PROPERTY $SUV^\infty$ AND PROPER SHAPE THEORY

BY

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ABSTRACT. A class of spaces called the $SUV^\infty$ spaces has arisen in the study of a possibly noncompact variant of cellularity. These spaces play a role in this new theory analogous to that of the $UV^\infty$ spaces in cellularity theory. Herein it is shown that the locally compact metric space $X$ is an $SUV^\infty$ space if and only if there exists a tree $T$ such that $X$ and $T$ have the same proper shape. This result is then used to classify the proper shapes of the $SUV^\infty$ spaces, two such being shown to have the same proper shape if and only if their end-sets are homeomorphic. Also, a possibly noncompact analog of property $UV^n$, called $SUV^n$, is defined and it is shown that if $X$ is a closed connected subset of a piecewise linear $n$-manifold, then $X$ is an $SUV^n$ space if and only if $X$ is an $SUV^\infty$ space. Finally, it is shown that a locally finite connected simplicial complex is an $SUV^\infty$ space if and only if all of its homotopy and proper homotopy groups vanish.

1. Introduction. An important notion in the study of embeddings of compact metric spaces has been that of property $UV^\infty$. (Definitions of this and other terms used in this introduction, as well as references, are included in the text.) This notion first arose in the study of cellularity, and is connected with shape theory through the result that the compact metric space $X$ has property $UV^\infty$ if and only if $X$ has the shape of a singleton.

In studying a generalization of cellularity, called quasi-cellularity, to noncompact sets, Hartley defined a possibly noncompact variant of property $UV^\infty$, called property $SUV^\infty$, and showed that property $SUV^\infty$ plays a role in quasi-cellularity theory analogous to that played by property $UV^\infty$ in cellularity theory.

Our main result connects property $SUV^\infty$ with proper shape theory. In particular, we show, in §3, that if $X$ is a locally compact metric space, then $X$ has property $SUV^\infty$ if and only if $X$ has the proper shape of a locally finite connected and simply connected 1-complex. We use this result, and some of the results of §2, to give a complete classification of proper shapes among the $SUV^\infty$ spaces.

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via the well-known theory of ends. To be precise, we show that the $SUV^\infty$ spaces $X$ and $Y$ have the same proper shape if and only if their end-sets are homeomorphic. In §4 we define a property, called $SUV^n$, which is analogous to property $UV^n$ for compact metric spaces, and show how this leads to a characterization of the finite-dimensional $SUV^\infty$ spaces. Finally, we show that a locally finite connected simplicial complex has property $SUV^\infty$ if and only if its homotopy and proper homotopy groups of all dimensions vanish.

We remark that all spaces considered in this paper are metrizable. In particular, ANR and AR mean, respectively, absolute neighborhood retract and absolute retract for metric spaces.

I would like to express my thanks to Professor B. J. Ball for his valuable remarks on a draft version of this paper. In particular, he pointed out the existence of Kuperberg's paper [9] and how the notion of unstable set could be used to simplify the statement of Lemma (2.1).

2. Trees and proper maps. A locally finite, connected and simply connected simplicial 1-complex is called a tree. In this section we shall classify (in Theorem (2.3)) the proper homotopy classes of maps of a (sufficiently nice) space into a tree and use this classification to study the relation of proper homotopy equivalence on the class of all trees. While our techniques are applicable to some other special types of spaces, we shall restrict ourselves to trees for convenience.

Perhaps we should recall that a map $f: X \to Y$ is proper if $f^{-1}(C)$ is compact for every compact set $C \subseteq Y$, and that maps $f: X \to Y$ and $g: X \to Y$ are properly homotopic if there exists a proper map $\Phi: X \times I \to Y$ such that $\Phi(x, 0) = f(x)$ and $\Phi(x, 1) = g(x)$ for all $x \in X$. ($I$ denotes the interval $[0, 1]$.) The latter determines an equivalence relation on the proper maps from $X$ to $Y$, and we write $f \simeq_p g$ provided $f$ and $g$ are properly homotopic. We also recall that spaces $X$ and $Y$ are said to be of the same proper homotopy type if there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $gf \simeq_p i_X$ and $g \simeq_p i_Y$. ($i_Z$ denotes the identity map on the space $Z$.) In this case we write $X \simeq_p Y$. If only $gf \simeq_p i_X$ is known to hold, we say that $Y$ properly homotopically dominates $X$ and we write $X \leq_p Y$ or $Y \geq_p X$.

