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COMPLEX APPROXIMATION FOR VECTOR-VALUED FUNCTIONS WITH AN APPLICATION TO BOUNDARY BEHAVIOUR(1)

BY

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ABSTRACT. This paper deals with the qualitative theory of uniform approximation by holomorphic functions. The first theorem is an extension to vector-valued mappings of N. U. Arakélian's theorem on uniform holomorphic approximation on closed sets. Our second theorem is on asymptotic approximation and yields, as in the scalar case, applications to cluster sets.

Let D be a proper domain of the Riemann sphere, D^* its one-point compactification, F a (relatively) closed subset of D, X a locally convex (complex) topological vector space, $H(D, X)$ the holomorphic mappings from D to X, $H_F(D, X)$ the uniform closure of restrictions to F of mappings in $H(D, X)$, and $A(F, X)$ the continuous mappings from F to X holomorphic on F^0.

Mergelian's celebrated theorem [11] gives necessary and sufficient conditions in order that $H_F(D, X) = A(F, X)$, for the case where $D = X = \mathbf{C}$, and F is compact. Recently, E. Briem, K. B. Laursen, and N. W. Pedersen [5] have extended Mergelian's theorem to the case where $D = \mathbf{C}$, F is compact, and X is a locally convex space.

In a different direction Arakélian ([1], [2], and [3]) has generalized Mergelian's theorem to the case where D is an arbitrary proper domain of the Riemann sphere, F a (relatively) closed subset of D, and $X = \mathbf{C}$. Theorem 1 of the present paper extends Arakélian's theorem to the case where X is a Fréchet space. We remark that the necessity of Arakélian's conditions holds for an arbitrary locally convex space.

The closed set $F \subset D$ is said to be a set of tangential, asymptotic, or Carleman approximation provided that for each $f \in A(F, \mathbf{C})$ and each positive continuous function ϵ on F, there is a function $g \in H(D, \mathbf{C})$ such that $|f(z) - g(z)| < \epsilon(z)$, $z \in F$. Arakélian [3] has given necessary and sufficient conditions for

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tangential approximation in case $F^0 = \emptyset$. For general closed F, the problem has been completely solved by Nersesian [12]. Our Theorem 2 gives a vector-valued theorem on asymptotic approximation for the case where $F^0 = \emptyset$. As in the scalar case (see [8]), Theorem 2 yields powerful corollaries on boundary behaviour of holomorphic functions. In the scalar case, asymptotic approximation has also found deep applications to value distribution theory [3].

The proof for Banach spaces is a modification of the proof in the scalar case and requires the use of the Hahn-Banach theorem at several points. We include it it in the Appendix as Theorem 3, partially for expository reasons, as the proof for the scalar case is not easily accessible and many details are left out. Our proof of Theorem 3 is modelled on Arakélian's proof for the case where $D = X = \mathbf{C}$ as presented by W. Fuchs [7] and Arakélian's proof where D is a general domain and $X = \mathbf{C}$ [2].

We use the following notations. S^2 is the Riemann sphere; ρ the chordal metric on S^2. For $E \subset S^2$, $\delta > 0$, $N_\rho(E, \delta) = \{z \in S^2: \rho(E, z) < \delta\}$, and $\bar{E}$ denotes the S^2-closure of E, ∂E the S^2-boundary. For $z \in \mathbf{C}$, $\delta > 0$, $B(z, \delta) = \{\zeta \in \mathbf{C}: |z - \zeta| < \delta\}$. Let D be a proper domain in S^2. A boundary curve α in D is a continuous curve $\alpha: [0, 1) \longrightarrow D$ such that $\alpha(t) \longrightarrow \partial D$, as $t \longrightarrow 1$. If E and X are topological spaces, $C(E, X)$ denotes the continuous mappings from E to X; we write $C(E) = C(E, \mathbf{C})$ and $A(E) = A(E, \mathbf{C})$ etc. For X a Fréchet space, $\{p_n\}$ will denote an increasing sequence of seminorms which generate the topology of X.

Theorem 1. *Let X be a Fréchet space, D a proper domain of the Riemann sphere, and F a (relatively) closed subset of D. Then*

$$H_F(D, X) = A(F, X) \tag{1}$$

if and only if $D^* \setminus F$ is connected and locally connected.

Proof. For the case where $X = \mathbf{C}$, the necessity was shown by Arakélian [2] (see also [9]). Now suppose $D^* \setminus F$ is not locally connected and connected. Then there is a function $f \in A(F, \mathbf{C})$ and an $\epsilon > 0$ such that

$$|f(z) - g(z)| < \epsilon, \qquad z \in F \tag{2}$$

is satisfied by no function $g \in H(D, \mathbf{C})$. Fix $x \in X$, $x \neq 0$, and choose a seminorm p such that $p(x) = 1$. Then by the Hahn-Banach theorem, there is an $x^* \in X^*$ for which $x^*(x) = 1$, and $|x^*(y)| \leq p(y)$, for all $y \in X$. Suppose, to obtain a contradiction, that (1) holds. Then there is a holomorphic mapping $G \in H(D, X)$ for which

$$p(f(z)x - G(z)) < \epsilon, \quad z \in F.$$

Thus

$$|x^*(f(z)x - G(z))| < \epsilon;$$

that is, $|f(z) - x^*(G(z))| < \epsilon$, which contradicts (2) and proves the necessity.

