A NEW CHARACTERIZATION OF TAME 2-SPHERES IN $E^3$

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ABSTRACT. It is shown in Theorem 1 that a 2-sphere $S$ in $E^3$ is tame from $A = \text{Int } S$ if and only if for each compact set $F \subset A$ there exists a 2-sphere $S'$ with complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that $F \subset A' \subset \overline{A'} \subset A$ and for each $x \in S'$ there exists a path in $\overline{B'}$ of diameter less than $\rho(F, S)$ which runs from $x$ to a point $y \in S$. Furthermore, the theorem holds when $A$ is replaced by $B$, $A'$ by $B'$, $B'$ by $A'$, and $\text{Int}$ by $\text{Ext}$. Two applications of this characterization are given. Theorem 2 states that a 2-sphere is tame from the complementary domain $C$ if for arbitrarily small $\epsilon > 0$, $S$ has a metric $\epsilon$-envelope in $C$ which is a 2-sphere. Theorem 3 answers affirmatively the following question: Is a 2-sphere $S \subset E^3$ tame in $E^3$ if there exists an $\epsilon > 0$ such that if $a, b \in S$ satisfy $\rho(a, b) < \epsilon$, then there exists a path in $S$ of spherical diameter $\rho(a, b)$ which connects $a$ and $b$?

1. Introduction. Bing's original characterization of a tame 2-sphere $S$ in $E^3$ states that tameness from the complementary domain $C$ is equivalent to the existence of an arbitrarily small $\epsilon$-homeomorphism from $S$ into $C$. Since any 2-sphere in $E^3$ can be homeomorphically approximated by a polyhedral 2-sphere, it may be assumed that the $\epsilon$-homeomorphism of Bing's theorem carries $S$ onto a polyhedral (hence tame) 2-sphere $S' \subset C$.

Theorem 1 was developed while the author was attempting to remove the restriction that $S'$ be "tied" so strongly to $S$ by the $\epsilon$-homeomorphism. The end result is Theorem 1, which modifies Bing's theorem in the following way: $S'$ is no longer required to be an $\epsilon$-homeomorphic image of $S$, but instead must have the property that each point of $S$ be a distance less than $\epsilon$ from $S'$, and furthermore, from each point $x$ of $S'$ there must exist a path of diameter less than $\epsilon$ which leads to a point $y \in S$ and which lies in the closure of the component of $C - S'$ whose boundary is $S \cup S'$. (This path need not depend continuously on $x$.)


AMS (MOS) subject classifications (1970). Primary 55A30; Secondary 57A05.

Key words and phrases. Tame surfaces, tame 2-spheres, surfaces in $E^3$, characterizations of tameness.

(1) This paper contains part of the author's Ph. D. dissertation, which has the same title.
Although the statement of Theorem 1 is somewhat more complicated than that of Bing’s characterization, it is sometimes easier to construct the 2-sphere $S'$ described in Theorem 1 than it is to apply Bing’s or other related characterizations. Theorems 2 and 3 are applications of Theorem 1 which illustrate this fact.

Most of the terms used in this paper are defined in the excellent survey article by Burgess and Cannon [3]. The exceptions are terms coined by the author, which are defined at the point of their first appearance. We use $\rho$ throughout for the usual metric on $E^3$ and $N(X, \epsilon)$ for the $\epsilon$-neighborhood of a set $X$ in $E^3$.

2. The principal result.

Theorem 1. Let $S$ be a 2-sphere embedded in $E^3$. Let $A$ and $B$ denote the interior and exterior of $S$ respectively, and let $\rho$ be the usual metric on $E^3$. Then the following statements are equivalent:

(1) $S$ is tame from $A$.
(2) For each compact set $F \subseteq A$, there exists a 2-sphere $S'$, embedded in $E^3$ and having complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that (i) $F \subseteq A' \subseteq A$, and (ii) for each $x \in S'$ there exists a path $\overline{xy}$ contained in $B'$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$.