Suppose $X$ is a space and $X' \subseteq X$. Then $X'$ is an unstable subset of $X$ if there exists a homotopy $H: X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$ and $H(x, t) \in X - X'$ for all $x \in X$ and $0 < t \leq 1$. The following lemma will be crucial to the proofs of Theorems (2.3) and (2.4) (cf. Theorem 2 of [9]).

(2.1) Lemma. Suppose $X$ is a compact metric absolute retract and $X'$ is an unstable subset of $X$. Then if $A$ is a compact metric space, $B$ is a closed subset of $A$, and $f: B \to X$ is a map, there exists a map $f^*: A \to X$ such that $f^*|B = f$ and $f^*(A - B) \subseteq X - X'$.
Proof. Let \( H: X \times I \rightarrow X \) be a homotopy such that \( H(x, 0) = x \) for all \( x \in X \) and \( H(x, t) \in X - X' \) for all \( x \in X \) and \( 0 < t \leq 1 \). Let \( \phi: A \rightarrow I \) be a map such that \( \phi(a) = 0 \) if and only if \( a \in B \), and let \( \tilde{f}: A \rightarrow X \) be an extension of \( f \). Now, define \( f^*: A \rightarrow X \) by \( f^*(a) = H(\tilde{f}(a), \phi(a)) \). Then \( f^* \) is easily seen to have the desired properties.

We now briefly discuss what will prove to be the key geometric idea behind our classification scheme—the notion of "ends". Suppose \( X \) is a semicompact (rim compact, locally peripherally compact) space. Then the Freudenthal compactification of \( X \), here denoted \( FX \), is the least upper bound of all compactifications \( Y \) of \( X \) such that \( \text{ind}(Y - X) = 0 \). In order that \( FX \) be metrizable, it is necessary and sufficient that \( X \) be separable and metrizable, and that \( QX \), the space of quasi-components of \( X \), be compact. For a more complete discussion of \( FX \), see [6] or [8]. We shall refer to \( FX - X \) as the space of ends of \( X \), here denoted by \( EX \). We note that \( EX \) is homeomorphic with a closed subset of the Cantor set.

Suppose \( X \) and \( Y \) are semicompact separable metric spaces, \( QX \) and \( QY \) are compact, and \( f: X \rightarrow Y \) is a proper map. Then \( f \) has a unique extension to a map of pairs \( Ff: (FX, EX) \rightarrow (FY, EY) \). If \( g: X \rightarrow Y \) is a proper map and \( f \cong_p g \), then \( Ff \mid EX = Fg \mid EX \). Also, the assignment \( f \rightarrow Ff \) is functorial; that is, \( F(i_X) = i_{FX} \) and, if \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) are proper maps, then \( F(gf) = (Fg)(Ff) \). These facts are discussed in some detail in §4 of [3] (see, particularly, Lemmas 4.2 and 4.3).

Now suppose \( T \) is a tree. Then the following facts are easily established:

(i) \( FT \) is a compact metric absolute retract;

(ii) \( FT \) is contractible to a point via a homotopy \( K: FT \times I \rightarrow FT \) such that \( K(x, t) \notin ET \) for all \( x \in FT \) and \( 0 < t \leq 1 \).

Thus

(2.2) Lemma. If \( T \) is a tree, then \( FT \) is a compact metric absolute retract and \( ET \) is an unstable subset of \( FT \).

We now give the main result of this section, which states that, under suitable restrictions on \( X \), the proper homotopy classes of maps of \( X \) into the tree \( T \) are in 1-1 correspondence with the space \( ET^{EX} \).

(2.3) Theorem. Suppose \( X \) is a locally compact separable metric space such that \( QX \) is compact. Suppose further that \( T \) is a tree and that \( g: X \rightarrow T \) and \( h: X \rightarrow T \) are proper maps. Then \( g \cong_p h \) if and only if \( Fg \mid EX = Fh \mid EX \).

Furthermore, if \( k_0: EX \rightarrow ET \) is a map, then there exists a proper map \( k: X \rightarrow T \) such that for all \( e \in EX \), \( k_0(e) = Fk(e) \).