To prove sufficiency it is enough to show that $A(F, X) \subset H_F(D, X)$ if $D^* \setminus F$ is connected and locally connected. Let $f \in A(F, X)$. We shall construct a Banach space $Y \subset X$, with a stronger topology such that f is also in $A(F, Y)$. Since the theorem is true for Banach spaces (Theorem 3), $f \in H_F(D, Y)$. However, since the topology on Y is stronger than the topology on X, $H_F(D, Y) \subset H_F(D, X)$, which will establish (1).

Let $\{p_n\}$ be a sequence of increasing seminorms which defines the topology for X. If $\{\omega_n\}$ is a sequence of positive numbers, then one easily shows that

$$\left\{x \in X: \sup_n \omega_n p_n(x) < \infty\right\} = Y \subset X$$

is a Banach space with a stronger topology than that of X. Let $\{D_n\}$ be an exhaustion of D by domains: $\overline{D}_n \subset D_{n+1}$, $n = 1, 2, \cdots$, and $D = \bigcup_{n=1}^{\infty} D_n$. Set $K_n = F \cap D_n$, and let K_n' be an exhaustion of F^0 by compact sets such that $K_n' \subset (K_{n+1}')^0$, $n = 1, 2, \cdots$, and $F^0 = \bigcup_{n=1}^{\infty}(K_n')$. We set

$$\omega_n' = \frac{1}{1 + \sup_{K_n} p_n(f(z))}, \quad \omega_n'' = \frac{1}{1 + \sup_{K_n'} p_n(f'(z))}.$$

Lemma 1. *For each j,*

$$\sup_{z \in K_j} \sup_n \omega_n' p_n(f(z)) = M_j < \infty, \tag{3}$$

and

$$\sup_{z \in K_j'} \sup_n \omega_n'' p_n(f'(z)) = M_j' < \infty. \tag{4}$$

Proof. If $z \in K_j$ and $n \geq j$, then $\omega_n' p_n(f(z)) \leq 1$. While if $n < j$, $p_n \circ f$ is continuous and thus bounded on K_j. This proves (3), and (4) is proved similarly.

Now set

$$\omega_n = \frac{\min(\omega_n', \omega_n'')}{n} \tag{5}$$

and let Y be the Banach space associated with the sequence $\{\omega_n\}$.

Lemma 2. *With respect to the strong topology on Y,*

(i) *$f: F \longrightarrow Y$ is continuous, and*

(ii) *$f: F^0 \longrightarrow Y$ is holomorphic.*

Proof. Lemma 1 implies that $f(F) \subset Y$ and $f'(F) \subset Y$, where f' is the derivative with respect to the topology of X.

To show (i) fix $z \in F$ and choose j so that $z \in K_j$. Then, for $\zeta \in K_{j+1}$,

$$\begin{aligned}\|f(z)-f(\zeta)\| &= \sup_n \omega_n p_n(f(z)-f(\zeta))\\ &\le \sup_{n\ge N} \omega_n p_n(f(z)-f(\zeta)) + \sup_{n<N} \omega_n p_n(f(z)-f(\zeta))\\ &\le N^{-1} \sup_{n\ge N} \omega_n' p_n(f(z)-f(\zeta)) + \sup_{n<N} \omega_n p_n(f(z)-f(\zeta))\\ &\le M_{j+1}/N + \sup_{n<N} \omega_n p_n(f(z)-f(\zeta)).\end{aligned}$$

Given $\epsilon > 0$, choose $N > 2M_{j+1}/\epsilon$, and choose $\delta > 0$ so that

$$\zeta \in B(z,\delta) \cap F \Rightarrow \sup_{n<N} \omega_n p_n(f(z)-f(\zeta)) < \epsilon/2.$$

Hence for such ζ, $\|f(z)-f(\zeta)\| < \epsilon$. This proves (i).

To show (ii), fix $z \in F^0$, and choose j and k such that $z \in K_j \cap K_k'$. Now choose $\delta > 0$ so that $B(z,\delta) \subset K_{j+1} \cap K_{k+1}'$ and set $\Gamma = \{\zeta: |\zeta - z| = \delta\}$. If $|h| < \delta/2$, then, for every n,

$$\begin{aligned}\omega_n' p_n\left(\frac{f(z+h)-f(z)}{h}\right) &= \omega_n' p_n\left(\frac{1}{h2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta - z}\,d\zeta\right)\\ &\le \frac{1}{2\pi}\int_\Gamma \frac{\omega_n' p_n(f(\zeta))\,|d\zeta|}{|[\zeta-(z+h)][\zeta-z]|}\\ &\le M_{j+1}\frac{1}{2\pi}\int_\Gamma \frac{|d\zeta|}{|[\zeta-(z+h)][\zeta-z]|} \le \frac{2}{\delta} M_{j+1}.\end{aligned}$$

Consequently

$$\sup_{|h|<\delta/2} \sup_n \omega_n' p_n\left(\frac{f(z+h)-f(z)}{h}\right) \le \frac{2}{\delta} M_{j+1}. \tag{6}$$

