A SIEGEL FORMULA FOR ORTHOGONAL GROUPS
OVER A FUNCTION FIELD

BY

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ABSTRACT. We obtain a Siegel formula for a quadratic form over a function field, by establishing the convergence of the corresponding Eisenstein-Siegel series directly, then via the Hasse principle, that of the associated Poisson formula.

Introduction. In this paper, we obtain a Siegel formula, as recast by Weil [7], for a quadratic form over a function field. The difficulty is that there is no criterion to guarantee the convergence of the integral
\[ \int_{G/k} \sum_{\xi \in X_k} \Phi(g \cdot \xi) |dg|_A, \]
which occurs in the formula (see §1 for the notation), as was the case for \( k \) a number field, cf. Weil [7], Igusa [2]. We establish convergence of the corresponding Siegel-Eisenstein series, then by the Hasse principle obtain the Siegel formula and the convergence of the above integral.

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1. Notation and the Siegel formula. Let \( k \) be a function field in one variable over a finite constant field, that is, a finitely generated extension of a finite prime field \( F_q \), of degree of transcendence one over \( F_q \). We shall assume that characteristic \((k) \neq 2\).

Let \( X \) be a vector space of dimension \( m \) and \( q(x) \) a nondegenerate quadratic form on \( X \), all defined over \( k \). Take \( G = SO(q) \) (a semisimple algebraic group, defined over \( k \), for \( m \geq 3 \)) to be the special orthogonal group of \( q \). The Siegel formula is given for the standard representation \( \rho: G \to \text{Aut}(X) \); it reads
\[ \int_{G/k} \left( \sum_{\xi \in X_k} \Phi(g \cdot \xi) \right) |dg|_A = 2 \sum_{i \in k} \int_{X/k} \Phi(x) \chi(q(x)x^i) |dx|_A + 2\Phi(0) \]

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where $G_A$, $G_k$ are the adelisation, the $k$-rational points (respectively) of $G$; 
$\Phi \in \mathcal{S}(X_A)$ is a Schwartz function on the adelisation of $X$ and $\chi$ is a fixed, non-trivial character of $k_A$, the adelisation of $k$, which is 1 on $k$.

2. Orbits, stabilisers. To analyse the integral $\int_{G_A/G_k} \sum_{\xi \in X_k} \Phi(g \cdot \xi) |dg|_A$, we recall results established by Weil [7, §14–29]. The orbits of $G$ in $X$ which contain points of $X_k$ are the sets $U(i) = \{x \in X| q(x) = i, x \neq 0\}$ where $i \in k$, $U(i)_k \neq \emptyset$, and $\{0\}$. This is precisely Witt’s theorem. Further, two points $x, y \in X_K$, not zero, belong to the same orbit of $G_K$ if and only if they belong to the same orbit of $G$. This is for any $K \supset k$.

For the nonempty $U(i)_k$, fix $\xi_i \in U(i)_k$ and let $H_i$ be the stabiliser of $G$ at $\xi_i$, i.e., $H_i = \{g \in G| g \cdot \xi_i = \xi_i\}$, an algebraic group defined over $k$. Hence $H_i = SO(m-1)$, of rank $m-1$, for $i \neq 0$;

$H_0 = SO(m-1) \cdot \text{unipotent}$, a semidirect product;

and in all cases, the Tamagawa numbers for $G$, $H_i$ are 2, Weil 15 [1].

Furthermore, the mapping $g \rightarrow g \cdot \xi_i$ of $G \rightarrow U(i)$ induces an isomorphism of $G/H_i$ onto $U(i)$. By Witt’s theorem there is a generic section for this map.

Also, as $k$ is an infinite field and $G$ is a reductive group, $G_k$ is Zariski dense in $G$ (Borel [1]). Whence, the mapping $g \rightarrow g \cdot \xi_i$ induces the identification $G_A/(H_i)_A = U(i)_A$ of the adelisations.