Proof. Suppose first of all that \( g \cong_p h \). Then (as noted in the remarks following Lemma (2.1)) by Lemma 4.3 of [3], \( Fg \mid EX = Fh \mid EX \).
Suppose then that $Fg \mid EX = Fh \mid EX$. Let $A = FX \times I$ and let $B = (FX \times \{0\}) \cup (EX \times I) \cup (FX \times \{1\})$. Define $f : B \to FT$ by

$$f(b) = \begin{cases} Fg(z) & \text{if } b = (z, 0) \in FX \times \{0\} \\ Fg(z) = Fh(z) & \text{if } b = (z, t) \in EX \times I \\ Fb(z) & \text{if } b = (z, 1) \in FX \times \{1\}. \end{cases}$$

Then $f$ is continuous and, by Lemmas (2.2) and (2.1), there exists a map $f^* : A \to FT$ such that $f^*|B = f$ and $f^*(A - B) \subset FT - ET$. Now, let $G : X \times I \to T$ be defined by $G(x, t) = f^*(x, t)$ for all $(x, t) \in X \times I$. Then $G$ is a proper homotopy joining $g$ and $b$.

Finally, suppose $k_0 : EX \to ET$ is a map. Since $EX$ is a compact subspace of $FX$, by Lemmas (2.2) and (2.1), there exists a map $\hat{k} : FX \to FT$ such that $\hat{k}(e) = k_0(e)$ for all $e \in EX$ and $\hat{k}(X) \subset FT - ET = T$. Define $k : X \to T$ by $k(x) = \hat{k}(x)$ for all $x \in X$. Then $k$ is proper and, if $e \in EX$, $Fk(e) = \hat{k}(e) = k_0(e)$.

Given a compact 0-dimensional metric space $D$, it is easy to construct a tree $T$ such that $ET \approx D$. (We use $\approx$ for the relation of homeomorphism.) This, along with the next result, implies that the proper homotopy types of trees are in 1-1 correspondence with the homeomorphism classes of closed subsets of the Cantor set.

(2.4) Theorem. Suppose $T_1$ and $T_2$ are trees. Then $T_1 \approx_p T_2$ if and only if $ET_1 \approx ET_2$.

Proof. Suppose first of all that $T_1 \approx_p T_2$. Then, by Corollary 4.9 of [3], $ET_1 \approx ET_2$. (The proof is an easy consequence of the facts remarked on after Lemma (2.1).)

Now suppose $ET_1 \approx ET_2$. Let $b : ET_1 \to ET_2$ be a homeomorphism and denote $b^{-1}$ by $g$. By Lemma (2.1), there exists a proper map $b^* : T_1 \to T_2$ such that if $e \in ET_1$, $Fb^*(e) = b(e)$, and a proper map $g^* : T_2 \to T_1$ such that if $e \in ET_2$, $Fg^*(e) = g(e)$. If $e \in ET_1$, then

$$Fi_{T_1}(e) = e = (gb)(e) = g(b(e)) = g(Fb^*(e)) = Fg^*(Fb^*(e)) = (Fg^*)(Fb^*)(e) = F(g^*b^*)(e).$$

Thus, $Fi_{T_1} \mid ET_1 = F(g^*b^*) \mid ET_1$ and, by Theorem (2.3), $i_{T_1} \approx_p g^*b^*$. Similarly $i_{T_2} \approx_p b^*g^*$, thereby showing that $T_1 \approx_p T_2$.

By the proof of Theorem (2.4), we also have the following.

(2.5) Corollary. Suppose $T_1$ and $T_2$ are trees. Then $T_1 \geq_p T_2$ if and only if $ET_2$ embeds in $ET_1$.

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(2.6) Corollary. Suppose $T_1$ and $T_2$ are trees. Then either $T_1 \succeq_p T_2$ or $T_2 \succeq_p T_1$.

Proof. By Corollary (2.5), it is only necessary to show that if $P$ and $Q$ are closed subsets of the Cantor set, then either $P$ embeds in $Q$ or $Q$ embeds in $P$. Suppose first that one of $P$ or $Q$, say $P$, is uncountable. Then $P$ contains a Cantor set, so $Q$ embeds in $P$. On the other hand, if $P$ and $Q$ are both countable, then each of $P$ and $Q$ is homeomorphic to a countable ordinal space, and it follows that one embeds in the other.

Finally, we remark that it is easy to construct trees $T_1$ and $T_2$ such that $T_1 \succeq_p T_2$ and $T_2 \succeq_p T_1$, but $T_1 \not\succeq_p T_2$. We simply construct $T_1$ so that $ET_1$ is homeomorphic to the Cantor set, and $T_2$ so that $ET_2$ is homeomorphic to the sum of the Cantor set and an isolated point.

3. Proper shape and property $SUV^\infty$. Suppose $M$ is a topological space and $X \subset M$. Then $X$ is said to have property $UV^\infty$ in $M$ if for each open set $U$ containing $X$, there is an open set $V$ containing $X$ such that $V \subset U$ and $V$ is contractible in $U$. This property has been particularly useful for compact $X$. See, for example, [1] or [12].