Let $|h| < \delta/2$ and f' be the derivative of f with respect to the topology of X. Then

$$\begin{aligned}\left\|\frac{f(z+h)-f(z)}{h} - f'(z)\right\| &= \sup_n \omega_n p_n\left(\frac{f(z+h)-f(z)}{h} - f'(z)\right)\\ &\le \sup_{n\ge N} \omega_n p_n(\ ) + \sup_{n<N} \omega_n p_n(\ ).\end{aligned}$$

As in the proof of (i), it is sufficient to show that, for given ϵ,

$$\sup_{n\ge N} \omega_n p_n\left(\frac{f(z+h)-f(z)}{h} - f'(z)\right) < \frac{\epsilon}{2}, \tag{7}$$

provided N is sufficiently large. Now (7) is less than or equal to

$$\sup_{n\geq N} \frac{\omega_n}{n} p_n\left(\frac{f(z+h)-f(z)}{h}\right) + \sup_{n>N} \frac{\omega_n}{n} p_n(f'(z))$$
$$\leq \frac{1}{N} \sup_{n\geq N} \omega_n' p_n\left(\frac{f(z+h)-f(z)}{h}\right) + \frac{1}{N} \sup_{n\geq N} \omega_n'' p_n(f'(z))$$
$$\leq \frac{1}{N}\frac{2}{\delta} M_{j+1} + \frac{1}{N} M_k'.$$

The last inequality follows from (6) and (4). This establishes (7), which completes the proof of Lemma 2 and of the theorem.

Having considered the possibility of uniform approximation, we now turn to the possibility of asymptotic approximation.

Theorem 2. *Let X be a Fréchet space, D a proper domain of the Riemann sphere, and F a (relatively) closed subset of D with empty interior. Then, for each mapping $f \in C(F, X)$ and positive function $\epsilon \in C(D, \mathbf{R}^+)$, there is a holomorphic mapping $g \in H(D, X)$ such that for each continuous seminorm p,*

$$p(f(z) - g(z)) = o(\epsilon(z)), \tag{8}$$

as $z \to \partial D$, $z \in F$, if and only if $D^*\backslash F$ is connected and locally connected.

Proof. In case $X = \mathbf{C}$, this theorem is essentially a corollary of Theorem 1. However in the general case, it seems that a detailed proof is required.

We remark that it is sufficient to show that for $n = 1, 2, \cdots$

$$p_n(f(z) - g(z)) = O(\epsilon(z)), \tag{9}$$

as $z \to \partial D$, $z \in F$. This is so, because for each p there is an n and a constant A such that $p < Ap_n$, and by choosing an appropriate $\epsilon(z)$ we can replace o by O.

We now prove (9). Let D, F, f, and ϵ be as in the hypothesis of the theorem. If F is compact, then the theorem is vacuously true.

Suppose F is not compact. Then we may assume $\lim \epsilon(z)$ is zero as z approaches the boundary of D.

If K is a compact set in D we may choose a compact set $K^1 \subset D$ and containing K such that $(D^*\backslash F)\backslash K^1$ is connected. Also set

$$K^2 = \{z \in F: \epsilon(z) \geq \epsilon(\zeta), \text{ for some } \zeta \in K^1\} \cup K^1.$$

Now let $\{K_n\}$ be a normal exhaustion of D by compact sets inductively constructed so that $K_0 \cap F \neq \emptyset$ and $K_{n-1}^2 \subset K_n^0$, $n = 1, 2, \cdots$. We may choose the K_n in such a manner so that the ϵ_n strictly decrease to zero, where $\epsilon_n = \min\{\epsilon(z): z \in K_n^2\}$. Set $F_0 = F \cap K_0^2$ and, for $n = 1, 2, \cdots$, set

$$F_n = K_{n-1}^2 \cup (F \cap K_n^2) = K_{n-1}^2 \cup \{z \in F: \epsilon(z) \geq \epsilon_n\}.$$

It is not difficult to verify that $D^*\backslash F_n$ is connected. Thus F_n is a "Mergelian set."

Set $f_0 = f$, for $z \in F_0$. Then by Mergelian's theorem [5], there is a mapping g_0 holomorphic on all of D such that $p_1(g_0(z) - f_0(z)) < \epsilon_1/4$, $z \in F_0$. Now set $K_{-1} = \emptyset$, $\epsilon_{-1} = +\infty$, $g_{-1} = g_0$ and suppose mappings g_j, $j = 0, 1, \cdots, n-1$, holomorphic on all of D, have been defined satisfying

$$p_j(f(z) - g_j(z)) \leq \epsilon_j/2, \qquad \text{if } \epsilon_j \leq \epsilon(z) \leq \epsilon_{j-1}, \tag{10}$$

$$p_{j+1}(f(z) - g_j(z)) \leq \epsilon_{j+1}/4, \quad \text{if } \epsilon(z) = \epsilon_j, \tag{11}$$

and

$$p_j(g_j(z) - g_{j-1}(z)) < \epsilon_j/2^{j+1}, \qquad z \in K^2_{j-1}. \tag{12}$$

We wish to define g_n, but first we must define f_n. Set $f_n = g_{n-1}$ on K^2_{n-1}. The function f_n is defined on the closed set K^2_{n-1} and we shall define it on all of F_n. Set $H_n = f_n - f$. Then H_n is defined on a closed set which does not meet the set

$$\{z \in F: \epsilon(z) = \epsilon_n\}. \tag{13}$$

Set $H_n = 0$ on the set (13). H_n is continuously defined on a closed subset of the normal space

$$\{z \in F: \epsilon_{n-1} \geq \epsilon(z) \geq \epsilon_n\} \tag{14}$$

and is p_n-bounded by $\epsilon_n/4$. By Dugundji's extension of Tietze's theorem [6], we extend H_n continuously to all of (14) so that the extension retains the same bound. Now $f_n = f + H_n$ is continuously defined on all of F_n and is holomorphic on the interior. By Mergelian's theorem, there is a function g_n holomorphic on all of D such that $p_{n+1}(g_n(z) - f_n(z)) < \epsilon_{n+1}/2^{n+1}$, $z \in F_n$. From the way in which f_n was defined, it follows that g_n satisfies (10), (11), and (12).