Take $\Phi \in \mathcal{S}(X_A)$; then for $i \in k$ so that $U(i)_k \neq \emptyset$,

$\int_{G_A/G_k} \sum_{\xi \in U(i)_k} \Phi(g \cdot \xi) |dg|_A = r(H_i) \int_{U(i)_A} \Phi|D_i|_A$,  

where $r(H_i)$ is the Tamagawa number of $H_i$, $|D_i|_A$ is the Tamagawa measure derived from $D_i = dg/dh_i$, $dg$, $dh_i$ invariant differential forms of maximal degree without zeros or poles for $G$, $H_i$, respectively, defined over $k$. The convergence factors may be taken to be 1, from the explicit nature of the stabilisers $H_i$.

By the Hasse principle for quadratic forms, $U(i)_k = \emptyset$ implies that $U(i)_A = \emptyset$. Thus we see that (1) is valid for all $i \in k$.

3. Asymptotic estimates. Let $\nu$ be a valuation on $k$, which is trivial on the field of constants, with $k_\nu$ as the completion. Then $k_\nu$ is nonarchimedean and denote by $\hat{0}_\nu$, $p_\nu$, and $q_\nu$ the maximal compact subring of $k_\nu$, the ideal of non-units of $\hat{0}_\nu$, and the number of elements of $\hat{0}_\nu/p_\nu$ (resp.). Let $X_\nu$, $X_\nu^0$ be the $k_\nu$-rational points of $X$ and $|dx|_\nu$, $|dx^0|_\nu$ be autodual measures on $k_\nu$ and $X_\nu$

For $X_\nu$ a nontrivial character of $k_\nu$, we identify $X_\nu$ with its dual by $(x, x') \rightarrow X_\nu(x \cdot x')$, where we write the elements of $X_\nu$ as row vectors, with respect to some $k$-basis. For $\Phi \in \mathcal{S}(X_\nu)$, the Schwartz-Bruhat space, the Fourier transform
is defined by $\Phi^*(x^*) = \int_{X^*_v} \Phi(x) (x, x^*) |dx_v|$, where $(x, x^*) = x^*(x)$. We choose as before $|dx_v|$ to be the autodual measure on $X^*_v$.

For $\Phi \in \mathcal{S}(X^*_v)$, we consider the function for $i^* \in k_v$ defined by

$$F^*_\Phi(i^*) = \int_{X^*_v} \Phi(x) X^*_v(q(x) i^*) |dx_v|.$$ 

The first sections of Weil [7] are devoted to proving general properties of such functions, in actually a more general setting. Namely, for $X, Y$ locally compact abelian groups and $f: X \to Y$ a continuous mapping, the principal result concerns the decomposition of the measure $dx$ on $X$, when $f$ satisfies a "condition (A)". If $\Lambda(X)$ denotes the subspace of $L^1(X)$ consisting of those continuous functions $\Phi$ with $\Phi^* \in L^1(X^*)$, then Fourier transformation gives a bijection of $\Lambda(X)$ with $\Lambda(X^*)$, so that $(\Phi^*)^*(x) = \Phi(-x)$ for every $x \in X$. Among other things, Weil proves that if $f$ satisfies "condition (A)", i.e.,

$$\int_{X^*} \Phi(x) (f(x), y^*) dx_v$$

is integrable on $Y^*$, uniformly so in $\Phi$ when $\Phi$ is restricted to a compact subset of $\mathcal{S}(X)$, then

(i) $F^*_\Phi$ belongs to $\Lambda(Y^*)$, and

(ii) there exists a unique family of measures $d\mu_y$ on $X$, each $d\mu_y$ being the image measure under $f^{-1}(y) \to X$, of a measure on $f^{-1}(y)$, such that $F^*_\Phi$ becomes the Fourier transform of $F^*_\Phi(y) = \int_X \Phi(x) d\mu_y(x)$.

We shall show that in the local and global cases, $f = q$, the quadratic form satisfies "condition (A)".

