|f\|_1 \leq CK/y_0^{1/p-1}
\]

for a fixed \( y_0 < y \). Thus if we fix \( t < 0 \) and let \( y \to \infty \), we find \( \hat{f}(t) = 0 \). Hence \( \hat{f}(t) \) is identically zero on \((0, \infty)\) and by continuity it is zero at \( t = 0 \). Also note \( \hat{f}_y(t) = 0 \) on \((-\infty, 0)\).

As we have noted, \( \hat{f}(t) = f_y(t) e^{-\pi y} \), so \( \hat{f}_y(t) = \hat{f}(t) e^{-\pi y} = f_y(t) e^{\pi (y_0 - y)} \), and letting \( y_0 = y/2 \), we have

\[
\int_0^\infty |\hat{f}_y(t)| dt \leq \|f_y\|_1 \int_0^\infty e^{\pi (y_0 - y)} dt
\]

\[
\leq \frac{KC}{y_0^{1/p-1} (y - y_0)}
\]

\[
= \frac{2^{1/p} KC}{y^{1/p}}.
\]

Hence for \( y > 0 \), \( \hat{f}_y \) belongs to \( L^1(-\infty, \infty) \) and we can apply the inversion theorem [7], to find

\[
f(z) = f_y(x) = (2\pi)^{-1} \int_0^\infty \hat{f}_y(t) e^{itx} dt
\]

\[
= (2\pi)^{-1} \int_0^\infty \hat{f}(t) e^{-\pi y} e^{itx} dt
\]

\[
= (2\pi)^{-1} \int_0^\infty \hat{f}(t) e^{itx} dt.
\]

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