A NEW CHARACTERIZATION OF TAME 2-SPHERES IN $E^3$

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ABSTRACT. It is shown in Theorem 1 that a 2-sphere $S$ in $E^3$ is tame from $A = \text{Int} \ S$ if and only if for each compact set $F \subset A$ there exists a 2-sphere $S'$ with complementary domains $A' = \text{Int} \ S'$, $B' = \text{Ext} \ S'$, such that $F \subset A' \subset A'' \subset A$ and for each $x \in S'$ there exists a path in $\overline{B'}$ of diameter less than $\rho(F, S)$ which runs from $x$ to a point $y \in S$. Furthermore, the theorem holds when $A$ is replaced by $B$, $A'$ by $B'$, $B'$ by $A'$, and Int by Ext. Two applications of this characterization are given. Theorem 2 states that a 2-sphere is tame from the complementary domain $C$ if for arbitrarily small $\epsilon > 0$, $S$ has a metric $\epsilon$-envelope in $C$ which is a 2-sphere. Theorem 3 answers affirmatively the following question: Is a 2-sphere $S \subset E^3$ tame in $E^3$ if there exists an $\epsilon > 0$ such that if $a, b \in S$ satisfy $\rho(a, b) < \epsilon$, then there exists a path in $S$ of spherical diameter $\rho(a, b)$ which connects $a$ and $b$?

1. Introduction. Bing's original characterization of a tame 2-sphere $S$ in $E^3$ states that tameness from the complementary domain $C$ is equivalent to the existence of an arbitrarily small $\epsilon$-homeomorphism from $S$ into $C$. Since any 2-sphere in $E^3$ can be homeomorphically approximated by a polyhedral 2-sphere, it may be assumed that the $\epsilon$-homeomorphism of Bing's theorem carries $S$ onto a polyhedral (hence tame) 2-sphere $S' \subset C$.

Theorem 1 was developed while the author was attempting to remove the restriction that $S'$ be "tied" so strongly to $S$ by the $\epsilon$-homeomorphism. The end result is Theorem 1, which modifies Bing's theorem in the following way: $S'$ is no longer required to be an $\epsilon$-homeomorphic image of $S$, but instead must have the property that each point of $S$ be a distance less than $\epsilon$ from $S'$, and furthermore, from each point $x$ of $S'$ there must exist a path of diameter less than $\epsilon$ which leads to a point $y \in S$ and which lies in the closure of the component of $C - S'$ whose boundary is $S \cup S'$. (This path need not depend continuously on $x$.)


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Although the statement of Theorem 1 is somewhat more complicated than that of Bing's characterization, it is sometimes easier to construct the 2-sphere \( S' \) described in Theorem 1 than it is to apply Bing's or other related characterizations. Theorems 2 and 3 are applications of Theorem 1 which illustrate this fact.

Most of the terms used in this paper are defined in the excellent survey article by Burgess and Cannon [3]. The exceptions are terms coined by the author, which are defined at the point of their first appearance. We use \( \rho \) throughout for the usual metric on \( E^3 \) and \( \mathcal{N}(X, \epsilon) \) for the \( \epsilon \)-neighborhood of a set \( X \) in \( E^3 \).

2. The principal result.

Theorem 1. Let \( S \) be a 2-sphere embedded in \( E^3 \). Let \( A \) and \( B \) denote the interior and exterior of \( S \) respectively, and let \( \rho \) be the usual metric on \( E^3 \). Then the following statements are equivalent:

1. \( S \) is tame from \( A \).
2. For each compact set \( F \subset A \), there exists a 2-sphere \( S' \), embedded in \( E^3 \) and having complementary domains \( A' = \text{Int} S' \), \( B' = \text{Ext} S' \), such that (i) \( F \subset A' \subset A \), and (ii) for each \( x \in S' \) there exists a path \( xy \) contained in \( B' \) having initial point \( x \in S' \), terminal point \( y \in S \), and whose diameter is less than \( \rho(F, S) \).

Statements (1) and (2) are also equivalent if we replace \( A \) by \( B \), \( A' \) by \( B' \), \( B' \) by \( A' \), and \( \text{Int} \) by \( \text{Ext} \) in these statements.

The original statement and proof of Theorem 1 appearing in (7) requires that the 2-sphere \( S' \) be tamely embedded. The referee has suggested the following simpler proof which uses the 0-ulc property to remove this restriction:

Proof. The proof will first deal with tameness from \( A \).