A fact which will play an important role is that if $\psi: k_v^n \to T$ is a non-degenerate second degree character of $k_v^n$, i.e., $\psi$ is continuous and satisfies $\psi(x + y) = \psi(x) \cdot \psi(y) \cdot (x, y)_b$ for some bicontinuous isomorphism $b: k_v^n \to (k_v^n)^*$, then its Fourier transform is given by

$$\psi^*(x^*) = \gamma(\psi) |b|^{-\frac{N}{2}} \psi(x^* b^{-1})^{-1},$$

where $\gamma(\psi) \in T$, a complex number of absolute value 1, and $|b|$ is the modulus of $b$ (Weil [6, p. 161]). Hence

$$\int_{k_v^n} \Phi(x) \psi(x) |dz_v| \leq \|\Phi^*\| L_1 |\det b|^{-\frac{N}{2}}.$$
For our case, take \( \psi(x) = \chi_{X_v}(q(x)) \), so that (2) reads \( |F^*_\Phi(i^*)| \leq \|\Phi\|_1 |i^*_v|^{-m/2} \).

Since, trivially, \( |F^*_\Phi(i^*)| \leq \|\Phi\|_1 \), we have

\[
|F^*_\Phi(i^*)| \leq \max (\|\Phi\|_1, \|\Phi^*_\Phi\|_1) \cdot \max (1, |i^*_v|) \cdot |i^*_v|^{-m/2}.
\]

Therefore, we have proved:

**Lemma 1.** Let \( C \) be a compact subset of \( \mathcal{S}(X_v) \). Then, there exists a positive constant \( c \), such that

\[
|F^*_\Phi(i^*)| \leq c \max (1, |i^*_v|) \cdot |i^*_v|^{-m/2}
\]

for all \( \Phi \in C, i^* \in k_v \).

It is easy to check that, for \( t \in k_v^X \),

\[
\int_{k_v^X} \max (|t|_v, |i|_v)^{-\sigma} \, dt \cdot |i|_v = \text{const} |i|_v^{1-\sigma}.
\]

This, combined with Lemma 1, shows that \( q: X_v \to k_v \) satisfies "condition (A)". Therefore, there exists a uniquely determined family of positive measures \( \{\mu_i\} : i \in k_v^X \) on \( X_v \), such that

(i) support \( \mu_i \subset \{x \in X_v \mid q(x) = i\} \);

(ii) for any continuous function \( \Phi \) with compact support on \( X_v \), the function \( F^*_\Phi(i) = \int_{X_v} \Phi(x) \, d\mu_i(x) \) defined on \( k_v \) is continuous and satisfies

\[
\int_{k_v} F^*_\Phi \, dt = \int_{X_v} \Phi(x) \, dx.
\]

Moreover,

(iii) if \( \Phi \in \mathcal{S}(X_v) \), \( F^*_\Phi \) is continuous, integrable and has as its Fourier transform

\[
F^*_\Phi(i^*) = \int_{X_v} \Phi(x) \chi_{X_v}(q(x)i^*) \, dx \quad (i^* \in k_v^X).
\]

As the sets \( U_v(i) = \{x \in X_v \mid q(x) = i, x \neq 0\} \) are in fact the fibres, for \( i \neq 0 \), these sets carry the measure \( \mu_i \). But the same is true for \( i = 0 \). To see this, use \( \Phi(\mu_x) \) in place of \( \Phi(x) \), for \( t \in k_v^X \). The uniqueness of the measures implies that \( \mu_0(\mu_x) = |i|_v^{-m} \mu_0(x) \), so that no part of the measure \( \mu_0 \) is carried by the set \( \{0\} \).