Statements (1) and (2) are also equivalent if we replace $A$ by $B$, $A'$ by $B'$, $B'$ by $A'$, and $\text{Int}$ by $\text{Ext}$ in these statements.

The original statement and proof of Theorem 1 appearing in (7) requires that the 2-sphere $S'$ be tamely embedded. The referee has suggested the following simpler proof which uses the 0-ulc property to remove this restriction:

Proof. The proof will first deal with tameness from $A$.
(1) $\Rightarrow$ (2): The proof is obvious.
(2) $\Rightarrow$ (1): Suppose that statement (2) holds. The proof that $S$ is tame from $A$ depends on two lemmas.

Lemma 1. The 2-sphere $S'$ of statement (2) may be chosen to be polyhedral.

Proof. We first use the validity of statement (2) to choose a 2-sphere $S''$, embedded in $E^3$ and having complementary domains $A'' = \text{Int } S''$, $B'' = \text{Ext } S''$, such that (i) $F \subseteq A'' \subseteq A$, and (ii) for each $x \in S''$ there exists a path $\overline{xy}$ contained in $B''$ having initial point $x \in S''$, terminal point $y \in S$, and whose diameter is less than $\epsilon = (1/2)\rho(F, S)$. We next choose disks $D_1, \ldots, D_n$ of diameter less than $\epsilon/3$ whose union is $S''$. Since the sets $A''$ and $B''$ are 0-ulc ([8, p. 66]; cf. also [3, Theorems 4.1.2 and 4.1.3]), there are arcs $\alpha_1 = \overline{p_1q_1}, \ldots, \alpha_n = \overline{p_nq_n}$ in $E^3$ of diameter less than $\epsilon$ such that, for each $i$, $p_i \in A''$, $q_i \in S$, and $\alpha_i \cap S''$ is a single point of $\text{Int } D_i$. Let $\delta > 0$ be smaller than any of the numbers $\rho(S'', S)$, $\epsilon/3$, $\rho(S'', F)$, $\rho(p_i, S'')$, and $\rho(S'' - D_i, \alpha_i) (i = 1, \ldots, n)$. Let $b: S'' \to S'$ be a
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homeomorphism from $S^\ast$ onto a polyhedral 2-sphere $S'$ in $E^3$ which moves no
point of $S^\ast$ as far as $\delta$ [2]. We claim that $S'$ satisfies the requirements of state-
ment (2) with respect to $S$ and $F$; this we see as follows. Since $\delta < \rho(S^\ast, S)$ and
$\delta < \rho(S^\ast, F)$, it follows that $F \subset A' \subset A \subset A$. Let $x \in b(D_i) \subset S'$ be an arbitrary
point of $S'$. Since $\delta < \rho(p_i, S^\ast)$, $p_i \in A'$. Thus $a_i \subset S' \not\subset \emptyset$. Since $\delta < \rho(S^\ast - D, a_i)$,
$a_i \subset S' \subset b(D_i)$. Thus there is a subarc $a$ of $a_i$ which lies in $B'$ and connects
$b(D_i)$ with $S$. There is an arc $\beta$ in $b(D_i)$ joining $x$ to $a$. Then $a \cup \beta$ contains
an arc from $x$ to $S$ which lies in $B'$ and has diameter less than $2\epsilon = \rho(F, S)$. This
completes the proof of Lemma 1.

**Lemma 2.** Let $S'$ be a polyhedral 2-sphere embedded in $E^3$ with complementary
domains $A' = \text{Int} S'$ and $B' = \text{Ext} S'$. Let $D$ be a polyhedral disk embedded in
$E^3$ with $Bd D \subset A'$ such that $D$ intersects $S^\ast$ transversely. Let $C$ be a compo-
nent of $\overline{B'} - D$ such that there is an arc $\overline{ab}$ from $Bd D$ to $C$ which except for its
endpoints $a \in Bd D$ and $b \in C$, misses $D \cup \overline{B'}$. Then given $\epsilon > 0$, there exists a
nonsingular polyhedral disk $D'$ such that

(a) $Bd D' = Bd D$ and
(b) $D' \subset A' \cap N[D \cup (S' - C), \epsilon]$.