Thus we define a sequence $\{g_n\}$ satisfying (10), (11), and (12). From (12) it is clear that $\{g_n\}$ converges to a mapping g holomorphic on all of D. Now fix p_n and let z be any point of F so close to the boundary of D that $\epsilon(z) \leq \epsilon_{n-1}$. Choose N so that $\epsilon_n(z) \leq \epsilon(z) \leq \epsilon_{N-1}$. Then for any $m > N$,

$$p_n(f(z) - g(z)) \leq p_n(f(z) - g_N(z)) + \sum_{j=N+1}^{m} p_j(g_j(z) - g_{j-1}(z)) + p_m(g_m(z) - g(z)).$$

By choosing m sufficiently large, the last term can be made as small as desired, and so from (10) and (12) it follows that $p_n(f(z) - g(z)) < \epsilon_N \leq \epsilon(z)$. Since this holds for any z for which $\epsilon(z) \leq \epsilon_{n-1}$, the proof is complete.

As a consequence of the preceding proof we have

Corollary 1. *Under the hypotheses of Theorem 2, for each seminorm p, there is a holomorphic mapping $g \in H(D, X)$ for which $p(f(z) - g(z)) < \epsilon(z)$, $z \in F$.*

As a further application of Theorem 2 we have the following result, first proved by Kaplan [10] for the case that $X = \mathbf{C}$.

Corollary 2. *Let X be a Fréchet space, D a proper domain of the Riemann sphere, $\{\alpha_n\}_{n=1}^{\infty}$ a family of disjoint boundary curves, and let $f \in C(\alpha_n, X)$, $n = 1, 2, 3, \cdots$, and $\epsilon \in C(D, \mathbf{R}^+)$. Then there exists a holomorphic mapping $g \in H(D, X)$ such that for each continuous seminorm p, and each $n = 1, 2, \cdots$, $p(f(z) - g(z)) = o(\epsilon(z))$, as $z \to \partial D$, along α_n.*

Proof. The proof is the same as for the case $X = \mathbf{C}$ (see [8]).

A holomorphic mapping $g \in H(D, X)$ is said to have asymptotic value $x \in X$ along a boundary path α in D provided $f(z) \to x$ as $z \to \partial D$ along α.

Corollary 3. *Let X be a Fréchet space, D a proper domain of the Riemann sphere, α_n, $n = 1, 2, \cdots$, a family of disjoint boundary curves, and x_n, $n = 1, 2, \cdots$, a sequence of elements of X. Then there is a holomorphic mapping $g \in H(D, X)$ which has asymptotic value x_n along α_n, for $n = 1, 2, \cdots$.*

Note that by an appropriate choice of $\epsilon(z)$, one also obtains a mapping which approaches prescribed values with a prescribed speed along prescribed boundary curves. For the case where $X = \mathbf{C}$, Corollary 3 was proved by Bagemihl and Seidel [4] and by Rudin [13].

We conclude this section with some open problems.

Conjecture 1. Theorem 1 fails if X is allowed to be a general locally convex space. A counterexample is required.

Conjecture 2. Theorem 1 holds for X locally convex, K compact, and D an open Riemann surface.

Conjecture 3. Theorem 2 holds for X Fréchet and D an open Riemann surface.

APPENDIX

Theorem 3. *If X is a Banach space then $H_F(D, X) = A(F, X)$ if and only if $D^* \backslash F$ is connected and locally connected.*

Proof. The following proof is based upon Arakélian's proof for the case where D and X are the plane as presented by W. Fuchs [7] and Arakélian's proof where D is a general domain and X the plane [2].

Four lemmas are required. We begin with a slightly modified lemma of Rudin.

Lemma 1 (Rudin [14, p. 383]). *Suppose σ is a compact and connected subset contained in a disc of diameter $2r$, $\Omega = S^2 \backslash \sigma$ is connected, and the diameter of σ is at least $.99r$. Then there is a function $g \in H(\Omega)$ and a constant b, with the following property: If*

$$Q(\zeta, z) = g(z) + (\zeta - b)g^2(z), \tag{15}$$

the inequalities

$$|Q(\zeta, z)| < 100/r, \tag{16}$$

$$|Q(\zeta, z) - 1/(z - \zeta)| < 1000\, r^2/|z - \zeta|^3, \tag{17}$$

hold for all $z \in \Omega$ *and for all* ζ *in the disc.*

Lemma 2. *Let* σ *be a compact, connected subset of* D *and* γ *be a boundary curve originating at a point of* σ, E_0 *a closed subset of* D *such that* $(\gamma \cup \sigma) \cap E_0 = \emptyset$. *If* $Q(z)$ *is a function holomorphic in* $S^2 \backslash \sigma$, $\eta > 0$, *then there exists a* ψ *in* $H(D)$ *such that*

$$|\psi(z) - Q(z)| < \eta \quad \text{for } z \in E_0. \tag{18}$$

Proof. The method of proof involves the standard technique of pole pushing. We include the proof for completeness.