To identify the measures \( \mu_i \), consider the gauge form \( D_{v,t}(x) = (dx/dq(x))_i \) on \( U_v(i) \). As \( q \) is submersive on \( X_v - \{0\} \), this is well defined and satisfies

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\[ \int_{X_v \setminus \{0\}} \Phi \, |dx|_v = \int_{k_v} |dt|_v \int_{U_v(i)} \Phi \, |D_{v, i}|_v, \]

where \(|D_{v, i}|_v\) is the measure on \(U_v(i)\) determined by \(D_{v, i}\). This holds for all continuous functions \(\Phi\) with compact support contained in \(X_v \setminus \{0\}\). But \(\{0\}\) has measure zero for \(|dx|_v\), so we can extend the above equality to:

\[ \int_{X_v} \Phi \, |dx|_v = \int_{k_v} |dt|_v \int_{U_v(i)} \Phi \, |D_{v, i}|_v, \]

whence by the uniqueness of the family \(\{\mu_i\}\), we have \(\mu_i = |D_{v, i}|_v \quad (i \in k_v)\).

It is convenient at this time to mention that the gauge form \(D_i(x) = (dx/dq(x))_i\) on \(U(i)\), for \(i \in k\), is also defined and is invariant under \(G\), so it differs from the earlier \(dg/db_i\) by a factor of \(k^x\). Thus the measures given by \(D_i(x)\) and \(dg/db_i\) are the same, since the product formula is valid for \(k^x\).

Note that in the estimate (3), for \(\Phi, \Phi^*\) the characteristic functions of \(X_0^0, X^0_v\) we have \(|F^*(x^*)| \leq \max(1, |x^*|^{-m/2})\).

4. A dominant series. We shall now prove the convergence of the Siegel-Eisenstein series. The method of proof is based on the following lemma and the methods used in [3], due to Igusa.

As always \(k\) denotes a function field of transcendence degree one over a finite field \(k_0\). We may assume that \(k_0\) is algebraically closed in \(k\). Put \(q = \text{card}(k_0)\) and let \(g\) denote the genus of \(k\). Choose a prime divisor \(P_\infty\) of \(k\) such that \(d = \deg(P_\infty) \geq 2g + 1\), whence \(l(P_\infty) = d + 1 - g \geq g + 2\). So, there exists \(x \in k\) with \((x)_\infty = P_\infty)\).

Denote by \(\mathcal{O}\) the \(k\)-normalization of \(k_0[x]\). The group of units of \(\mathcal{O} = k^x\), hence finite. Also, every \(b \neq 0 \in \mathcal{O}\) has \(|b|_\infty \geq 1\).

Lemma 2. Let \(\lambda, \alpha\) denote real numbers, \(\lambda \geq 1, \alpha > 1\). Then

\[ \sum_{a \in \mathcal{O}} \max(\lambda, |a|_\infty)^{-\alpha} \leq c \lambda^{1-\alpha} \]

where \(c\) is independent of \(\lambda\).

Proof. We have

\[ \sum_{a \in \mathcal{O}} \max(\lambda, |a|_\infty)^{-\alpha} = \sum_{e=0}^{\infty} \text{card}(L(P^e_\infty) - L(P^e_\infty - 1)) \max(\lambda, q^d e)^{-\alpha}. \]

Write \(\lambda = q^{d\delta}\), so that \(0 \leq [\delta] \leq \delta < [\delta] + 1\). So
$$\sum_{a \in \mathbb{Q}} \max(\lambda, |a|_\infty)^{-\alpha} = \begin{cases} A & \text{if } [\delta] \geq 1, \\ B & \text{if } [\delta] = 0, \end{cases}$$

where

$$A = \text{card}(L(p^{[\delta]}))_\lambda^{-\alpha} + \sum_{\mathfrak{p} \neq [\delta] + 1} (q^{d(e+1-n) - q^{d(e-1)+1-n}})^{-\alpha} \mathfrak{p}^{-\alpha},$$

$$B = q^{\lambda^{-\alpha}} + (q^{d(e+1-n) - q^{d(e-1)+1-n}})^{-\alpha} \sum_{\mathfrak{p} \neq 2} \mathfrak{p}^{-\alpha}.$$