Property $UV^\infty$ is an embedding property, rather than an intrinsic one. For example, if $X$ denotes the "topologist's sin($1/x$) curve" in $R^2$, then $X$ has property $UV^\infty$ in $R^2$ but $X$ does not have property $UV^\infty$ in $X$. However, suppose $X$ is a closed subset of the ANR $M$ and $X$ has property $UV^\infty$ in $M$. It follows that if $X'$ is a closed subset of the ANR $M'$ and $X \approx X'$, then $X'$ has property $UV^\infty$ in $M'$ [12]. Thus it is customary to say $X$ has property $UV^\infty$, for which we shall write $X \in UV^\infty$, if $X$ embeds as a closed $UV^\infty$ subset of some ANR.

Now suppose $M$ is a space and $X \subset M$. Then $X$ is said to have property $SUV^\infty$ in $M$ if for each closed neighborhood $U$ of $X$, there exist a closed neighborhood $V$ of $X$ lying interior to $U$, a tree $T$, maps $f: V \rightarrow T$ and $g: T \rightarrow U$, and a proper homotopy $H: V \times I \rightarrow U$ such that $H_0$ is the inclusion and $H_1 = gf$. Since $f$ is onto and $H_1$ is proper, it easily follows that $f$ and $g$ are proper. As in the case of property $UV^\infty$, property $SUV^\infty$ is not intrinsic. However, having property $SUV^\infty$ is invariant under (closed) embeddings into locally compact ANR's [7], and we say that $X$ has property $SUV^\infty$, and write $X \in SUV^\infty$, provided that $X$ embeds as a closed $SUV^\infty$ subset of some locally compact ANR. It follows easily that if $X \in SUV^\infty$, then $X \in UV^\infty$ and, for $X$ compact, $X \in UV^\infty$ if and only if $X \in SUV^\infty$. However, there exist noncompact $UV^\infty$ spaces (e.g. the plane) which do not have property $SUV^\infty$.

For $X$ compact and finite dimensional, $X \in UV^\infty$ if and only if $X$ embeds as a cellular subset of a finite-dimensional manifold [10] (cf. also [11]). In [7] a
(possibly noncompact) version of cellularity, called quasi-cellularity, is defined
and studied. A compact subset of an n-manifold is quasi-cellular if and only if
it is cellular, and it is shown in [7] that for X locally compact and finite dimen-
sional, $X \in SUV^\infty$ if and only if $X$ embeds as a quasi-cellular subset of some
finite-dimensional manifold. Thus there is a strong parallel between properties
$UV^\infty$ and $SUV^\infty$ in the compact and noncompact cases, respectively. In this
section we show a further analog as described below.

For $X$ compact, $X \in UV^{\infty}$ if and only if $X$ has the shape of a point [4], [13].
We are now ready to state and prove our main result, namely that for $X$ locally
compact, $X \in SUV^{\infty}$ if and only if $X$ has the proper shape of a tree. (This will
be slightly strengthened in Corollary (3.5).) Following this we will apply the
results of §2 to give the proper shape classification of the $SUV^{\infty}$ spaces. For
notations and terminology associated with the theory of proper shape, see [2] or
[3].

(3.1) Theorem. Suppose $X$ is a locally compact metrizable space. Then
$X \in SUV^{\infty}$ if and only if there exists a tree $T_0$ such that $Sh_p X = Sh_p T_0$.

Proof. First suppose that there exists a tree $T_0$ such that $Sh_p X = Sh_p T_0$.
Let $Q$ denote the Hilbert cube and $K = Q \times [0, 1)$. Since $Sh_p X = Sh_p T_0$, $X$ is
connected, and it follows that $X$ embeds as a closed subset of $K$. Hence, we may
suppose that $X$ is a closed subspace of $K$ and that there exist proper fundamental
nets $f_\lambda = \{ f_{\lambda, A} | A \in \Lambda \}$ and $g_\delta = \{ g_{\delta, A} | \delta \in \Delta \}$ from $X$ to $T_0$ in $(K, T_0)$ and $T_0$ to $X$ in
$(T_0, K)$, respectively, such that $g_{\delta, A} \prec_p f_{\lambda, A}$ and $f_{\lambda, A} \prec_p g_{\delta, A}$ (see §5 of [3]).