(1) \( \Rightarrow \) (2): The proof is obvious.

(2) \( \Rightarrow \) (1): Suppose that statement (2) holds. The proof that \( S \) is tame from \( A \) depends on two lemmas.

**Lemma 1.** The 2-sphere \( S' \) of statement (2) may be chosen to be polyhedral.

Proof. We first use the validity of statement (2) to choose a 2-sphere \( S'' \), embedded in \( E^3 \) and having complementary domains \( A'' = \text{Int} S'', B'' = \text{Ext} S'' \), such that (i) \( F \subset A'' \subset A \), and (ii) for each \( x \in S'' \) there exists a path \( xy \) contained in \( B'' \) having initial point \( x \in S'' \), terminal point \( y \in S \), and whose diameter is less than \( \epsilon = (1/2)\rho(F, S) \). We next choose disks \( D_1, \ldots, D_n \) of diameter less than \( \epsilon/3 \) whose union is \( S'' \). Since the sets \( A'' \) and \( B'' \) are 0-ulc ([8, p. 66]; cf. also [3, Theorems 4.1.2 and 4.1.3]), there are arcs \( \alpha_1 = \overline{p_1q_1}, \ldots, \alpha_n = \overline{p_{n_1}q_n} \) in \( E^3 \) of diameter less than \( \epsilon \) such that, for each \( i \), \( p_i \in A'' \), \( q_i \in S \), and \( \alpha_i \cap S'' \) is a single point of \( \text{Int} D_i \). Let \( \delta > 0 \) be smaller than any of the numbers \( \rho(S'', S), \epsilon/3, \rho(S'', F), \rho(p_i, S''), \rho(S'' - D_i, \alpha_i) \) \( (i = 1, \ldots, n) \). Let \( b: S'' \to S' \) be a
homeomorphism from $S^\alpha$ onto a polyhedral 2-sphere $S'$ in $E^3$ which moves no point of $S^\alpha$ as far as $\delta [2]$. We claim that $S'$ satisfies the requirements of statement (2) with respect to $S$ and $F$; this we see as follows. Since $\delta < \rho(S^\alpha, S)$ and $\delta < \rho(S^\alpha, F)$, it follows that $F \subset A' \subset A$. Let $x \in b(D_i) \subset S'$ be an arbitrary point of $S'$. Since $\delta < \rho(p_i, S^\alpha)$, $p_i \in A'$. Thus $a_i \cap S' \neq \emptyset$. Since $\delta < \rho(S^\alpha - D_i, a_i)$, $a_i \cap S' \subset b(D_i)$. Thus there is a subarc $a$ of $a_i$ which lies in $B'$ and connects $b(D_i)$ with $S$. Therefore $S'$ satisfies the requirements of statement (2) with respect to $S$ and $F$. This completes the proof of Lemma 1.

**Lemma 2.** Let $S'$ be a polyhedral 2-sphere embedded in $E^3$ with complementary domains $A' = \text{Int } S'$ and $B' = \text{Ext } S'$. Let $D$ be a polyhedral disk embedded in $E^3$ with $\text{Bd } D \subset A'$ such that $D$ intersects $S'$ transversely. Let $C$ be a component of $\overline{B'} - D$ such that there is an arc $\overline{ab}$ from $\text{Bd } D$ to $C$ which except for its endpoints $a \in \text{Bd } D$ and $b \in C$, misses $D \cup \overline{B'}$. Then given $\epsilon > 0$, there exists a nonsingular polyhedral disk $D'$ such that

(a) $\text{Bd } D' = \text{Bd } D$ and
(b) $D' \subset A' \cap N[D \cup (S' - C), \epsilon]$.

**Proof.** Since $D$ intersects $S'$ transversely, $D \cap S'$ is the union of finitely many disjoint simple closed curves. The proof is by induction on the number $n$ of those curves. For $n = 0$, we may set $D' = D$. Suppose inductively that the lemma is true for fewer than $n$ curves of intersection and that $n > 0$. Since $n > 0$, $D \cap S' \neq \emptyset$; and we may choose a component $J$ of $D \cap S'$ such that the interior of the subdisk $D_J$ of $D$ bounded by $J$ misses $S'$. We consider two cases.