**Proof.** Since $D$ intersects $S^\ast$ transversely, $D \cap S^\ast$ is the union of finitely
many disjoint simple closed curves. The proof is by induction on the number $n$ of
those curves. For $n = 0$, we may set $D' = D$. Suppose inductively that the lemma
is true for fewer than $n$ curves of intersection and that $n > 0$. Since $n > 0$, $D \cap S^\ast \not\subset \emptyset$;
and we may choose a component $J$ of $D \cap S^\ast$ such that the interior of the subdisk
$D_J$ of $D$ bounded by $J$ misses $S^\ast$. We consider two cases.

**Case 1.** $D_J \subset \overline{B'}$. In this case there is a disk $S_J$ in $S^\ast$ which is bounded by
$J$ and misses $C$; i.e., $S_J \subset (\overline{B'} - C)$. By standard cut and paste techniques it is
possible to cut off $D$ near $S_J$ in $A'$. (Consider the disks in $D$ bounded by curves
in $S_J \cap D$. Those which are not contained in the interior of any other such disk
are replaced by disks in $A'$ parallel to subdisks of $S_J$.) Call the new disk thus
obtained $D_1$ and note that $D_1$ meets $S^\ast$ transversely and in fewer components
than did $D$. Let $C_1$ denote the component of $\overline{B'} - D_1$ which contains $C$. Note
that if $D_1 - D$ is chosen sufficiently close to $S_J$, then the arc $\overline{ab}$, except for its
endpoints, misses $D_1 \cup \overline{B'}$ and for some $\epsilon_1 > 0$, $A' \cap N[D_1 \cup (S^\ast - C_1), \epsilon_1] \subset A' \cap N[D \cup (S^\ast - C), \epsilon]$. By inductive hypothesis, there is a nonsingular polyhedral
disk $D'$ such that

(a) $Bd D' = Bd D_1 = Bd D$ and
(b) $D' \subset A' \cap N[D_1 \cup (S^\ast - C_1), \epsilon_1] \subset A' \cap N[D \cup (S^\ast - C), \epsilon]$, as desired.

**Case 2.** $D_J \subset \overline{A'}$. In this case let $S_J$ be the disk in $S^\ast$ bounded by $J$ whose
interior is separated in $A'$ from the arc $\overline{ab}$ by the disk $D_J$. Let $S_J'$ be a polyhedral
disk in $S^\ast$ which is very close to $S_J$ homeomorphically and which contains $S_J$ in
its interior. Let $S'$ be a polyhedral disk in $\bar{A} - D$ which is very close to $D_j$ homeomorphically, lies, except for its boundary, in $A'$, and has the same boundary as $S_j$. Let $S_1$ be the polyhedral 2-sphere $(S' - S_j) \cup S''$. Let $A_1 = \text{Int } S_1$, $B_1 = \text{Ext } S_1$, and $C_1 =$ component of $\bar{B}_1 - D$ which contains $C$. It is easy to check that $\text{Bd } D \subset A_1 \subset A'$ (since $\text{Int } \bar{a}_b \subset A_1$); that $D$ intersects $S_1$ transversely and in fewer components than it intersected $S$; that $\bar{a}_b$ is an arc from $\text{Bd } D$ to $C_1$ which, except for its endpoints, misses $D \cup \bar{B}_1$; and finally, that if $S''$ is close enough to $D_j$, then $A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$ for some $\epsilon_1 > 0$. Again the inductive hypothesis applies and supplies a polyhedral disk $D'$ such that

(a) $\text{Bd } D' = \text{Bd } D$ and
(b) $D' \subset A_1 \cap N[D \cup (S_1 - C_1), \epsilon_1] \subset A' \cap N[D \cup (S' - C), \epsilon]$, as desired.