Let γ be parameterized by $z\colon [0, 1) \longrightarrow D$. By finite induction one chooses $0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < 1$ such that if $t \geq t_n$ then $\rho(z(t), \partial D) < 1/n$. Again by induction we choose domains G_n such that

(i) $\overline{G_n} \cap E_0 = \emptyset$ for all n,

(ii) $G_1 \supset \sigma \cup z([0, t_1))$,

(iii) $G_n \supset z([t_{n-1}, t_n])$, $n > 1$,

(iv) $G_n \subset N_\rho(\partial D, 1/(n-2))$, $n > 2$.

Using Runge's theorem we construct inductively a sequence of rational functions ψ_n with the following properties:

(i) ψ_n has a unique pole at $z(t_n)$,

(ii) $|Q(z) - \psi_1(z)| < \eta/2$ for $z \in S^2 \backslash G_1$,

(iii) $|\psi_{n-1}(z) - \psi_n(z)| < \eta/2^n$ for $z \in S^2 \backslash G_n$ and $n > 1$.

On compact subsets of D, ψ_n converges uniformly to a function ψ in $H(D)$. If $z \in E_0$, then

$$|Q(z) - \psi(z)| \leq |Q(z) - \psi_1(z)| + \sum_{n=2}^{\infty} |\psi_{n-1}(z) - \psi_n(z)| < \eta.$$

This completes the proof of the lemma.

Lemma 3. *Let* D *be a domain of the finite plane. Then there exists a function* $\rho \in C^\infty(D)$ *such that* $0 < \rho(z) < \rho(z, \partial D)$, $|\bar\partial \rho| < 1$ for $z \in D$.

Proof. See Arakélian [2, Lemma 4].

If ϕ is a function from D into X which has continuous partial derivatives we say that $\phi \in C^1(D, X)$.

Lemma 4. *Let D be a domain of the finite plane and f a continuous function from D to X. For $\epsilon > 0$, there exist $\delta \in C^1(\mathbf{C})$ and $\phi \in C^1(D, X)$ such that $\delta|_{\mathbf{C}\setminus D} = 0$, and for $z \in D$,*

$$0 < \delta(z) < \tfrac{1}{2}\rho(z, \partial D), \qquad |\bar{\partial}\delta| < \min\{\epsilon, \delta^{1/2}(z)\}, \tag{19}$$

$$\|\phi(z) - f(z)\| < \epsilon, \qquad \|\bar{\partial}\phi(z)\| < \epsilon/\delta(z). \tag{20}$$

Furthermore, if for some point $z \in D$, the function f is holomorphic in the disc $|\zeta - z| < \delta(z)$, then $\phi(z) = f(z)$.

Proof. Taking $\epsilon < 1$, we shall construct the function $\delta(z)$. For the compact set $K_\lambda \subset \mathbf{C}$, $K \ = \{z \in D\colon \rho(z, \partial D) \geq \lambda\}$, the condition $z \in K_\lambda$ and $|\zeta - z| \leq \lambda/2$ implies that $\zeta \in K_{\lambda/2}$. Thus for δ a constant, $\delta \leq \lambda/2$, we may set

$$\omega(\lambda, \delta) = \max_{z \in K_\lambda, |\zeta - z| \leq \delta} \|f(\zeta) - f(z)\|.$$

Note that $\omega(\cdot, \delta)$ is nonincreasing, $\omega(\lambda, \cdot)$ is nondecreasing and $\omega(\lambda, 0) = 0$. Thus we may define $\delta_1(\lambda)$ as the unique value of δ for which $\omega(\lambda, \delta) + \delta = \epsilon\lambda/2$. Note that $\delta_1(\lambda)$ is strictly increasing. Now define $\delta_2(\lambda)$ by

$$\delta_2^{1/2}(\lambda) = \tfrac{1}{2} \int_0^\lambda \delta_1^{1/2}(t)\, dt, \qquad 0 < \lambda < 1.$$

Evidently, we have

$$\delta_2(\lambda) < \delta_1(\lambda) \leq \epsilon\lambda/2 < \lambda/2. \tag{21}$$

Also, a direct calculation gives

$$\delta_2'(\lambda) = \delta_2^{1/2}(\lambda)\delta_1^{1/2}(\lambda) < \min\{\epsilon/2, \delta_2^{1/2}(\lambda)\}.$$

For $z \in D$, we set $\delta(z) = \delta_2(\rho(z))$, where ρ is the function in Lemma 3. (19) follows since

$$0 < \delta(z) = \delta_2(\rho(z)) < \rho(z)/2 < \rho(z, \partial D)/2$$

and

$$|\bar{\partial}\delta(z)| = |\delta_2'(\rho(z))||\bar{\partial}\rho(z)|.$$

For $z \in \mathbf{C}\setminus D$, set $\delta(z) = 0$. We must verify that $\delta \in C^1(\mathbf{C})$. This follows from (21) and the fact that δ_2' is continuous and $\rho \in C^1(D)$.