**Case 1.** $D_J \subset \overline{B'}$. In this case there is a disk $S_J$ in $S'$ which is bounded by $J$ and misses $C$; i.e., $S_J \subset (\overline{B'} - C)$. By standard cut and paste techniques it is possible to cut off $D$ near $S_J$ in $A'$. (Consider the disks in $D$ bounded by curves in $S_J \cap D$. Those which are not contained in the interior of any other such disk are replaced by disks in $A'$ parallel to subdisks of $S_J$.) Call the new disk thus obtained $D_1$ and note that $D_1$ meets $S'$ transversely and in fewer components than did $D$. Let $C_1$ denote the component of $\overline{B'} - D_1$ which contains $C$. Note that if $D_1 - D$ is chosen sufficiently close to $S_J$, then the arc $\overline{ab}$, except for its endpoints, misses $D_1 \cup \overline{B'}$ and for some $\epsilon_1 > 0$, $A' \cap N[D_1 \cup (S' - C_1), \epsilon_1] \subset A'$ and $N[D \cup (S' - C), \epsilon]$. By inductive hypothesis, there is a nonsingular polyhedral disk $D'$ such that

(a) $\text{Bd } D' = \text{Bd } D_1 = \text{Bd } D$ and
(b) $D' \subset A' \cap N[D_1 \cup (S' - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$, as desired.

**Case 2.** $D_J \subset \overline{A'}$. In this case let $S_J$ be the disk in $S'$ bounded by $J$ whose interior is separated in $A'$ from the arc $\overline{ab}$ by the disk $D_J$. Let $S'_J$ be a polyhedral disk in $S'$ which is very close to $S_J$ homeomorphically and which contains $S_J$ in
its interior. Let $S_j^w$ be a polyhedral disk in $\overline{A^j} - D$ which is very close to $D_j$ homeomorphically, lies, except for its boundary, in $A'$, and has the same boundary as $S_j$. Let $S_1$ be the polyhedral 2-sphere $(S' - S_j) \cup S_j^w$. Let $A_1 = \text{Int } S_1$, $B_1 = \text{Ext } S_1$, and $C_1 = \text{component of } \overline{B_1} - D$ which contains $C$. It is easy to check that $\text{Bd } D \subset A_1 \subset A'$ (since $\text{Int } \overline{ab} \subset A_1$); that $D$ intersects $S_1$ transversely and in fewer components than it intersected $S$; that $ab$ is an arc from $\text{Bd } D$ to $C_1$ which, except for its endpoints, misses $D \cup \overline{B_1}$; and finally, that if $S_j^w$ is close enough to $D_j$ then $A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$ for some $\epsilon_1 > 0$. Again the inductive hypothesis applies and supplies a polyhedral disk $D'$ such that

(a) $\text{Bd } D' = \text{Bd } D$ and 
(b) $D' \subset A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$, as desired.

Cases 1 and 2 complete the inductive proof of Lemma 1.

We now return to the proof that $(2) \Rightarrow (1)$.

Bing has proved [1] that a 2-sphere $S$ in $E^3$ is tame from a complementary domain $A$ if $A$ is 1-ULC. His proof is easily seen to be valid if the following condition replaces the 1-ULC condition.

**Condition *.** Suppose $E$ is a disk in $S$ and $D$ is a polyhedral disk in $E^3$ such that $\text{Bd } D \subset A$ and $D \cap S \subset \text{Int } E$. Suppose further that $\text{Bd } D$ can be joined to $S - E$ by an arc $a = uv$ which lies, except for its endpoints $u \in \text{Bd } D$ and $v \in S - E$, in $E^3 - (S \cup D)$. Then, given $\epsilon > 0$, $\text{Bd } D$ bounds a disk $D'$ in $A \cap N[D \cup E, \epsilon]$.

Theorem 1 will be established once we prove that statement (2) of Theorem 1 implies that Condition * is satisfied. Suppose therefore that $E$, $D$, $a$, and $\epsilon$ are given as in Condition *.

We first wish to apply statement (2). To this end, choose a point $w \in \text{Int } a$, and let $a_1 = uv$ and $a_2 = wv$ denote the two arcs into which $w$ divides $a = uv$. Choose a positive number $\delta$ such that

$$\delta < \min \{\epsilon/2, \rho(a_2, D \cup E), \rho(S, \text{Bd } D \cup a_1)\}.$$ 

Let $F = A - N(S, \delta)$. Statement (2) of Theorem 1 implies that there is a 2-sphere $S'$ in $E^3$ having complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that (i) $F \subset A' \subset A^T \subset A$, and (ii) for each $x \in S'$ there exists a path $\overline{xy}$ contained in $B'$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$.