Cases 1 and 2 complete the inductive proof of Lemma 1.

We now return to the proof that $(2) \implies (1)$. Bing has proved [1] that a 2-sphere $S$ in $E^3$ is tame from a complementary domain $A$ if $A$ is 1-ULC. His proof is easily seen to be valid if the following condition replaces the 1-ULC condition.

Condition *. Suppose $E$ is a disk in $S$ and $D$ is a polyhedral disk in $E^3$ such that $\text{Bd } D \subset A$ and $D \cap S \subset \text{Int } E$. Suppose further that $\text{Bd } D$ can be joined to $S - E$ by an arc $a = uv$ which lies, except for its endpoints $u \in \text{Bd } D$ and $v \in S - E$, in $E^3 - (S \cup D)$. Then, given $\epsilon > 0$, $\text{Bd } D$ bounds a disk $D'$ in $A \cap N[D \cup E, \epsilon]$. Theorem 1 will be established once we prove that statement (2) of Theorem 1 implies that Condition * is satisfied. Suppose therefore that $E$, $D$, $a$, and $\epsilon$ are given as in Condition *.

We first wish to apply statement (2). To this end, choose a point $w \in \text{Int } a$, and let $a_1 = \bar{w}w$ and $a_2 = w\bar{w}$ denote the two arcs into which $w$ divides $a = \bar{w}w$. Choose a positive number $\delta$ such that

$$\delta < \min \{\epsilon/2, \rho(a_2, D \cup E), \rho(S, \text{Bd } D \cup a_1)\}.$$  

Let $F = A - N(S, \delta)$. Statement (2) of Theorem 1 implies that there is a 2-sphere $S'$ in $E^3$ having complementary domains $A' = \text{Int } S'$, $B' = \text{Ext } S'$, such that (i) $F \subset A' \subset \bar{A} \subset A$, and (ii) for each $x \in S'$ there exists a path $\bar{x}y$ contained in $\bar{B}'$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less than $\rho(F, S)$. We now wish to apply Lemma 2. Lemma 1 shows that we may choose $S'$ to be polyhedral and to meet $D$ transversely. Note that $\text{Bd } D \subset F \subset A'$. Let $C$ be the component of $\bar{B}' - D$ which contains $S - E$. Let $x$ be the first point of $a = \bar{w}w$ which lies in $S'$. We must necessarily have $x \in \text{Int } a_1$ since $\delta < \rho(S, a_1)$. We claim that $x \in C$. Indeed, let $\bar{x}y'$ be a path contained in $\bar{B}'$ which connects
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$x$ to $S$ and has diameter less than $\delta$. Since $x \in \alpha_2$ and $\delta < \rho(\alpha_2, D \cup E), y \in S - E \subseteq C$. Thus $x \in \overline{xy} \subseteq C$, as claimed. We have established therefore that there is an arc $\overline{xy}$ from $\text{Bd} D$ to $C$ which, except for its endpoints, misses $D \cup \overline{B'}$. Thus Lemma 2 applies and yields a polyhedral disk $D'$ such that

(a) $\text{Bd} D' = \text{Bd} D$ and
(b) $D' \subset A' \cap N[D \cup (S' - C), \delta]$.

Since $A' \subseteq A$, we will be done once we have shown that $N[D \cup (S' - C), \delta] \subseteq N[D \cup E, \epsilon]$, or, since $\delta < \epsilon/2$, that $(S' - C) \subseteq N[D \cup E, \delta]$. Let $z \in S' - C$ and let $\beta$ be an arc in $\overline{D'}$ of diameter less than $\delta$ which joins $z$ to $S$. If $\beta$ misses both $D$ and $E$, then $z$ is in the same component of $\overline{B'} - D$ as is $S - E$; i.e., $z \in C$, a contradiction. Hence $\beta \cap (D \cup E) \neq \emptyset$, and $z \in N[D \cup E, \delta]$. This completes the proof that statement (2) of Theorem 1 implies Condition *.

