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SEMICELLULARITY, DECOMPOSITIONS AND MAPPINGS IN MANIFOLDS

BY

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ABSTRACT. If X is an arbitrary compact set in a manifold, we give algebraic criteria on X and on its embedding to determine that X has an arbitrarily small, closed neighborhood each component of which is a p-connected, piecewise linear manifold which collapses to a q-dimensional subpolyhedron from some p and q. This property generalizes cellularity. The criteria are in terms of UV properties and Alexander-Spanier cohomology. These criteria are then applied to decide when the components of a given compact set in a manifold are elements of a decomposition such that the quotient space is the n-sphere. Conversely, algebraic criteria are given for the point inverses of a map between manifolds to have arbitrarily small neighborhoods of the type mentioned above; these criteria are considerably weaker than for an arbitrary compact set.

1. Introduction. The concept of cellularity, introduced by M. Brown [7], has been important to the study of embeddings of compact sets in manifolds. This paper deals with the following more general property. We say that a compact set X in a manifold is (p, q)-*semicellular* if X has an arbitrarily small closed neighborhood each component of which is a p-connected piecewise linear manifold which collapses to a q-dimensional subpolyhedron. In §§1–5, we give algebraic criteria for semicellularity analogous to D. R. McMillan's criterion for cellularity [19]. The latter part of this paper presents applications of these results to questions about extending decompositions and about mappings on manifolds.

The problem of extending decompositions was first raised by R. H. Bing [5] and R. J. Bean [4]. It asks: Given a decomposition G of S^n (the n-sphere), under what conditions is there a "nice" (in some precise sense) decomposition H of S^n such that each nondegenerate element of G is an element of H and $S^n/H \cong S^n$.

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There are variations of this question for manifolds other than S^n, too. Bing and Bean answer the question where $n = 3$ and "nice" means monotone upper semicontinuous. B. J. Ball and R. B. Sher [3] have considered this question where $n \neq 4$ and "nice" means cell-like. This paper considers the question in higher dimensions where "nice" means UV^p. (The terms used above are defined later.)

The converse of this problem is also studied. The result could be called a semicellularity criterion for maps analogous to R. C. Lacher's cellularity criterion for maps [16]. It gives geometric conclusions for the point inverses of a map between manifolds given algebraic conditions on them.

The term manifold as used here allows nonempty boundary. If M is a manifold, Int M denotes the interior of M (those points with a neighborhood homeomorphic to Euclidean space); and Bd M denotes the boundary of M (those points with a neighborhood homeomorphic to Euclidean half-space and without a neighborhood homeomorphic to Euclidean space). If $A \subset B$, Cl A denotes the closure of A in B. A *q-spine* for a triangulated manifold is a q-dimensional subpolyhedron to which the manifold collapses. The standard n-ball and n-sphere are denoted B^n and S^n, respectively.

For an arbitrary compact set X in a manifold, we use "UV properties" in place of the usual homeotopy-connectedness. We say that X has *property* UV^k provided given a neighborhood U of X, there is a neighborhood V of X in U such that each singular i-sphere $(i \leq k)$ in V is null-homotopic in U. Also X is *cell-like*, or X has *property* UV^∞, provided given a neighborhood U of X, there is a neighborhood V of X in U such that V is contractible in U. Both of these properties are independent of the particular embedding of X in a manifold [1]. One property which definitely depends on the embedding is *property* CC: Given a neighborhood U of X, there is a neighborhood V of X in U such that each loop in $V - X$ is null-homotopic in $U - X$. For cohomology we use Alexander-Spanier cohomology with integral coefficients [23] denoted $\bar{H}^*(X)$. To denote ordinary (singular, integral) homology and cohomology, we drop the bar over the H.

The main theorem is the following.

Theorem 1. *Let X be a compact proper subset of the interior of a piecewise-linear m-manifold M. If X has CC, each component of X has UV^p, and $\bar{H}^i(X) = 0$ for $i \geq q + 1$, then X has an arbitrarily small compact neighborhood each component of which is a p-connected polyhedral m-manifold that has a q-spine provided p, q and m satisfy one of the following inequalities:*

(a) $p = 1$, $q \geq 2$ *and* $m \geq 5$;

(b) $2 \leq p \leq q - 1 \leq m - 5$;

(c) $p = 2$ *and* $q = m - 3 \geq 3$;

(d) $p > q - 1$, $p \geq 1$ *and* $m \geq 5$.