We now wish to apply Lemma 2. Lemma 1 shows that we may choose $S'$ to be polyhedral and to meet $D$ transversely. Note that $\text{Bd } D \subset F \subset A'$. Let $C$ be the component of $\overline{B'} - D$ which contains $S - E$. Let $x$ be the first point of $a = uv$ which lies in $S'$. We must necessarily have $x \in \text{Int } a_2$ since $\delta < \rho(S, a_1)$. We claim that $x \in C$. Indeed, let $\overline{xy}$ be a path contained in $\overline{B'}$ which connects
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$x$ to $S$ and has diameter less than $\delta$. Since $x \in A_2$ and $\delta < \rho(\alpha_2, D \cup E)$, $y \in S - E \subseteq C$. Thus $x \in \overline{xy} \subseteq C$, as claimed. We have established therefore that there is an arc $\overline{xy}$ from $\text{Bd} D$ to $C$ which, except for its endpoints, misses $D \cup \overline{E}$. Thus Lemma 2 applies and yields a polyhedral disk $D'$ such that

(a) $\text{Bd} D' = \text{Bd} D$ and
(b) $D' \subset A' \cap N[D \cup (S' - C), \delta]$.

Since $A' \subset A$, we will be done once we have shown that $N[D \cup (S' - C), \delta] \subset N[D \cup E, \epsilon]$, or, since $\delta < \epsilon/2$, that $(S' - C) \subset N[D \cup E, \delta]$. Let $z \in S' - C$ and let $\beta$ be an arc in $\overline{B}'$ of diameter less than $\delta$ which joins $z$ to $S$. If $\beta$ misses both $D$ and $E$, then $z$ is in the same component of $\overline{B}' - D$ as is $S - E$; i.e., $z \in C$, a contradiction. Hence $\beta \cap (D \cup E) \neq \emptyset$, and $z \in N[D \cup E, \delta]$. This completes the proof that statement (2) of Theorem 1 implies Condition *. As noted earlier, this also completes the proof that $S$ is tame from $A$.

We can use the foregoing proof to deal with tameness from $B$ by simply forming the one-point compactification $S^3$ of $E^3$ and then removing a point from $A$ to form $E^3$ again. $\square$

3. Applications of Theorem 1. The following theorem is an almost immediate result of Theorem 1. If $X, Y$ are subsets of $E^3$, define the metric $\epsilon$-envelope of $Y$ in $X$ to be the set $\{x \in X|\rho(x, Y) = \epsilon\}$.

**Theorem 2.** Let $S$ be a 2-sphere embedded in $E^3$ with $A = \text{Int} S$, $B = \text{Ext} S$. Let $\rho$ be the usual $E^3$ metric. Suppose that for each $\alpha > 0$ there exists a real $\epsilon$ with $0 < \epsilon < \alpha$ such that the metric $\epsilon$-envelope of $S$ in $A$ is a 2-sphere embedded in $E^3$. Then $S$ is tame from $A$. The implication also holds if $A$ is replaced by $B$ in the last two statements.

**Proof.** $S$ will be proven tame from $A$ by showing that it satisfies condition (2) of Theorem 1. The proof for tameness from $B$ is similar.

Let $F \subset A$ be a compact set. By hypothesis there exists a real number $\epsilon$ such that $0 < \epsilon < \rho(F, S)$ and the set $S' = \{x \in A|\rho(x, S) = \epsilon\}$ is a 2-sphere in $A$. Let $A' = \text{Int} S'$, $B' = \text{Ext} S'$. We first show that $F \subset A' \subset \overline{A'} \subset A$. Since $S' \subset A$, $\overline{B}$ is contained in one of the complementary domains of $S'$. Since $B$ is unbounded, we must have $B \subset \overline{B} \subset B'$. It follows immediately that $A' \subset \overline{A'} \subset A$. To show that $F \subset A'$, let $x$ be a point of $F$. Since $\rho(x, S) > \epsilon$, $x \notin S'$. Let $\beta$ be any path in $E^3$ which starts at $x$ and ends at some point $z \in S \subset \overline{B} \subset B'$. Since $\rho(x, S) > \epsilon$ and $\rho(x, S) = 0$, there is some point $y$ on $\beta$ such that $\rho(y, S) = \epsilon$ by the intermediate value property. Then $y \in S'$, which implies that every path from $x$ to $z$ in $B'$ must intersect $S'$. Since $B'$ is path-connected and $x \notin S'$, we conclude that $x \in A'$.