So, setting $\langle \delta \rangle = \delta - [\delta],$

$$A = \lambda^{-\alpha} \left\{ q^{-d(\delta)} \sum_{\mathfrak{p} \neq [\delta]} \left( \frac{q^{1-q^{-d}}(1-q^{-d})^q}{1-q^{-(\alpha-1)d}} \right) \right\},$$

$$B = \lambda^{-\alpha} \left\{ q^{-d(\delta)} + q^{-d(\delta)}(1-q^{-d})^q \sum_{\mathfrak{p} \neq [\delta]} \left( \frac{q^{1-q^{-d}}(1-q^{-d})^q}{1-q^{-(\alpha-1)d}} \right) \right\}.$$

Fix this choice of generator $x.$ The ideal class group of $k$ for this $C$ is finite and let $r_1, \ldots, r_b$ be coset representatives, which may be taken to be integral ideals. Set $S_{\infty} = \{ P_{\infty} \}.$

Proposition 1. Let $n$ be a given integer $> 0,$ and $\epsilon > 0$ be fixed. Suppose that for each valuation $v$ on $k,$ $\sigma_v$ is a given real number, such that $\sigma_v > n,$ for all $v,$ $\sigma_v \geq n + 1 + \epsilon,$ for almost all $v.$ Then

$$\sum_{i=\langle i_1, \ldots, i_n \rangle \in k^n} \prod_{i=1}^{i_n} \max(1, |i_1|_v, \ldots, |i_n|_v)^{-\sigma_v}$$

is convergent.

Proof. The convergence is clear for $n = 0,$ so suppose $n \geq 1$ and use induction. Let $E \subseteq \{ 1, 2, \ldots, n \}$ be a subset and

$$k_E = \{ i \in k^n \mid i_p \neq 0 \text{ for } p \in E, \ i_p = 0 \text{ for } p \notin E \}.$$

Then we have the disjoint union $k^n = \bigcup_{E} k_E.$ By induction, the partial sums over $k_E$ are convergent for every $E \neq \{ 1, 2, \ldots, n \}.$ So it remains to show that the partial sum over $(k^E)^n$ is convergent.

By hypothesis, there is a finite set of valuations $S$ on $k,$ $S \supset S_{\infty}$ such that $\sigma_v > n$ for all $v$ and $\sigma_v \geq \gamma = 1 + n + \epsilon$ for all $v \notin S.$ We can enlarge $S$ without changing $\gamma,$ so suppose $S$ contains all the prime factors of $r_1, \ldots, r_b.$ Further, as a function of $\sigma_v, \max(1, |i_1|_v, \ldots, |i_n|_v)^{-\sigma_v}$ is monotone decreasing, so it suffices to prove convergence when $\sigma_v = \alpha > n,$ $v \in S,$ $\sigma_v = \beta,$ $v \notin S.$
Let \( i = (i_1, \ldots, i_n) \in (k^n)^{\times} \). Then \( i_p \hat{C} = a_p^i b \) for integral ideals \( b, a_1, \ldots, a_n \). Choosing them to be relatively prime, this set is uniquely determined by \( i \).

Moreover, there is a unique index \( j \) so that \( b r_j = C b \), for some \( b \neq 0 \in \hat{C} \). Setting \( a_p = b i_p \) we have \( a_p \hat{C} = b r_j i_p = a_p^i j \subset \hat{C} \) so \( a_p \neq 0 \in \hat{C} \). By the choice of \( S \)

\[
\prod_v \max(1, |i_1|_v, \ldots, |i_n|_v) = \prod_{v \in S} \max(|b|_v^\alpha, |a_1|_v, \ldots, |a_n|_v) - \alpha \times \prod_{v \notin S} \max(|b|_v^\beta, |a_1|_v, \ldots, |a_n|_v^\beta).
\]

But, as the prime factors of the \( r_j \) are in \( S \), \( \max_{v \in S} (|b|_v, |a_1|_v, \ldots, |a_n|_v) = 1 \).