The mapping ϕ is defined by the vector-valued integral

$$\phi(z) = \{\pi\delta^2(z)\}^{-1} \iint_{|\zeta - z| < \delta(z)} f(\zeta)\, ds_\zeta, \tag{22}$$

where ds_ζ is Lebesgue planar measure. It is clear from (22) that if f is holomorphic in the disc $|\zeta - z| < \delta(z)$, then $x^*(\phi(z)) = x^*(f(z))$, for each $x^* \in X^*$,

which implies by the Hahn-Banach theorem, that $\phi(z) = f(z)$.

Furthermore

$$\|\phi(z) - f(z)\| \leq \max_{|\zeta - z| \leq \delta(z)} \|f(\zeta) - f(z)\| \leq \omega(\rho(z), \delta(z))$$

$$= \omega(\rho(z), \delta_2(\rho(z))) \leq \omega(\rho(z), \delta_1(\rho(z))) < \epsilon\rho(z)/2 < \epsilon/2 < \epsilon.$$

To complete the proof, there remains only to verify the second inequality of (20).

$$\bar{\partial}\phi(z) = -\frac{2\bar{\partial}\delta(z)}{\pi\delta^3(z)} \iint_{|\zeta - z| \leq \delta(z)} f(\zeta)\, ds_\zeta + \frac{1}{\pi\delta^2(z)} \bar{\partial} \iint_{|\zeta - z| \leq \delta(z)} f(\zeta)\, ds_\zeta.$$

Let $\psi(z, \delta) = \iint_{|\zeta - z| < \delta} f(\zeta)\, ds_\zeta$. Then

$$\bar{\partial}\psi(z, \delta) = \left[\frac{\partial\psi}{\partial\delta}(z, \delta)\bar{\partial}\delta(z) + \frac{\partial\psi}{\partial z}(z, \delta)\bar{\partial}z\right]_{\delta = \delta(z)}.$$

Since $\bar{\partial}z = 0$, we have

$$\bar{\partial}\phi(z) = -\frac{2\bar{\partial}\delta(z)}{\delta(z)}\phi(z) + \frac{1}{\pi\delta^2(z)}\left(\frac{\partial}{\partial\delta}\iint_{|\zeta - z| \leq \delta} f(\zeta)\, ds_\zeta\right)\bar{\partial}\delta(z)$$

$$= -\frac{2}{\delta(z)}\bar{\partial}\delta(z) \cdot \phi(z) + \frac{1}{\pi\delta^2(z)}\left[\frac{\partial}{\partial\delta}\int_0^\delta \int_0^{2\pi} f(z + ue^{i\theta})u\, d\theta\, du\right]\bar{\partial}\delta(z)$$

$$= \frac{2\bar{\partial}\delta(z)}{\delta(z)}\left[-\phi(z) + \frac{1}{2\pi}\int_0^{2\pi} f(z + \delta(z)e^{i\theta})\, d\theta\right]$$

$$= \frac{2\bar{\partial}\delta(z)}{\delta(z)}\left[f(z) - \phi(z) + \frac{1}{2\pi}\int_0^{2\pi} \{f(z + \delta(z)e^{i\theta}) - f(z)\}\, d\theta\right].$$

Thus,

$$\|\bar{\partial}\phi(z)\| \leq \frac{2\epsilon}{\delta(z)}\left[\epsilon + \frac{1}{2\pi}\int_0^{2\pi} \omega(\rho(z), \delta(z))d\theta\right] < \frac{3\epsilon^2}{\delta(z)},$$

and the proof of Lemma 4 is complete.

Having established the required lemmas, we now proceed with the proof of the theorem. Let $D^* \backslash F$ be connected and locally connected. If $f \in A(F, X)$, then we shall prove that $f \in H_F(D, X)$. Since X is an absolute retract [6], we may assume that $f \in C(D, X)$. We may also assume $\infty \notin D$. Given $\epsilon > 0$, let ϕ and δ be the functions from Lemma 4, and T the support of $\bar{\partial}\phi$. Also, let $\{D_n\}$ be a normal exhaustion of D. That is, for each n, D_n is a domain bounded by finitely many disjoint smooth Jordan curves, $\bar{D}_n \subset D_{n+1}$, and $D = \bigcup_{n=1}^\infty D_n$. Using the Hahn-Banach theorem, one easily obtains the following generalization of Pompeiu's formula [7, p. 14]

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\iint_{D_n} \frac{\bar{\partial}\phi(\zeta)}{\zeta - z}\, ds_\zeta, \tag{23}$$

for $z \in D_n$.