The division into cases of the inequalities is determined by the proof. The cases are mutually exclusive. In case (d) the conclusion is that each component of the neighborhood is an m-cell. The converse of the theorem is mostly true. The conclusion does imply that each component of X has UV^p and $\overline{H}^i(X) = 0$ for $i \geq q + 1$, but X does not necessarily have CC unless $q \leq m - 3$.

2. **The proof under inequality (a).** Given any neighborhood U of X, there is a compact neighborhood N of X in U each component of which is a polyhedral m-manifold. Since N has only finitely many components and the argument proceeds one component at a time, we will, for notational simplicity, assume that N is connected. By choosing N so that each loop in $N - X$, and specifically in Bd N, is null-homotopic in $U - X$ and using a standard surgery argument on Bd N (see [8], [6] and [10]), we may also assume that Bd N is simply connected. Furthermore, Int $N - X$ is simply connected since any loop there bounds a singular disk in $U - X$ which can be "capped off at the boundary". Similarly by choosing N so that each loop in N is null-homotopic in U, we may assume N is simply connected. Now, if $q \geq m - 1$, we are done; so henceforth assume $q \leq m - 2$.

By duality

$$H_{m-i}(\text{Int } N, \text{Int } N - X) = \overline{H}^i(X) = 0 \quad \text{for } i \geq q + 1,$$

and hence

$$H_j(\text{Int } N, \text{Int } N - X) = 0 \quad \text{for } 0 \leq j \leq m - q - 1.$$

Since $H_1(\text{Int } N, \text{Int } N - X) = 0$ implies that Int $N - X$ is 0-connected, both Int N and Int $N - X$ are 1-connected by the first paragraph. Consequently, (Int N, Int $N - X$) is $(m - q - 1)$-connected by the Hurewicz theorem.

Now $m - q - 1 \leq m - 3$, so we can apply Stallings engulfing theorem [24] to Int N in order to obtain a piecewise-linear homeomorphism $h: N \to N$ such that h is the identity in a neighborhood of Bd N and the $(m - q - 1)$-skeleton P of N is contained in $h(N - X)$. Then $h^{-1}(P) \cap X = \emptyset$. Let Q be the complementary skeleton to P and $R = h^{-1}(Q)$. Then X is contained in a regular neighborhood of R [13, pp. 49–57], [25, L. 8.1]. This is the neighborhood desired since $\dim R = \dim Q = \dim N - \dim P - 1 = q$ and any loop in it bounds a singular disk N which can be homotoped off $h^{-1}(P)$ by general position.

3. **The proof under inequality (b).** In this case we can apply case (a) to obtain an arbitrarily small neighborhood of X each component of which is a 1-connected manifold with a q-spine. We can make each spine p-connected by coning over its p-skeleton, so the problem is to embed the cone in the neighborhood. The next two lemmas allow us to overcome this difficulty.

Lemma 2. *Let $f\colon Q \to N$ be a piecewise-linear, general position map from a polyhedron Q into a piecewise-linear manifold N such that* $\dim S(f) \le r$ *where* $r \ge 0$, *and* $H_k(Q) = 0$ *for* $1 \le k \le p$ *where* $p \ge 1$. *Then* $H_k(f(Q)) = 0$ *for* $r + 2 \le k \le p$.

Proof. Let U be a second derived neighborhood of $S = S(f)$ in Q. Then $V = f(U)$ is a second derived neighborhood of $T = f(S)$ in $f(Q)$. Since $\dim S \le r$ and $\dim T \le r$, $H_k(U) = H_k(V) = 0$ for $k \ge r + 1$. Consider the following diagram where $r + 2 \le k \le p$.

$$\begin{array}{ccccccccc}
\to H_k(U - S) & \longrightarrow & H_k(Q - S) & \longrightarrow & H_k(Q) & \to & H_{k-1}(U - S) & \to \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & \\
\to H_k(V - T) & \to & H_k(f(Q) - T) & \to & H_k(f(Q)) & \to & H_{k-1}(V - T) & \to
\end{array}$$

The horizontal rows come from Mayer-Vietoris sequences. The vertical arrows are induced by f. By the five lemma

$$H_k(f(Q)) \cong H_k(Q) = 0 \quad \text{for } r + 2 \le k \le p.$$

Lemma 3. *Let R be a connected, q-dimensional polydron such that* $H_k(R) = 0$ *for* $r + 2 \le k \le p$ *where* $0 \le r \le q - 3$. *Then there exists a q-dimensional, p-connected polyhedron $Q \supset R$ such that* $\dim (Q - R) \le r + 3$.