As noted earlier, this also completes the proof that $S$ is tame from $A$. We can use the foregoing proof to deal with tameness from $B$ by simply forming the one-point compactification $S^3$ of $E^3$ and then removing a point from $A$ to form $E^3$ again. □

3. Applications of Theorem 1. The following theorem is an almost immediate result of Theorem 1. If $X, Y$ are subsets of $E^3$, define the metric $\epsilon$-envelope of $Y$ in $X$ to be the set $\{x \in X | \rho(x, Y) = \epsilon\}$.

**Theorem 2.** Let $S$ be a 2-sphere embedded in $E^3$ with $A = \text{Int} S, B = \text{Ext} S$. Let $\rho$ be the usual $E^3$ metric. Suppose that for each $\alpha > 0$ there exists a real $\epsilon$ with $0 < \epsilon < \alpha$ such that the metric $\epsilon$-envelope of $S$ in $A$ is a 2-sphere embedded in $E^3$. Then $S$ is tame from $A$. The implication also holds if $A$ is replaced by $B$ in the last two statements.

**Proof.** $S$ will be proven tame from $A$ by showing that it satisfies condition (2) of Theorem 1. The proof for tameness from $B$ is similar.

Let $F \subseteq A$ be a compact set. By hypothesis there exists a real number $\epsilon$ such that $0 < \epsilon < \rho(F, S)$ and the set $S' = \{x \in A | \rho(x, S) = \epsilon\}$ is a 2-sphere in $A$. Let $A' = \text{Int} S', B' = \text{Ext} S'$. We first show that $F \subseteq A' \subseteq \overline{A'} \subseteq A$. Since $S' \subseteq A, \overline{B}$ is contained in one of the complementary domains of $S'$. Since $B$ is unbounded, we must have $B \subseteq \overline{B} \subseteq B'$. It follows immediately that $A' \subseteq \overline{A'} \subseteq A$. To show that $F \subseteq A'$, let $x$ be a point of $F$. Since $\rho(x, S) > \epsilon$, $x \notin S'$. Let $\beta$ be any path in $E^3$ which starts at $x$ and ends at some point $z \in S \subseteq \overline{B} \subseteq B'$. Since $\rho(x, S) > \epsilon$ and $\rho(x, S) = 0$, there is some point $y$ on $\beta$ such that $\rho(y, S) = \epsilon$ by the intermediate value property. Then $y \in S'$, which implies that every path from $x$ to $z \in B'$ must intersect $S'$. Since $B'$ is path-connected and $x \notin S'$, we conclude that $x \in A'$. Therefore $F \subseteq A' \⊆ \overline{A'} \subseteq A$.

It remains to show that for each $x \in S'$ there exists a path $\overline{xy}$ contained in $\overline{B'}$ having initial point $x \in S'$, terminal point $y \in S$, and whose diameter is less
than \( \rho(F, S) \). Let \( x \in S' \). Since \( S \) is compact and \( \rho(x, S) = \epsilon \), there is a point \( y \in S \) such that \( \rho(x, y) = \epsilon \). Consider the path \( \overline{xy} \) formed by the straight line segment running from \( x \) to \( y \). The diameter of this path is \( \epsilon < \rho(F, S) \), and all we need to show is that it lies in \( B' \). Suppose not. Then some point \( w \) strictly between \( x \) and \( y \) on \( \overline{xy} \) must lie in \( A' \). Since \( y \in S \subseteq B' \), there is a point \( z \) strictly between \( w \) and \( y \) on \( \overline{xy} \) which lies on \( S' \). But this results in a contradiction because \( z \in S' \Rightarrow \rho(z, S) = \epsilon \) but \( z \) is strictly between \( x \) and \( y \) on \( \overline{xy} \) which implies that \( \rho(z, y) < \epsilon \Rightarrow \rho(z, S) < \epsilon \). \( \square \)