Hence, applying the product formula for \( b \in \hat{C} \), the above becomes

\[
\prod_{v \in S} |b|_v^\gamma \max(|b|_v, |a_1|_v, \ldots, |a_n|_v)^{-\alpha}.
\]

For \( v \in S - S_\infty \),

\[
\text{ord}_p (b) = \text{ord}_p (b) + \text{ord}_p (r_j), \quad \text{ord}_p (a_i) = \text{ord}_p (a_i) + \text{ord}_p (r_j)
\]

whence \( \max_{v \in S - S_\infty} (|b|_v, |a_1|_v, \ldots, |a_n|_v) = N_p^{\text{ord}_p (b)} \), since \( b, a_1, \ldots, a_n \) are relatively prime. Here \( N_p = \text{card}(\hat{C}/p) \). Setting \( c_p = \max \{\text{ord}_p (r_j), 1 \leq j \leq b\} \),

\[
c' (\prod_{v \in S - S_\infty} N_p^c p)^a, \text{ we find}
\]

\[
\prod_{v \in S - S_\infty} \max(1, |i_1|_v, \ldots, |i_n|_v)^{-\gamma} \leq c' (\prod_{v \in S} |b|_v)^{\alpha - \beta} \max(|b|_\infty, |a_1|_\infty, \ldots, |a_n|_\infty)^{-\alpha}.
\]

Therefore, it suffices to show that the sum on the right, for \( (a_1, \ldots, a_n) \in \hat{C}^n \) and \( C b \) over the set of principal ideals, \( b \neq 0 \) of \( \hat{C} \), is convergent.

Since \( |b|_\infty \geq 1 \) for \( b \neq 0 \in \hat{C} \) and \( \alpha > n \), we can apply Lemma 2 repeatedly, to show

\[
\sum_{(a_1, \ldots, a_n) \in \hat{C}^n} \max(|b|_\infty, |a_1|_\infty, \ldots, |a_n|_\infty)^{-\alpha} \leq c^n |b|_\infty^{-n}
\]

where \( c \) is a fixed constant, independent of \( b \). Hence, it suffices to show that the series \( \sum_{(a_1, \ldots, a_n) \in \hat{C}^n} (\prod_{v \in S - S_\infty} |b|_v)^{\alpha - \beta} |b|_\infty^{-\alpha} \) is convergent. By the product formula, this is

\[
\sum_{(a_1, \ldots, a_n) \in \hat{C}^n} \left( \prod_{v \in S - S_\infty} |b|_v \right)^{\alpha - n} \left( \prod_{v \notin S} |b|_v \right)^{\beta - n}
\]

\[
= \sum_{(a_1, \ldots, a_n) \in \hat{C}^n} \left( \prod_{p \in S - S_\infty} N_p^{\text{ord}_p (b)} \right)^{\alpha - n} \left( \prod_{p \notin S} N_p^{\text{ord}_p (b)} \right)^{\beta - n}
\]

\[
< \sum_{b \neq 0} \text{all integral ideals} \left( \prod_{p \notin S} N_p^{\text{ord}_p (b)} \right)^{\alpha - n} \left( \prod_{p \notin S} N_p^{\text{ord}_p (b)} \right)^{\beta - n}.
\]
But, by the Euler product, this differs by only an elementary factor from
\[ \sum_{\nu \in \Sigma} (N_\nu)^{-\sigma}. \] But for \( \sigma = \beta - n > 1 \) this is convergent.

5. The Siegel formula. The character \( \chi \) of \( k_A \) puts it into duality with \( k_A^* \) by \( (i, i^*) \mapsto \chi(ii^*) \), for \( i, i^* \in k_A \). Identifying \( X_A \) with its dual by \( (x, y) \mapsto \chi(x^t y) \), for \( x, y \in X_A \), the autodual measure \( |dx|_A \) on \( X_A \) is then the Haar measure for which \( X_A/X_k \) has measure 1.

For every \( \Phi \in \tilde{S}(X_A) \), define
\[ F_\Phi^*(i^*) = \int_{X_A} \Phi(x) \chi(q(x)i^*) |dx|_A, \]
for \( i^* \in k_A \).