Proof. Let $P = R \cup C(R^{r+1})$ be the polyhedron obtained from R by attaching the cone over the $(r + 1)$-skeleton of R. By the Mayer-Vietoris sequence we have

$$H_k(R) \oplus H_k(C(R^{r+1})) \to H_k(P) \xrightarrow{\partial^*} H_{k-1}(R^{r+1}).$$

Because $H_k(R) = 0$ for $r + 2 \le k \le p$ and $C(R^{r+1})$ is contractible, ∂^* is injective for $r + 2 \le k \le p$. For $k = r + 2$, $H_{r+1}(R^{r+1})$ is a free Abelian group of some rank, say n. For $k \ge r + 3$, $H_{k-1}(R^{r+1}) = 0$ and so $H_k(P) = 0$ for $r + 3 \le k \le p$.

Now, since P is $(r + 1)$-connected, $H_{r+2}(P) \cong \pi_{r+2}(P)$ and, therefore, there is a set of free generators for $H_{r+2}(P)$ such that each one is a polyhedral singular $(r + 2)$-sphere $g_l\colon S^{r+2} \to P$ for $l = 1, 2, \cdots, n$. Let $\{B_1, B_2, \cdots, B_n\}$ be a disjoint collection of $(r + 3)$-balls, and denote their union by B. Let $S = \operatorname{Bd} B$ and define a map $g\colon S \to P$ by $g \mid \operatorname{Bd} B_l = g_l$. Then let $Q = P \cup_g B$ be the polyhedron obtained by attaching B to P by g.

We now claim that Q is p-connected. Let U be a second derived neighborhood of S in B. Then $V = P \cup_g U$ is a second derived neighborhood of P in Q. Since the point-set interiors of V in Q and of each B_l in Q are 1-connected and each intersection is 0-connected, Q is 1-connected by Van Kampen's theorem and induction.

Because of the Hurewicz theorem, we now need only to prove that $H_k(Q) = 0$ for $2 \le k \le p$ and for this consider the following commutative diagram.

$$\begin{array}{ccccccccc}
H_{k+1}(B, S) & \cong & H_k(S) & \longrightarrow & H_k(B) & \longrightarrow & H_k(B, S) & \cong & H_{k-1}(S) \\
G_* \downarrow \cong & & g_* \downarrow & & & & G_* \downarrow \cong & & g_* \downarrow \\
H_{k+1}(Q, P) & \xrightarrow{\partial^*} & H_k(P) & \xrightarrow{i_*} & H_k(Q) & \xrightarrow{j_*} & H_k(Q, P) & \xrightarrow{\partial^*} & H_{k-1}(P)
\end{array}$$

The isomorphisms in the top line occur because $H_k(B) = 0$ for $k \ge 1$. The vertical arrows labeled G_* are induced by the map $G: (B, S) \to (Q, P)$ defined as inclusion on Int B and as $g: S \to P$ on Bd $B = S$. It can be shown to induce isomorphisms in every dimension because the inclusion $B - S \subset Q - P$ is a homeomorphism. Then $H_k(Q, P) \cong H_{k-1}(S) = 0$ for $k \ge 2$ and $k \ne r + 3$. Therefore, $H_k(Q) \cong H_k(P) = 0$ for $2 \le k \le p$ and $k \ne r + 2, r + 3$. For $k = r + 2$, we have the short exact sequence

$$0 \to \ker i_* \to H_{r+2}(P) \xrightarrow{i_*} H_{r+2}(Q) \xrightarrow{j_*} 0.$$

But $\ker i_* = \operatorname{im} \partial_* = \operatorname{im} g_*$ by exactness and the presence of isomorphisms on the other two sides of the square. Since the maps g_l: Bd $B_l \to P$ generate $H_{r+2}(P)$, g_* is surjective and $H_{r+2}(Q) = 0$. For $k = r + 3$, we have the short exact sequence

$$0 \to H_{r+3}(P) \xrightarrow{i_*} H_{r+3}(Q) \xrightarrow{j_*} \operatorname{im} j_* \to 0.$$

However, $\operatorname{im} j_* = \ker \partial_* \cong \ker g_*$. Since the maps g_l freely generate $H_{r+2}(P)$, g_* is injective, so $\operatorname{im} j_* = 0$. Thus, $H_{r+3}(Q) \cong H_{r+3}(P) = 0$. In sum, $H_k(Q) = 0$ for $1 \le k \le p$ which was what was needed to complete the proof of the lemma.