The converse of Theorem 2 is clearly not true. Theorem 2 gives rise to the following question: If the metric \( \epsilon \)-envelope of a set \( X \) in \( E^3 \) is a 2-sphere \( S \), is \( S \) tame? Partial answers can be obtained if \( X \) lies in one of the complementary domains \( C \). In this case it is clear that for each point \( x \in S \), there exists a round tangent ball in \( S \cup C \) which touches \( S \) only at \( x \). Loveland [6, p. 396] has asked if this makes \( S \) tame, and Cannon [4, pp. 444–445] proved that \( S \) is tame from \( E^3 - C \) under this condition. If \( X \subseteq \text{Int } S \) and \( \epsilon > \text{diam } X \) then each point of \( S \) is visible from a point \( x \in X \). Cobb [5] shows that \( S \) is then tame in \( E^3 \). His proof appears in [3, pp. 326–327].

Define the spherical diameter of a set \( X \subseteq E^3 \) to be the diameter of the smallest closed round ball containing \( X \). (For a given set, the ratio \( r \) of the spherical diameter to the usual diameter satisfies \( 1 < r < \sqrt{3/2} \).)

All of the commonly known wild spheres appear to have the property that one can find two points \( x, y \) on the sphere which are arbitrarily close together such that any arc on the sphere having \( x \) and \( y \) as endpoints must have a spherical diameter greater than \( \rho(x, y) \). Must every wild sphere have this property? It seems reasonable that such a property might result from the rather severe entanglement in \( E^3 \) which is characteristic of wild spheres. The following lemmas are used in Theorem 3 which answers the question affirmatively. The proof of Lemma 3 is a simple geometrical argument and is therefore omitted.

**Lemma 3.** Let \( P \) be a solid, closed, rectangular parallelepiped. Let \( R \) be the union of all closed, round balls having a diameter which is an edge, a face diagonal, or a principal diagonal of \( P \). Let \( a, b \) be distinct points in \( P \). Then any path \( \alpha \) from \( a \) to \( b \) which has spherical diameter equal to \( \rho(a, b) \) must lie in \( R \).

**Lemma 4.** Suppose \( M \) is a Euclidean polyhedron in \( E^3 \) which is connected but not simply connected. Let \( \beta \) be a positive real number. Then there exist two simple closed curves \( K \subseteq M, H \subseteq E^3 - M \) such that neither is null-homotopic in the complement of the other. Furthermore \( H \) may be chosen to lie in the \( \beta \)-neighborhood of \( M \).

**Proof.** Let \( U_\beta \) denote the \( \beta \)-neighborhood of \( M \). Let \( N \subseteq U_\beta \) be a regular
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neighborhood of $M$. Each component of $\text{Bd} N$ is a p.l. 2-manifold without boundary. Since $M$ is not simply connected, neither is $N$. Therefore, some component $C$ of $\text{Bd} N$ is not simply connected. The fundamental theorem of compact surfaces states that $C$ is either a 2-sphere or the connected sum of a finite number of tori. The first possibility is ruled out, so there is a subset $\hat{C}$ of $C$ which is a torus $T$ minus the interior of a disk $D \subset T$. Select two polygonal simple closed curves $H$ and $K$ on $\hat{C}$ which intersect each other transversely and at a single point on $\hat{C}$. A simple linking argument shows that either $H$ pushed slightly into $U_{\beta} - N$ links $K$ homologically in $E^3$ or $K$ pushed slightly into $U_{\beta} - N$ links $H$ homologically in $E^3$. Interchanging the names of $H$ and $K$ if necessary, we may assume the former. $K$ can be homotopically pushed into $M$ via a collapsing of $N$ into $M$. At this point neither of $K, H$ is null-homotopic in the complement of the other, although $K$ may not simple. However, some subset of $K$ is a simple closed curve satisfying the non null-homotopic condition. Taking $K$ to be this curve completes the proof. □

**Theorem 3.** Let $S \subset E^3$ be a 2-sphere with $\rho$ the usual $E^3$ metric. Suppose there exists an $\epsilon > 0$ such that any two points, $a, b \in S$ satisfying $\rho(a, b) < \epsilon$ can be joined by a path in $S$ of spherical diameter $= \rho(a, b)$. Then $S$ is tame in $E^3$.