To show that $X \in SUV^{\infty}$ it is sufficient to show that $X$ has property $SUV^{\infty}$
in $K$. To this end, let $U$ be a closed neighborhood of $X$. Since $g_{\delta, A} \prec_p f_{\lambda, A}$, there
exist a closed neighborhood $V$ of $X$ and indices $\lambda_0, \delta_0 \in \Lambda$, $\delta_0 \in \Delta$, such that if
$\lambda \geq \lambda_0$ and $\delta \geq \delta_0$, then $g_{\delta, \lambda} \prec_p f_{\lambda, \lambda} \prec_p i$, in $U$. Without loss of generality, we may
assume that $V$ is connected and lies in the interior of $U$. Let $H: V \times I \to U$ be a
proper homotopy such that $H_0$ is the inclusion and $H_1 = g_{\delta_0} f_{\lambda_0} | V$. Let $T =
H_0 (V)$. Since $V$ is connected, $T$ is homeomorphic to a tree and thus it follows
from the commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f_{\lambda_0}} & T \\
\downarrow{H_1} & & \downarrow{g_{\delta_0}} \\
U & \rightarrow & R_{f_{\lambda_0}}
\end{array}
\]

that $X$ has property $SUV^{\infty}$ in $K$. (It is worth noting here that the hypothesis
$f_{\lambda, A} \prec_p g_{\delta, A}$ remained unused. This will be remarked upon in Corollary (3.5)
below.)
Now suppose $X \in \text{SUV}^\infty$. Then $X$ is connected. Hence, $X$ embeds in $K$ and we may, in fact, suppose that $X$ is embedded in $K$ in such a way that the closure of $X$ in $Q \times I$ is $FX$. Let $T_0$ be a tree such that $ET_0 = EX$. We shall construct proper fundamental nets $f : X \to T_0$ in $(K, T_0)$ and $g : T_0 \to X$ in $(T_0, K)$ such that $fg \cong_p i_X$ in $(K, K)$ and $gf \cong_p i_{T_0}$ in $(T_0, T_0)$, thus showing that $Sh X = Sh T_0$ ($\S$ of [3]).

By Theorem (2.3), there exists a proper map $f : X \to T_0$ such that $Ff$ agrees with the identity on $EX$. Let $\{f^\delta\} = f : X \to T_0$ denote a proper fundamental net from $X$ to $T_0$ in $(K, T_0)$ generated by $f$ and having a single element.

Let $\Delta = \{U \subset K \mid U$ is a closed neighborhood of $X$ in $K$ and $EU = EX\}$. Then $\Delta$ is a cofinal system of closed neighborhoods of $X$ in $K$ and is a directed set under inclusion. Now fix $\delta = U \in \Delta$. Since $X \in \text{SUV}^\infty$, there exist a closed neighborhood $V_\delta$ of $X$ lying in the interior of $U$, a tree $T_\delta$, proper maps $f_\delta : V_\delta \to T_\delta$ and $g_\delta : T_\delta \to U$, and a proper homotopy $H_\delta : V_\delta \times I \to U$ such that $H_\delta^0$ is the inclusion and $H_\delta^1 = g_\delta f_\delta$. We may, without loss of generality, assume that $V_\delta \in \Delta$. Then, restricting domains and ranges, we have the commutative diagram

\[ EV_\delta \xrightarrow{FH_\delta} ET_\delta \]

But $EU = EV_\delta$ and, since $H_\delta^1 \cong_p i_{V_\delta}$ in $U$, $FH_\delta^1 = i_{EU} : EV_\delta \to EU$. It follows that $Ff_\delta$ (restricted) is a homeomorphism from $EV_\delta$ onto $ET_\delta$ with inverse $Fg_\delta$. By Theorem (2.3), there exists a proper map $h_\delta : T_0 \to T_\delta$ such that $Fh_\delta^\delta | ET_\delta = Ff_\delta^\delta | ET_\delta$. Now, let $g_\delta : T_0 \to K$ be defined by $g_\delta(x) = g_\delta(h_\delta(x))$ for all $x \in T_0$.

Note that if $e \in ET_\delta$, then

\[ Fg_\delta(e) = F(g_\delta h_\delta^\delta)(e) = (Fg_\delta h_\delta^\delta)(e) = (Fg_\delta h_\delta^\delta)(e) = e. \]