For almost all \( \nu \), the usual Haar measure on \( k \) is autodual, \( C_\nu \) is the kernel of \( \chi_\nu \) and \( m(\chi_\nu^0) = 1 \). Recall that \( X_A \) is the inductive limit of \( X_s = X_0^0 \times X_1 \), where \( X_0^0 = \bigcap \nu \in S \chi_\nu^0 \), \( X_1 = \bigcap \nu \in S \chi_\nu^1 \), for \( S \) running over the family of finite sets of valuations on \( k \). Therefore, for every compact subset \( C \) of \( \tilde{S}(X_A) \), there exist an \( S \) and a compact subset \( C_1 \) of \( \tilde{S}(X_1) \), such that every \( \Phi \in C \) is of the form \( \Phi_0 \otimes \Phi_1 \), where \( \Phi_0 \) is the characteristic function of \( X_0^0 \), \( \Phi_1 \) is in \( C_1 \).

Put \( \sigma_\nu = m/2 \) for all \( \nu \). Then, by Lemma 1 and Fubini's theorem, there is a positive constant \( c \) such that
\[ \sum_{i^* \in k} |F_\Phi^*(i^*)| \leq c \sum_{i^* \in k} \prod_{\nu \in S} \max(1, |i^*_\nu|^{\nu})^{-\sigma_\nu} \]
for every \( \Phi \in C \). By Proposition 1, the right-hand side is convergent for \( m \geq 5 \).

Also, the mapping
\[ (X_A) \times k_A \rightarrow \tilde{S}(X_A), \]
\[ \psi \mapsto (\Phi, i^*) \rightarrow F_{\Phi}^{i^*} \]
where \( F_{\Phi}^{i^*}(x) = \Phi(x) \chi(q(x)i^*) \) is continuous. Hence, by Weil's criterion [7, p. 8], the continuous mapping \( q: X_A \rightarrow k_A \) satisfies "condition (A)" and the following Poisson formula:
\[ \sum_{i^* \in k} F_{\Phi}^{i^*}(i) = \sum_{i \in k} (F_{\Phi}^{i})^{i^*}(i). \]

Here \( (F_{\Phi}^{i})^{i^*}(i) = F_{\Phi}^{i^*}(-i) \) for every \( i \in k_A \).

Lemma 3. \( F_{\Phi}^{i}(i) = \int_{U(i)_A} \Phi |D_i|_{1A}, \) for every \( i \in k_A \).

Proof. It suffices to show this for \( \Phi \) restricted to a subset of \( \tilde{S}(X_A) \) which spans a dense subspace of \( \tilde{S}(X_A) \). Take \( \Phi = \prod \nu \Phi_\nu \), where \( \Phi_\nu \in \tilde{S}(X_\nu) \) for every
$v$ and $\Phi_v = \text{the characteristic function of } \chi_v^0$, for all but finitely many $v$. Then $F_v^*$ decomposes into the product of $F_{\Phi_v}$, defined by $F_{\Phi_v}(i_v) = (F_{\Phi_v}^*)^*(-i_v)$, whence, by the results of §3, $F_{\Phi_v}(i_v) = \int_{U_v(i_v)} \Phi_v \cdot |D_{v,i_v}|^2_v$, for every $i_v \in k_v$.

This implies the desired result. Therefore, (4) now reads

$$\sum_{i \in k} \int_{X_A} \Phi(x)\chi(q(x)i^*)\left|dx\right|_A = \sum_{i \in k} \int_{U(i)_A} \Phi |D_i|^2_A.$$ 

Combining this with (1) and the exceptional orbit $\{0\}$, we obtain the Siegel formula,

**Theorem.**

$$\int_{G_k/G_k} \left( \sum_{\xi \in X_k} \Phi(\xi \cdot \xi) \right) |dg|_A = 2 \sum_{i \in k} \int_{X_A} \Phi(x)\chi(q(x)i^*)\left|dx\right|_A + 2\Phi(0),$$

which is valid for $m \geq 5$. Here $G$ is the special orthogonal group, acting on $X$, of dimension $m$.

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