**Proof.** The proof deals with tameness from $A = \text{Int} S$. That $S$ is tame from $B = \text{Ext} S$ can be proved similarly. Let $F \subset A$ be compact with $\eta = \rho(F, S)$. Consider the solid, closed cubes in $E^3$ of edge length $\epsilon < \min \{\eta / 4, \epsilon / \sqrt{6}\}$ whose vertices have coordinates of the form $(me, ne, pe)$ where $m, n, p$ are integers. The word "cube" will refer to one of these cubes unless otherwise stated. The non-empty union $T$ of all cubes lying entirely in $A$ contains $F$, and $S$ is accessible from each point of $\text{Bd} T$ via a path in $A - \overline{T}$ of diameter $< \eta$. The object is to change $T$ into a polyhedral 3-cell $B$ which retains these two properties. Then $\text{Bd} B$ will be a 2-sphere $S'$ satisfying statement (2) of Theorem 1, proving that $S$ is tame from $A$. $T$ will first be modified into $T''$ so that each component of $T''$ becomes a polyhedral 3-cell. $B$ is then easily constructed by connecting the components with slightly thickened polygonal arcs in $A - \text{Int} T''$. Each component $T_i$ of $T$ has a connected complement and therefore can fail to be a 3-cell by either not being simply connected or by not being a 3-manifold-with-boundary. The first of these difficulties is corrected by removing from each $T_i$ neighborhoods of constriction points of $T_i$ as shown in Figure 1. Such a point $x$ is a vertex of exactly two cubes $M, N$ lying in $T_i$ with $M \cap N = \{x\}$. The resulting modified version $T'$ of $T$ retains the two properties of $T$ mentioned previously. For purposes of continuity, the proof that each component $T'_i$ of $T'$ is simply connected will be deferred until later. Figure 2 shows how each $T'_i$ is then made into a polyhedral 3-manifold with boundary, hence a polyhedral 3-cell $B'_i$, by attaching to the concave "troughs" of $\text{Bd} T'_i$ small cubes of edge length $\epsilon / m$. 

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where \( m > 2 \) satisfies \( e/m < \rho(T, S) \). The \( B_j \) are disjoint and their union \( T'' \) retains the two properties of \( T \) mentioned previously. Connecting the \( B_j \) with fattened polygonal arcs in \( A - \text{Int } T'' \) provides the desired 3-cell \( B \), and the proof is complete except for the argument below.

**Figure 1**

**Figure 2**

Proof that each \( T'_j \) is simply connected. Assume not. Choose \( \beta < \rho(T, S) \).