We shall now verify that $g = g_\delta \mid \delta \in \Delta$ is a fundamental net from $T_0$ to $X$ in $(T_0, K)$. To this end, let $U$ be a closed neighborhood of $X$. We wish to find an index $\delta_0 \in \Delta$ such that if $\delta \geq \delta_0$, then $g_\delta \cong_p g_\delta^0$ in $U$. Clearly there is no loss of generality in supposing that $U \in \Delta$. Now, since $X \in \text{SUV}^\infty$ there exist a closed neighborhood $V$ of $X$ lying in the interior of $U$, a tree $T$, proper maps $f : V \to T$ and $g : T \to U$, and a proper homotopy $H : V \times I \to U$ such that $H_0$ is the inclusion and $H_1 = \overline{g f}$. We may suppose $V \in \Delta$, and we take $\delta_0 = V$. Supposing now that $\delta \geq \delta_0$, we have that $g_\delta(T_0) \cup g_\delta^0(T_0) \subset V$ and that $Fg_\delta$ and $Fg_\delta^0$ agree on $ET_0$. By this latter statement and Theorem (2.3), $\overline{g_\delta \cong_p g_\delta^0}$.
in $T$. Since $g^f = H_1 \simeq_p i_V$ in $U$, we have $g_\delta \simeq_p g^f g_0 \simeq_p g^f$ in $U$. Thus

$$g_\delta \simeq_p g^f g_\delta \simeq_p g^f g_0 \simeq_p g_\delta,$$

all in $U$, completing the proof that $g$ is a fundamental net.

Now suppose $\delta \in \Delta$. Then, if $e \in ET_0$, $F(f^* g_\delta)(e) = (Ff^*)(Fg_\delta(e)) = (Ff^*)(e) = e$. Hence, by Theorem (2.3), $f^* g_\delta \simeq_0 i_{T_0}$, so that $f^* \simeq_0 i_{T_0}$ in $(T_0, T_0)$.

It remains only to show that $g^f \simeq_p i_X$ in $(K, K)$. Let $U$ be a closed neighborhood of $X$ in $K$. It is necessary to find $\delta_0 \in \Delta$ and a closed neighborhood $V$ of $X$ in $U$ such that for all $\delta \geq \delta_0$, $g_\delta f^* | V \simeq_p i_V$ in $U$. Again, there is no loss of generality in supposing that $U \in A$. Let $V = V_U$ and choose $\delta_0 \in \Delta$ such that if $\delta \geq \delta_0$, then $g_\delta(T_0) \subset V$. Recall that the diagram

$$\begin{array}{ccc}
V & \xrightarrow{i_U} & T_U \\
\downarrow & & \downarrow \\
H_1 & \simeq_p & g_U
\end{array}$$

is commutative. Now, suppose $\delta \geq \delta_0$. If $e \in EV$, then

$$F(i_U g^f | V)(e) = (Ff^*) (Fg_\delta)(e) = (Ff^*)(e) = F^e.$$  

Hence, by Theorem (2.3), $i_V \simeq_p g_U i_U \simeq_p g_U f^* | V$ in $T_U$. Thus $i_V \simeq_p g_U f^* | V$ all in $U$.

Now, using Theorem (3.1), the classification results of §2 can be immediately translated into results on $SUV^\infty$ spaces. For convenience, we make a listing of those restatements here. (Recall that if $X \in SUV^\infty$, then $X$ is locally compact and metrizable.)

(3.2) Corollary. Suppose $X_1$ and $X_2$ are $SUV^\infty$ spaces. Then $Shp X_1 = Shp X_2$ if and only if $EX_1 \simeq EX_2$.

(3.3) Corollary. Suppose $X_1$ and $X_2$ are $SUV^\infty$ spaces. Then $Shp X_1 \geq Shp X_2$ if and only if $EX_2$ embeds in $EX_1$.

In particular, if $X \in SUV^\infty$, then $Shp X \geq Shp [0, \infty)$.

(3.4) Corollary. Suppose $X_1$ and $X_2$ are $SUV^\infty$ spaces. Then either $Shp X_1 \geq Shp X_2$ or $Shp X_2 \geq Shp X_1$.

Finally, we return to the remark made following the first portion of the proof.
of Theorem (3.1). By this remark, if $T_0$ is a tree and $S_{b^p} X \leq S_{b^p} T_0$, then $X \in SUV^\infty$. Thus we have the following.

(3.5) Corollary. Suppose $X$ is a locally compact metrizable space. Then the following are equivalent.

(i) $X \in SUV^\infty$.
(ii) There exists a tree $T_0$ such that $S_{b^p} X = S_{b^p} T_0$.
(iii) There exists a tree $T_0$ such that $S_{b^p} X \leq S_{b^p} T_0$.