Therefore $F \subset A' \subset \overline{A'} \subset A$.

It remains to show that for each $x \in S'$ there exists a path $\overline{xy}$ contained in $B'$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less
than \( \rho(F, S) \). Let \( x \in S' \). Since \( S \) is compact and \( \rho(x, S) = \epsilon \), there is a point \( y \in S \) such that \( \rho(x, y) = \epsilon \). Consider the path \( xy \) formed by the straight line segment running from \( x \) to \( y \). The diameter of this path is \( \epsilon < \rho(F, S) \), and all we need to show is that it lies in \( B' \). Suppose not. Then some point \( w \) strictly between \( x \) and \( y \) on \( xy \) must lie in \( A' \). Since \( y \in S \subseteq B' \), there is a point \( z \) strictly between \( w \) and \( y \) on \( xy \) which lies on \( S' \). But this results in a contradiction because \( z \in S' \Rightarrow \rho(z, S) = \epsilon \) but \( z \) is strictly between \( x \) and \( y \) on \( xy \) which implies that \( \rho(z, y) < \epsilon \Rightarrow \rho(z, S) < \epsilon \). \( \square \)

The converse of Theorem 2 is clearly not true. Theorem 2 gives rise to the following question: If the metric \( \epsilon \)-envelope of a set \( X \) in \( E^3 \) is a 2-sphere \( S \), is \( S \) tame? Partial answers can be obtained if \( X \) lies in one of the complementary domains \( C \). In this case it is clear that for each point \( x \in S \), there exists a round tangent ball in \( S \cup C \) which touches \( S \) only at \( x \). Loveland [6, p. 396] has asked if this makes \( S \) tame, and Cannon [4, pp. 444–445] proved that \( S \) is tame from \( E^3 - C \) under this condition. If \( X \subseteq \text{Int} \, S \) and \( \epsilon > \text{diam} \, X \) then each point of \( S \) is visible from a point \( x \in X \). Cobb [5] shows that \( S \) is then tame in \( E^3 \). His proof appears in [3, pp. 326–327].

Define the spherical diameter of a set \( X \subseteq E^3 \) to be the diameter of the smallest closed round ball containing \( X \). (For a given set, the ratio \( r \) of the spherical diameter to the usual diameter satisfies \( 1 < r \leq \sqrt{3/2} \).)

All of the commonly known wild spheres appear to have the property that one can find two points \( x, y \) on the sphere which are arbitrarily close together such that any arc on the sphere having \( x \) and \( y \) as endpoints must have a spherical diameter greater than \( \rho(x, y) \). Must every wild sphere have this property? It seems reasonable that such a property might result from the rather severe entanglement in \( E^3 \) which is characteristic of wild spheres. The following lemmas are used in Theorem 3 which answers the question affirmatively. The proof of Lemma 3 is a simple geometrical argument and is therefore omitted.

**Lemma 3.** Let \( P \) be a solid, closed, rectangular parallelepiped. Let \( R \) be the union of all closed, round balls having a diameter which is an edge, a face diagonal, or a principal diagonal of \( P \). Let \( a, b \) be distinct points in \( P \). Then any path \( \alpha \) from \( a \) to \( b \) which has spherical diameter equal to \( \rho(a, b) \) must lie in \( R \).

**Lemma 4.** Suppose \( M \) is a Euclidean polyhedron in \( E^3 \) which is connected but not simply connected. Let \( \beta \) be a positive real number. Then there exist two simple closed curves \( K \subseteq M, H \subseteq E^3 - M \) such that neither is null-homotopic in the complement of the other. Furthermore \( H \) may be chosen to lie in the \( \beta \)-neighborhood of \( M \).