Let ζ_0 be such that $d(\zeta_0, D\backslash F) \leq \delta(\zeta_0)$. Therefore there is a point $\zeta' \in (D\backslash F) \cap B(\zeta_0, 1.001\,\delta(\zeta_0))$. Consequently, since $D^*\backslash F$ is connected and locally connected ζ' can be joined to the boundary of D by a curve $\gamma_0 \subset D\backslash F$, such that for each m there is an $n > m$ such that

$$\gamma_0 \cap D_m = \emptyset \quad \text{if } \zeta_0 \in D\backslash D_n. \tag{24}$$

Moreover, one can find an arc σ_0 of γ_0 contained in $B(\zeta_0, 2\delta(\zeta_0))$ and of diameter $0.99\,\delta(\zeta_0)$. An application of Lemma 1 establishes the existence of a function $Q_0(\zeta, z)$ linear in ζ, $Q_0(\zeta, \cdot) \in H(S^2\backslash\sigma_0)$,

$$|Q_0(\zeta, z)| < \frac{100}{\delta(\zeta_0)}, \qquad \left|Q_0(\zeta, z) - \frac{1}{\zeta - z}\right| < \frac{1000\,\delta^2(\zeta_0)}{|\zeta - z|^3},$$

where $\zeta \in B(\zeta_0, 2\delta(\zeta_0))$ and $z \in S^2\backslash\sigma_0$. By restricting ζ to a sufficiently small disc B_{ζ_0} with center at ζ_0 we arrive at the following inequalities:

$$|Q_0(\zeta, z)| < 200/\delta(\zeta), \tag{25}$$

$$|Q_0(\zeta, z) - 1/(\zeta - z)| < 2000\,\delta^2(\zeta)/|\zeta - z|^3, \tag{26}$$

where $\zeta \in B_{\zeta_0}$ and $z \in S^2\backslash\sigma_0$.

The family $\{B_\zeta : \zeta \in T\} = \mathfrak{B}$ is an open cover of $\bar{D}_n \cap T$ for all n. By finite induction and the Heine-Borel theorem we can extract a sequence B_n with associated functions $Q_n(\zeta, z)$, $n = 1, 2, \cdots$, such that $T \cap \bigcup_1^n B_j$ is strictly increasing and for a subsequence of indices n_k, $\bar{D}_k \cap T \subset \bigcup_{j=1}^{n_k} B_j$.

For $\zeta \in T$, we define $Q(\zeta, z) = Q_j(\zeta, z)$, where j is the smallest index such that $\zeta \in B_j$. Let $G_n = T \cap (B_n \backslash \bigcup_1^{n-1} B_j)$. Then we have

$$Q(\cdot, z) \text{ is linear on } G_n. \tag{27}$$

(28) If $\zeta \in G_n$, $Q(\zeta, \cdot)$ is holomorphic on $\Omega_\zeta = S^2\backslash\sigma_n \supset F$, where σ_n is an arc of a curve which joins a point near ζ to the boundary of D,

$$|Q(\zeta, z)| \leq 200/\delta(\zeta), \tag{29}$$

$$|Q(\zeta, z) - 1/(\zeta - z)| < 2000\,\delta^2(\zeta)/|\zeta - z|^3, \tag{30}$$

for $\zeta \in T$ and $z \in \Omega_\zeta$.

The integral

$$I = \frac{1}{\pi} \iint_{T \cap D_n} \bar{\partial}\phi(\zeta) \left\{Q(\zeta, z) - \frac{1}{\zeta - z}\right\} ds_\zeta, \qquad z \in F,$$

exists by (27), and the fact that $T \cap D_n$ is contained in a finite union of the G_k's. We shall show that

(31)
$$\|I\| < A\epsilon,$$

where A is a constant independent of n and of $z \in F$. For $V_z = \{\zeta \in D: |\zeta - z| < 4\delta(\zeta)\}$,

$$\|I\| \leq \frac{1}{\pi} \iint_{T\cap D_n} \frac{\epsilon}{\delta(\zeta)} \left| Q(\zeta, z) - \frac{1}{\zeta - z} \right| ds_\zeta$$
$$= \iint_{T\cap D_n \cap V_z} + \iint_{T\cap D_n \setminus V_z} \leq A\epsilon[I_1 + I_2],$$

by (29) and (30), where

(32)
$$I_1 = \iint_{V_z} \frac{ds_\zeta}{\delta(\zeta)|\zeta - z|}, \qquad I_2 = \iint_{\mathbf{C}\setminus V_z} \frac{\delta(\zeta)}{|\zeta - z|^3}\, ds_\zeta.$$

Because of (19) and the mean-value theorem,

(33)
$$|\delta(\zeta) - \delta(z)| \leq 2\epsilon|\zeta - z|, \quad \zeta, z \in D.$$

Hence if $\epsilon < 1/16$, we obtain for $\zeta \in V_z$, $\delta(z)/2 < \delta(\zeta) < 2\delta(z)$, which implies that $V_z \subset B(z, 8\delta(z))$. Therefore

$$I_1 < \frac{2}{\delta(z)} \iint_{B(z,8\delta(z))} \frac{ds_\zeta}{|\zeta - z|} = 32\pi.$$

Analogously $\mathbf{C}\setminus V_z \subset \mathbf{C}\setminus B(z, 2\delta(z))$ and so

$$I_2 < 2\pi + \iint_{B(z,1)\setminus B(z,2\delta(z))} \frac{\delta(\zeta)}{|\zeta - z|^3}\, ds_\zeta = 2\pi + \int_0^{2\pi} I_3(\theta)\, d\theta.$$

Furthermore, taking (19) into account, and the estimate $\delta(\zeta) \leq |\zeta - z|$, $\zeta \in \mathbf{C}\setminus B(z, 2\delta(z))$, and integrating by parts,

$$I_3 = -\int_{2\delta(z)}^1 \delta(z + \rho e^{i\theta})\, d(1/\rho)$$
$$< \frac{\delta(z + 2\delta(z)e^{i\theta})}{2\delta(z)} + \int_{2\delta(z)}^1 \frac{\delta^{1/2}(z + \rho e^{i\theta})}{\rho}\, d\rho < 1 + \int_0^1 \frac{d\rho}{\sqrt{\rho}} = 3.$$

This completes the proof of (31); which says that the function

$$h_n(z) = \frac{1}{2\pi} \int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{T\cap \bar{D}_n} \bar{\partial}\phi(\zeta) Q(\zeta, z)\, ds_\zeta$$

satisfies

(34)
$$|\phi(z) - h_n(z)| < A\epsilon, \qquad z \in F \cap D_n,$$

where A is a constant independent of n.