By Lemma 4, there exist polygonal simple closed curves \( H_1 \subset E^3 - T' \) and \( K_1 \subset T'_1 \) such that \( H_1 \) lies in the \( \beta \)-neighborhood \( N_j \) of \( T'_1 \) and neither curve is null-homotopic in the complement of the other. \( H_1 \) can be chosen to miss \( T \) by moving it, if necessary, in \( N_j - T'_1 \) so that it fails to intersect any of the neighborhoods which earlier were removed from each \( T'_i \). \( \beta \) has been chosen small enough to assure that each cube in the union \( V \) of all cubes intersecting \( H_1 \) will contain points of \( S \), and only the boundary of each cube will intersect \( T \). Because \( H_1 \subset \text{Int } V \), \( H_1 \) can be moved slightly in \( V - T \) so that it intersects no edge of any cube. Thus \( H_1 \) is now the union of \( m \) polygonal arcs laid end-to-end with each arc \( \gamma_i \) satisfying the following: \( \gamma_i \subset Q_i \cup Q_{i+1} \), where \( Q_i, Q_{i+1} \) are cubes from \( V \) which intersect in a common face, and the endpoints of \( \gamma_i \) lie in \( Q_i, Q_{i+1} \) respectively. For each \( i, Q_i \) and \( Q_{i+1} \) each contain points of \( S \), so \( H_1 \) can be further moved in \( V - T \) so that the arcs \( \gamma_i \) change into straight line segments \( \alpha_i \) satisfying \( \alpha_i \subset Q_i \cup Q_{i+1} \) with the endpoints \( x_i, x_{i+1} \) of \( \alpha_i \) being points of \( S \) lying in \( Q_i, Q_{i+1} \) respectively. Since \( \rho(x_i, x_{i+1}) < \beta \), the hypothesis of the theorem implies that there is a path \( \Gamma_i \subset S \) of spherical diameter \( \rho(x_i, x_{i+1}) \) joining \( x_i \) and \( x_{i+1} \). Because the closed path \( \Gamma = (\bigcup_{i=1}^m \Gamma_i) \subset S \) is null-homotopic in \( S \), hence in \( E^3 - K_1 \), there is some \( k \) such that the path \( G_k = \alpha_k \cup \Gamma_k \) is not null-homotopic in \( E^3 - K_1 \). The spherical diameter of \( G_k \) is \( \rho(x_k, x_{k+1}) \). By Lemma 3, \( G_k \) lies in the 3-cell \( R \) formed by the union of all closed round balls whose diameter is an edge, a face diagonal, or a principal diagonal of \( Q_1 \cup Q_{k+1} \). Figure 3 indicates that all cubes except \( Q_k \) and \( Q_{k+1} \) (hence all cubes in \( T \)) fall into 6 classes depending upon
the nature of their intersection with $R$. The darkened edges of each representative cube $L$ are those which miss Int $R$, and $E_L$ denotes the union of these edges. If two such cubes $L, M$ meet in a common edge or face, then $E_L \cup E_M$ is arc-connected.

Figure 3

The previous removal of neighborhoods in $T$ of constriction points assures that the closed curve $K_1 \subset T' \subset T$ mentioned previously can be chosen to intersect no vertex of any cube, and therefore can be traversed by passing through a sequence of cubes in $T$, each cube $N_i$ intersecting its predecessor in a common face or edge. Since $E_{N_i} \cup E_{N_{i+1}}$ is arc-connected, a further adjustment of $K_1$ in $T$ makes $K_1 \cap N_i \subset E_{N_i}$ for each $i$. $K_1$ now misses Int $R$. Now $G_k \subset R$, $G_k$ misses $K_1$, and is not null-homotopic in $E^3 - K_1$. But $R$ is a 3-cell, so $G_k$ can be contracted in $R$ radially inward to a point without hitting $K_1$. This contradiction completes the argument that $T_1'$ is simply connected. □

The converse of this theorem is clearly false. It might be possible to strengthen the theorem by showing that there is some constant $K > 1$ such that tameness is implied if the path from $x$ to $y$ mentioned in the hypothesis has a spherical diameter equal to $K \rho(x, y)$. This leads to the problem of finding the least upper bound for such a constant. The proof of Theorem 3 breaks down if $K > 1$ because then it can no longer be guaranteed that $E_M \cap E_N$ is connected when the cubes $M, N$
intersect in a common edge or face. The theorem might be true if "diameter" replaces "spherical diameter", but neither a proof nor a counterexample has been found.

Another question related to Theorem 3 is the following: If $S$ is a 2-sphere in $E^3$ and $C$ is one of its complementary domains, define a chord of $C$ to be a straight line segment lying in $C$ having its endpoints in $S$. Is $S$ tame from $C$ if there exists an $\epsilon > 0$ such that for each chord of $C$ of length $l < \epsilon$ there is an arc in $S$ of spherical diameter $= l$ which connects the endpoints of the chord?

REFERENCES

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