**Proof.** Let \( U_\beta \) denote the \( \beta \)-neighborhood of \( M \). Let \( N \subseteq U_\beta \) be a regular
neighborhood of $M$. Each component of $\partial N$ is a p.l. 2-manifold without boundary. Since $M$ is not simply connected, neither is $N$. Therefore, some component $C$ of $\partial N$ is not simply connected. The fundamental theorem of compact surfaces states that $C$ is either a 2-sphere or the connected sum of a finite number of tori. The first possibility is ruled out, so there is a subset $\hat{C}$ of $C$ which is a torus $T$ minus the interior of a disk $D \subset T$. Select two polygonal simple closed curves $H$ and $K$ on $\hat{C}$ which intersect each other transversely and at a single point on $\hat{C}$. A simple linking argument shows that either $H$ pushed slightly into $U_{\beta} - N$ links $K$ homologically in $E^3$ or $K$ pushed slightly into $U_{\beta} - N$ links $H$ homologically in $E^3$. Interchanging the names of $H$ and $K$ if necessary, we may assume the former. $K$ can be homotopically pushed into $M$ via a collapsing of $N$ into $M$. At this point neither of $K, H$ is null-homotopic in the complement of the other, although $K$ may not simple. However, some subset of $K$ is a simple closed curve satisfying the non null-homotopic condition. Taking $K$ to be this curve completes the proof. □

**Theorem 3.** Let $S \subset E^3$ be a 2-sphere with $\rho$ the usual $E^3$ metric. Suppose there exists an $\epsilon > 0$ such that any two points, $a, b \in S$ satisfying $\rho(a, b) < \epsilon$ can be joined by a path in $S$ of spherical diameter $= \rho(a, b)$. Then $S$ is tame in $E^3$.

**Proof.** The proof deals with tameness from $A = \text{Int} S$. That $S$ is tame from $B = \text{Ext} S$ can be proved similarly. Let $F \subset A$ be compact with $\eta = \rho(F, S)$. Consider the solid, closed cubes in $E^3$ of edge length $\epsilon < \min[\eta/4, \epsilon/\sqrt{6}]$ whose vertices have coordinates of the form $(me, ne, pe)$ where $m, n, p$ are integers. The word “cube” will refer to one of these cubes unless otherwise stated. The non-empty union $T$ of all cubes lying entirely in $A$ contains $F$, and $S$ is accessible from each point of $\partial T$ via a path in $A - \overline{T}$ of diameter $< \eta$. The object is to change $T$ into a polyhedral 3-cell $B$ which retains these two properties. Then $\partial B$ will be a 2-sphere $S'$ satisfying statement (2) of Theorem 1, proving that $S$ is tame from $A$. $T$ will first be modified into $T''$ so that each component of $T''$ becomes a polyhedral 3-cell. $B$ is then easily constructed by connecting the components with slightly thickened polygonal arcs in $A - \text{Int} T''$. Each component $T_i$ of $T$ has a connected complement and therefore can fail to be a 3-cell by either not being simply connected or by not being a 3-manifold-with-boundary. The first of these difficulties is corrected by removing from each $T_i$ neighborhoods of constriction points of $T_i$ as shown in Figure 1. Such a point $x$ is a vertex of exactly two cubes $M, N$ lying in $T_i$ with $M \cap N = \{x\}$. The resulting modified version $T'$ of $T$ retains the two properties of $T$ mentioned previously. For purposes of continuity, the proof that each component $T'_j$ of $T'$ is simply connected will be deferred until later. Figure 2 shows how each $T'_j$ is then made into a polyhedral 3-manifold with boundary, hence a polyhedral 3-cell $B'_j$, by attaching to the concave "troughs" of $\partial B'_j$ small cubes of edge length $\epsilon/m$. 

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where \( m > 2 \) satisfies \( e/m < \rho(T, S) \). The \( B_j \) are disjoint and their union \( T'' \) retains the two properties of \( T \) mentioned previously. Connecting the \( B_j \) with fattened polygonal arcs in \( A - \text{Int } T'' \) provides the desired 3-cell \( B \), and the proof is complete except for the argument below.

**Figure 1**

Proof that each \( T'_j \) is simply connected. Assume not. Choose \( \beta < \rho(T, S) \).