Recalling that for $\zeta \in G_j$, $Q(\zeta, z) = g_j(z) + (\zeta - b_j)g_j^2(z)$, where g_j is holomorphic in $S^2\setminus\sigma_j$, we invoke Lemma 2 to obtain a function $\psi_j(z)$ holomorphic in D such that

$$|g_j(z) - \psi_j(z)| < \eta_j, \qquad z \in F \cup \overline{D}_{k(j)},$$

where $k(j)$ is the largest index for which $\overline{D}_k$ does not meet the boundary curve γ_j of which σ_j is an arc, and η_j is chosen so small that setting

$$G(\zeta, z) = \psi_j(z) + (\zeta - b_j)\psi_j^2(z), \tag{35}$$

$\zeta \in G_j$, $z \in D$, we have

$$|G(\zeta, z) - Q(\zeta, z)| < \delta(\zeta)/(1 + |\zeta|^3),$$

for $\zeta \in G_j$ and $z \in F \cup \overline{D}_{k(j)}$. From (35), $G(\cdot, z)$ is integrable over $T \cap D_n$, and we have

$$\left\| \frac{1}{\pi} \iint_{T\cap D_n} \bar{\partial}\phi(\zeta)\{Q(\zeta, z) - G(\zeta, z)\}\, ds_\zeta \right\| \le \frac{1}{\pi} \iint_{T\cap D_n} \frac{\epsilon}{\delta(\zeta)} \frac{\delta(\zeta)}{1 + |\zeta|^3}\, ds_\zeta$$

$$\le \frac{\epsilon}{\pi} \iint_C \frac{ds_\zeta}{1 + |\zeta|^3} = A\epsilon,$$

for $z \in F$, where A is a constant independent of z and of n.

Thus we have shown that the function

$$H_n(z) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{T\cap D_n} \bar{\partial}\phi(\zeta) G(\zeta, z)\, ds_\zeta,$$

holomorphic in D_n, satisfies

$$\|f(z) - H_n(z)\| < A\epsilon, \qquad z \in F \cap D_n, \tag{36}$$

where A is independent of n and z.

To complete the proof we need only show that as $n \to \infty$, H_n converges to a mapping g, holomorphic on all of D. Let K be a compact set in D. Choose n_0 such that $K \subset D_{n_0}$. By (24) there is an N_1 such that $j \ge N_1 \Rightarrow k(j) \ge n_0$. Now choose $N_2 > N_1$ such that $T \setminus D_{N_2} \subset \bigcup_{j=N_1}^{\infty} G_j$. Suppose $N_2 < m < n$. Then, for $z \in K$,

$$H_n(z) - H_m(z) = \frac{1}{2\pi i} \left(\int_{\partial D_n} - \int_{\partial D_m} \right) \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{T\cap(D_n \setminus D_m)} \bar{\partial}\phi(\zeta) G(\zeta, z)\, ds_\zeta.$$

The formula of Pompeiu then gives:

$$\begin{aligned} H_n(z) - H_m(z) &= \frac{1}{\pi} \iint_{T\cap(D_n \setminus D_m)} \bar{\partial}\phi(\zeta) \left\{ \frac{1}{\zeta - z} - G(\zeta, z) \right\} ds_\zeta \\ &= \frac{1}{\pi} \iint_{T\cap(D_n \setminus D_m)} \bar{\partial}\phi(\zeta) \left\{ \frac{1}{\zeta - z} - Q(\zeta, z) \right\} ds_\zeta \\ &\quad + \frac{1}{\pi} \iint_{T\cap(D_n \setminus D_m)} \bar{\partial}\phi(\zeta)\{Q(\zeta, z) - G(\zeta, z)\}\, ds_\zeta \\ &= I_1 + I_2. \end{aligned}$$

The choice of N_2 yields the following estimate:

$$\|I_1\| \leq \frac{1}{\pi} \iint_{D_n \setminus D_m} \frac{\epsilon}{\delta(\zeta)} \frac{2000\, \delta^2(\zeta)}{|\zeta - z|^3} ds_\zeta \leq \frac{2000}{\pi} \iint_{D_n \setminus D_m} \frac{ds_\zeta}{|\zeta - z|^3};$$

$$\|I_2\| \leq \frac{1}{\pi} \iint_{D_n \setminus D_m} \frac{\epsilon}{\delta(\zeta)} \frac{\delta(\zeta)}{1 + |\zeta|^3} ds_\zeta < \frac{1}{\pi} \iint_{D_n \setminus D_m} \frac{ds_\zeta}{1 + |\zeta|^3}.$$

These estimates show that I_1 and I_2 converge to zero, uniformly on K, as $n, m \to \infty$. Thus H_n converges, uniformly on compact subsets of D, and the proof of Theorem 3 is complete.

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