To prove Theorem 1 in case (b), let $n = \max\{p + q - m + 3, 1\}$ and choose neighborhoods $U_{np}, U_{np-1}, \cdots, U_1, U_0$ of X by property UV^p and case (a) such that U_0 is arbitrarily small, Int U_i contains closure (U_{i+1}), each singular k-sphere in U_{i+1} is null-homotopic in U_i for $1 \le k \le p$, and U_{np} is a compact neighborhood each component of which is a polyhedral m-manifold that has a q-spine. Let R denote the union of these q-spines.

We will prove by induction that for $k = 1, 2, \cdots, n$, there is a polyhedron Q_k of dimension q each component of which is p-connected, and a piecewise-linear map $f_k: Q_k \to U_{(n-k)p}$ such that $\dim S(f_k) \le n - k - 1$ and $R \subset f_k(Q_k)$.

For $k = 1$, let $Q_1 = R \cup C(R^p)$ where $C(R^p)$ is the union of the cones over p-skeleta of components of R. Define $f_1: Q_1 \to U_{(n-1)p}$ by extending the identity on R over successive skeleta of Q_1 using the fact that singular spheres in U_{i+1} are null-homotopic in U_i and general position. Then

$$\begin{aligned}
\dim S(f_1) &\le \max\{\dim S(f_1 \mid Q_1 - R), \dim(f_1(Q_1 - R) \cap R)\} \\
&\le n - 2.
\end{aligned}$$

Now assume the claim for $1 \leq k-1 \leq n-1$. Let $R_{k-1} = f_{k-1}(Q_{k-1})$ and construct Q_k by applying Lemma 2 and Lemma 3 with $r = n-k$ to each component of R_{k-1}. Construct f_k by extending the identity on R_{k-1} over successive skeleta of Q_k as before. Then

$$\dim S(f_k) \leq (n-k+3) + q - m \leq n-k-1$$

since $q \leq m-4$. By induction $R \subset R_{k-1}$ and by construction $R_{k-1} \subset f_k(Q_k)$, so $R \subset f_k(Q_k)$.

Therefore, there is an embedding $f_n: Q_n \to U_0$ where Q_n is a q-dimensional polyhedron each component of which is p-connected. Since $R \subset f_n(Q_n)$ and a regular neighborhood of R contains X, there is a regular neighborhood of $f_n(Q_n)$ containing X which is the neighborhood of X desired to complete the proof.

4. **Proofs of the remaining cases.** For case (c), essentially the same proof works, but an improved version of Lemma 3 is needed because of the lack of codimension. The following lemma satisfies this need. Its proof is omitted since it is quite similar to the last part of the proof of Lemma 3.

Lemma 4. *Let R be a connected q-dimensional polyhedron, $q \geq 2$. If $\pi_1(R)$ is a finitely generated free group and $H_2(R) = 0$, then there exists a 2-connected q-dimensional polyhedron Q containing R such that $Q - R$ is at most 2-dimensional.*

To prove the theorem in case (c), choose $U_4 \subset U_3 \subset U_2 \subset U_1 \subset U_0$ as before. Let R_1 be the union of the spines for components of U_4 and $Q_1 = R_1 \cup C(R_1^2)$ where $C(R_1^2)$ is the union of the cones over the 2-skeleta of components of R_1. Construct $f_1: Q_1 \to U_2$ with $\dim S(f_1) \leq 0$ and let $R_2 = f_1(Q_1)$. This much is the same as in case (b); now, though, we want to apply Lemma 4 to each component R of R_2. By Lemma 2, $H_2(R) = 0$. We have $\pi_1(R)$ is finitely generated and free since R is obtained from a 2-connected polyhedron by identifying finitely many pairs of points. Hence, Lemma 4 yields a q-dimensional polyhedron Q_2 each component of which is 2-connected such that $\dim (Q_2 - R_2) \leq 2$. We now extend the inclusion $R_2 \subset U_2$ to a map $f_2: Q_2 \to U_0$ (using UV^2 as before) such that $\dim S(f_2) \leq 2 + q - m = -1$. Then f_2 is an embedding and there is a regular neighborhood of $f_2(Q_2)$ which satisfies the conclusion of the theorem.