4. Other characterizations of $SUV^\infty$ spaces. Suppose $M$ is a space and $f: S^k \times J^+ \to V \subset U \subset M$ is a proper map, where $S^k$ is the $k$-sphere and $J^+$ denotes the discrete space of positive integers. Then $f$ is said to be properly $S^k$-inessential in $U$ if there exists a proper map $F: S^k \times J^+ \times I \to U$ such that

1. for all $x \in S^k$ and $i \in J^+$, $F(x, i, 0) = f(x, i)$, and
2. for all $i \in J^+$, $F|_{S^k \times \{i\} \times \{1\}}$ is a constant map.

Now, if $X \subset M$, then $X$ is said to have property $k - SUV$ in $M$ if for each closed neighborhood $U$ of $X$, there exists a closed neighborhood $V$ of $X$ lying in the interior of $U$ such that each proper map from $S^k \times J^+$ into $V$ is properly $S^k$-inessential in $U$. If $n$ is a positive integer and $X$ has property $k - SUV$ in $M$ for $1 \leq k \leq n$, then $X$ is said to have property $SUV^n$ in $M$ (cf. §3 of [1]). As expected, property $SUV^n$ is not intrinsic; however, the proof that property $SUV^\infty$ is invariant under embeddings into locally compact ANR’s is easily seen to show also that property $SUV^n$ is invariant under such embeddings. Hence, we write $X \in SUV^n$ if $X$ embeds as a closed $SUV^n$ subspace of some locally compact ANR.

It follows easily that $X \in SUV^\infty$ implies $X \in SUV^n$ for all positive integers $n$. We now state a partial converse, corresponding to Proposition 3.4 of [1].

(4.1) Theorem. Suppose $n$ is a positive integer, $X$ is a closed connected subset of the piecewise linear $n$-manifold $M$, and $X \in SUV^n$. Then $X \in SUV^\infty$.

Proof. We may suppose that $X$ is noncompact, for otherwise the result follows from Proposition 3.4 of [1]. Suppose $U$ is a closed neighborhood of $X$ in $M$.

Denote $U$ by $V_{n+1}$. By hypothesis, there exist closed neighborhoods $V_{n+1} \supset V_n \supset V_{n-1} \supset \cdots \supset V_1$ of $X$ in $M$ such that if $k \in \{1, 2, \ldots, n\}$, then each proper map from $S^k \times J^+$ into $V_k$ is properly $S^k$-inessential in $V_{k+1}$. We may also suppose that $V_1$ is a connected piecewise linear manifold with boundary. We shall show that there exist a closed tree $T \subset V_1$, a map $f: V_1 \to T$, and a proper homotopy $H: V_1 \times I \to U$ such that $H_0$ is the inclusion and $H_1 = g|_T$, where $g: T \to U$ is the inclusion of $T$ into $U$. It follows that $X$ has property $SUV^\infty$ in $M$, and hence that $X \in SUV^\infty$.

Let $T_0$ be a closed tree in $V_1$ embedded so that the inclusion of $T_0$ into
$V_1$ induces the identity map on $ET_0 = EV_1$. By Theorem (2.3), there exists a proper map $b: V_1 \to T_0$ such that if $e \in EV_1$, then $Fb(e) = e$. Let $T = b(V_1)$. Then, since $V_1$ is connected and $b$ is proper, $T$ is (topologically) a tree and $T$ is a closed subset of $U$. Let $f: V_1 \to T$ be defined by $f(x) = b(x)$ for all $x \in V_1$.

It now remains to construct $H$. Let $Q = V_1 \times \{0\} \cup V_1 \times \{1\} \subset V_1 \times I$, and define $H': Q \to U$ by

$$H'(x, t) = \begin{cases} x & \text{if } t = 0 \\ f(x) & \text{if } t = 1. \end{cases}$$

Then $H'$ is a proper map. Let $K^0 = \{v_1, v_2, v_3, \cdots\}$ denote the 0-skeleton of $V_1$. We begin the construction of $H$ by extending $H'$ to $H^0: Q \cup (K^0 \times I) \to U$ as described in the following paragraph.

Let $V_1 = \bigcup_{i=1}^{\infty} C_i$, where each $C_i$ is a compact subpolyhedron of $V_1$ and $\emptyset = C_1 \subset C_2 \subset C_3 \subset \cdots$. If $i \in J^+$, let $j_i$ be the largest positive integer such that there exists a path from $v_i$ to $f(v_i)$ in $V_1 - C_j$. The existence of $j_i$ for each $i \in J^+$ follows from the path-connectivity of $V_1$. Let $\alpha_i: I \to V_1$ be a path such that $\alpha_i(0) = v_i$, $\alpha_i(1) = f(v_i)$, and $\alpha_i(I) \cap C_j_i = \emptyset$. Define $H^0: Q \cup (K^0 \times I) \to U$ by

$$H^0(x, t) = \begin{cases} H'(x, t) & \text{if } (x, t) \in Q \\ \alpha_i(t) & \text{if } (x, t) = (v_i, t) \in K^0 \times I. \end{cases}$$