By Lemma 4, there exist polygonal simple closed curves \( H_1 \subset E^3 - T' \) and \( K_1 \subset T'_j \) such that \( H_1 \) lies in the \( \beta \)-neighborhood \( N_j \) of \( T'_j \) and neither curve is null-homotopic in the complement of the other. \( H_1 \) can be chosen to miss \( T \) by moving it, if necessary, in \( N_j - T'_j \) so that it fails to intersect any of the neighborhoods which earlier were removed from each \( T'_j \). \( \beta \) has been chosen small enough to assure that each cube in the union \( V \) of all cubes intersecting \( H_1 \) will contain points of \( S \), and only the boundary of each cube will intersect \( T \). Because \( H_1 \subset \text{Int } V \), \( H_1 \) can be moved slightly in \( V - T \) so that it intersects no edge of any cube. Thus \( H_1 \) is now the union of \( m \) polygonal arcs laid end-to-end with each arc \( \gamma_i \) satisfying the following: \( \gamma_i \subset Q_i \cup Q_{i+1} \), where \( Q_i \) and \( Q_{i+1} \) are cubes from \( V \) which intersect in a common face, and the endpoints of \( \gamma_i \) lie in \( Q_i \) and \( Q_{i+1} \) respectively. For each \( i \), \( Q_i \) and \( Q_{i+1} \) each contain points of \( S \), so \( H_1 \) can be further moved in \( V - T \) so that the arcs \( \gamma_i \) change into straight line segments \( \alpha_i \) satisfying \( \alpha_i \subset Q_i \cup Q_{i+1} \) with the endpoints \( x_i, x_{i+1} \) of \( \alpha_i \) being points of \( S \) lying in \( Q_i, Q_{i+1} \) respectively. Since \( \rho(x_i, x_{i+1}) \leq \sqrt{\beta} < \epsilon \), the hypothesis of the theorem implies that there is a path \( \Gamma_i \subset S \) of spherical diameter \( \rho(x_i, x_{i+1}) \) joining \( x_i \) and \( x_{i+1} \). Because the closed path \( \Gamma = (\bigcup_{i=1}^m \Gamma_i) \subset S \) is null-homotopic in \( S \), hence in \( E^3 - K_1 \), there is some \( k \) such that the path \( G_k = \alpha_k \cup \Gamma_k \) is not null-homotopic in \( E^3 - K_1 \). The spherical diameter of \( G_k \) is \( \rho(x_k, x_{k+1}) \). By Lemma 3, \( G_k \) lies in the 3-cell \( R \) formed by the union of all closed round balls whose diameter is an edge, a face diagonal, or a principal diagonal of \( Q_k \cup Q_{k+1} \). Figure 3 indicates that all cubes except \( Q_k \) and \( Q_{k+1} \) (hence all cubes in \( T \)) fall into 6 classes depending upon...
the nature of their intersection with \( R \). The darkened edges of each representative cube \( L \) are those which miss \( \text{Int } R \), and \( E_L \) denotes the union of these edges. If two such cubes \( L, M \) meet in a common edge or face, then \( E_L \cup E_M \) is arc-connected.

![Diagram of six classes of cubes](image)

**Figure 3**

The previous removal of neighborhoods in \( T \) of constriction points assures that the closed curve \( K_1 \subset T' \subset T \) mentioned previously can be chosen to intersect no vertex of any cube, and therefore can be traversed by passing through a sequence of cubes in \( T \), each cube \( N_i \) intersecting its predecessor in a common face or edge. Since \( E_{N_i} \cup E_{N_{i-1}} \) is arc-connected, a further adjustment of \( K_1 \) in \( T \) makes \( K_1 \cap N_i \subset E_{N_i} \) for each \( i \). \( K_1 \) now misses \( \text{Int } R \). Now \( G_k \subset R \), \( G_k \) misses \( K_1 \), and is not null-homotopic in \( E^3 - K_1 \). But \( R \) is a 3-cell, so \( G_k \) can be contracted in \( R \) radially inward to a point without hitting \( K_1 \). This contradiction completes the argument that \( T_1 \) is simply connected. \( \Box \)

The converse of this theorem is clearly false. It might be possible to strengthen the theorem by showing that there is some constant \( K > 1 \) such that tameness is implied if the path from \( x \) to \( y \) mentioned in the hypothesis has a spherical diameter equal to \( K \rho(x, y) \). This leads to the problem of finding the least upper bound for such a constant. The proof of Theorem 3 breaks down if \( K > 1 \) because then it can no longer be guaranteed that \( E_M \cap E_N \) is connected when the cubes \( M, N \).
intersect in a common edge or face. The theorem might be true if "diameter" replaces "spherical diameter", but neither a proof nor a counterexample has been found.

Another question related to Theorem 3 is the following: If $S$ is a 2-sphere in $E^3$ and $C$ is one of its complementary domains, define a chord of $C$ to be a straight line segment lying in $C$ having its endpoints in $S$. Is $S$ tame from $C$ if there exists an $\epsilon > 0$ such that for each chord of $C$ of length $l < \epsilon$ there is an arc in $S$ of spherical diameter $= l$ which connects the endpoints of the chord?

REFERENCES


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