In case (d), each component of X is cell-like by a result of Lacher [16, Corollary 4.4 (2)]. Therefore, X has an arbitrarily small neighborhood each component of which is an m-cell by [19] and [22].

5. **Remarks and similar theorems.** For a set which is cellular, we know about the boundary of its cell neighborhoods. Similarly, for a compactum which is semicellular, we remark that if N is a p-connected m-manifold that has a q-spine,

then Bd N is n-connected where $n = \min\{p, m - q - 2\}$ by a simple general position argument.

Because of a result of J. Hollingsworth and R. B. Sher [12], we observe that in Theorem 1 if $\bar{H}^i(X) = 0$ for $i \geq q + 1$ and $q = 2$ or 3, then we need not assume M to be piecewise-linear, since in this situation X will have a piecewise-linear neighborhood regardless. Similarly, if each component of X has UV^p and $p \geq 4$, some theorems of Lacher [16, Theorem 3.2] again allow us to prove that $\bar{H}^4(X; Z_2) = 0$ and the hypothesis that M be piecewise-linear is unnecessary.

We also mention that the hypothesis that each component of X have UV^1 in case (a) can be replaced by the weaker hypothesis that there exists a neighborhood V of X such that every loop in V is null-homotopic in M. This hypothesis together with CC implies UV^1.

We now give two more theorems which are variations of Theorem 1. The proofs are not new although the statements may be.

Theorem 5. *Let X be a compact proper subset of the interior of a piecewise-linear m-manifold M with $m \geq 4$. If each component of X has UV^1 and given any (connected) manifold U in M with* Bd $U \cap X = \emptyset$, $U - (X \cap U)$ *is connected, then X has an arbitrarily small, compact neighborhood each component of which is a polyhedral m-manifold that has an $(m-2)$-spine.*

Proof. (Compare this to Theorem 2 of [19].) Let N_1 and N_2 be arbitrarily small compact neighborhoods of X each component of which is a polyhedral m-manifold such that $N_2 \subset \operatorname{Int} N_1$ and every loop in N_2 is null-homotopic in $\operatorname{Int} N_1$. Let M_1 be a component of N_1 and $M_2 = M_1 \cap N_2$. As in the proof of Theorem 1, case (a), we show that the pair $(\operatorname{Int} M_1, \operatorname{Int} M_1 - X)$ is 1-connected. It is 0-connected since $\operatorname{Int} M_1$ is. Now let

$$f\colon (I, \operatorname{Bd} I) \longrightarrow (\operatorname{Int} M_1, \operatorname{Int} M_1 - X)$$

represent an element of $\pi_1(\operatorname{Int} M_1, \operatorname{Int} M_1 - X)$. Cover $f^{-1}(X)$ by a finite number of disjoint subintervals J_i such that $f(J_i) \subset M_2$ and $f(\operatorname{Bd} J_i) \subset M_2 - X$. The points $f(\operatorname{Bd} J_i)$ can be joined by a path $g_i\colon J_i \longrightarrow M_2 - X$ by hypothesis. Each loop $g_i \cdot (f \mid J_i)^{-1}$ is null-homotopic in $\operatorname{Int} M_1$, so f is homotopic in $\operatorname{Int} M_1$ to a path in $\operatorname{Int} M_1 - X$, namely the path obtained by replacing $f \mid J_i$ by g_i for each subinterval J_i. The proof is finished now exactly as it was in Theorem 1, case (a).

Theorem 6. *Let $X = f(Q)$ be a topological embedding of a compact piecewise-linear q-manifold Q in the interior of a piecewise-linear m-manifold M. If X has CC, X is 1-connected and $q \leq m - 3$, $m \geq 6$, then X has an arbitrarily small, compact polyedral manifold neighborhood which collapses to a piecewise-linear embedding $g(Q)$.*

Proof. Given a neighborhood U of X, there is a compact polyhedral neighborhood V of X such that $V \subset \operatorname{Int} U$, V is a manifold, V deformation retracts to X and Bd V is simply connected [9, Theorem 4.6]. The topological embedding f of Q can be approximated by a piecewise-linear embedding g of Q into V so close that $f \simeq g$ in V [21]. Since $f: Q \rightarrow V$ is a homotopy equivalence, so is $g: Q \rightarrow V$. Since V and Bd V are simply connected V collapses to $g(Q)$ [18, Theorem 10.3] and the proof is complete.