To show that $H^0$ is proper, it clearly suffices to show that if $i_0 \in J^+$, then $C_{i_0}$ intersects only finitely many of the sets $\{\alpha_1(I), \alpha_2(I), \cdots\}$. Suppose, by way of contradiction, that this is not the case, and let $i_1 < i_2 < i_3 < \cdots$ be positive integers such that $\alpha_{i_k} \cap C_{i_0} \neq \emptyset$ for all $k \in J^+$. We may suppose (by taking a subsequence if necessary) that the sequence $\{v_{i_k}\}_{k=1}^{\infty}$ converges to $e \in EV_1$. But then the sequence $\{f(v_{i_k})\}_{k=1}^{\infty}$ also converges to $e$. This implies that there exists a positive integer $N$ such that if $k \geq N$, then $v_{i_k}$ and $f(v_{i_k})$ lie in the same component (hence path component) of $V - C_{i_0}$, thus contradicting the choice of $\alpha_{i_N}, \alpha_{i_{N+1}}, \cdots$.

Now, let $K^i$ denote the $i$-skeleton of $V_1$, where $i \in \{0, 1, \cdots, n\}$. Suppose, inductively, that $H'$ has been extended to a proper map $H^i: Q \cup (K^i \times I) \to U$ such that the image of $H^i$ lies in $V_{j+i}$, where $j \in \{0, 1, \cdots, n - 1\}$. Since proper maps from $S^{i+1} \times I$ into $V_{j+i}$ are properly $S^{i+1}$-inessential in $V_{j+i}$, $H^i$ can be extended to a proper map $H^{i+1}: Q \cup (K^{i+1} \times I) \to U$ such that the image of $H^{i+1}$ lies in $V_{j+i+2}$. The extension $H^1: V_1 \times I \to U$ is the desired map $H$. 

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Suppose now that $X$ is a locally compact separable metric space of finite dimension. Let $d_X$ be the minimum of the set $\{ q \in J^+ | X \text{ embeds in some piecewise linear manifold of dimension } q \}$. Since $X$ embeds in Euclidean space of dimension $2 \dim X + 1$, $d_X$ exists and $d_X \leq 2 \dim X + 1$.

(4.2) Corollary. Suppose $X$ is a finite-dimensional locally compact connected metric space, and $X \in SUV^d$. Then $X \in SUV^\infty$.

We conclude this section by using a recent result of E. M. Brown to characterize the $SUV^\infty$ spaces within the class of connected locally finite simplicial complexes. If $K$ is a connected locally finite simplicial complex, $[a]$ is an end of $K$, $n \in J^+ \cup \{\infty\}$, then $\pi_n(K, a)$ denotes the $n$th proper homotopy group of $K$ based at $[a]$ as defined in [5].

(4.3) Theorem. Suppose $K$ is a connected locally finite simplicial complex. Then $K \in SUV^\infty$ if and only if

(i) for each $n \in J^+$, $\pi_n(K) = 0$, and

(ii) for each $n \in J^+ \cup \{\infty\}$ and end $[a]$ of $K$, $\pi_n(K, a) = 0$.

Proof. Suppose first that $K \in SUV^\infty$. Then there exists a tree $T$ such that $Sb_p K = Sb_p T$. By Theorem 3.12 of [3], $K \simeq_p T$. The result now follows from the easily verified fact that $\pi_n(T) = 0$ for all $n \in J^+$ and $\pi_n(T, b) = 0$ for all $n \in J^+ \cup \{\infty\}$ and end $[b]$ of $T$.

Now suppose that (i) and (ii) hold. Let $T$ be a tree such that $ET \simeq EK$. By Theorem (2.3), there exists a proper map $f: K \to T$ such that $Ff$ (restricted) is a homeomorphism of $EK$ onto $ET$. Then, if $n \in J^+$, $\pi_n(T) = 0$ and, if $n \in J^+ \cup \{\infty\}$ and $[b]$ is an end of $T$, $\pi_n(T, b) = 0$. Hence, by the "proper Whitehead theorem" of [5], $f$ is a proper homotopy equivalence. Then $K \simeq_p T$ and, by Theorem 3.12 of [3], $Sb_p K = Sb_p T$. By Theorem (4.1), $K \in SUV^\infty$.

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