This theorem points out that Theorem 1 is only "best possible" when little is known about X. When more is known about X, it should be possible to obtain more precise information about the spine. Theorem 6 is one step in this direction.

Attempts to extend Theorem 1 to include the case $p = 0$ are probably pointless since, for dim $X \leq m - 2$, if X has CC, then the components of X have UV^1, and for dim $X = m - 1$, then $RP^{m-1} \subset RP^m$ (m-odd) provides one counterexample. (RP^k denotes k-dimensional projective space.) On the other hand, there is hope of extending the result in the direction of requiring less codimension: that is $q = m - 2$ or $q = m - 3$, $m \geq 4$. For $p = 1$, $m \geq 5$ this has been done (Theorem 1 (a)) and for $p = 2$, this has been partly done (Theorem 1 (c)). Notice that getting neighborhoods with a q-spine is not the problem (Theorem 1 (a) and Theorem 5); the difficulty is to obtain the p-connectivity. Extending Theorem 1 (d) to include $m = 3$ and $m = 4$ is hindered by the Poincaré conjecture. One still obtains cell-likeness for the components of X and if $m = 3$ and some neighborhood of X embeds in E^3, the conclusion still holds [19].

6. Extending UV^p decompositions.

We now apply the semicellularity criterion (Theorem 1) to the problem of extending decompositions. The standard terminology for decompositions will be used without definition; refer to [11] or [14]. A decomposition is a *UV^p-decomposition* if each element is a compact set with property UV^p. A UV^0-decomposition is more frequently known as a *monotone* decomposition.

The theorems are stated in terms of maps rather than decompositions for convenience. A map $f: M \rightarrow N$ between manifolds is a *proper* map if f is closed and $f^{-1}(y)$ is compact for every $y \in N$. A proper map is a *UV^p-map* provided each point inverse has property UV^p. A proper, surjective, UV^p-map induces an upper semicontinuous, UV^p decomposition G_f defined by $G_f = \{f^{-1}(y) \mid y \in N\}$. This correspondence allows us to state as corollaries analogous results in terms of decompositions.

Theorem 7. *Let X be a compact subset of* Int M *where M is a compact, $(p+1)$-connected, piecewise-linear n-manifold,* Bd M *is nonempty and p-connected, $p \geq 1$ and $n \geq 6$. If X has CC, each component of X has UV^p and $\bar{H}^i(X) = 0$ for $i \geq n - p - 1$, then there is a surjective, UV^p-map $f: M \rightarrow B^n$ such that*

$f(\mathrm{Bd}\, M) = S^{n-1}$, $f(\mathrm{Int}\, M) = \mathrm{Int}\, B^n$, $f(X)$ is a tame zero-dimensional set and for each $y \in N$, $f^{-1}(y)$ is a point, a component of X or an $(n-p-2)$-dimensional polyhedron disjoint from X.

The following two lemmas will be needed in the proof of Theorem 7 and in subsequent theorems.

Lemma 8. *Let W be a compact piecewise-linear m-manifold such that both W and* Bd W *are q-connected* (Bd $W \neq 0$) *where $2q + 3 \leq m$. Then the following statements are true.*

(i) *W has a q-connected, $(m-q-1)$-spine.*

(ii) *If $H_{q+1}(W, \mathrm{Bd}\, W) = 0$ and $q \geq 1$, $m \geq 6$, then W has a q-connected, $(m-q-2)$-spine.*

(iii) *If W is $(q+1)$-connected and $q \geq 1$, $m \geq 6$, then W has a $(q+1)$-connected, $(m-q-2)$-spine.*

The proof is a standard handle cancellation argument and is implicit in most proofs of the high dimensional Poincaré theorem or h-cobordism theorem; see [13], [18]. The details were written out in [8]. Consequently, we omit the proof here.

Lemma 9. *Let W be a compact, piecewise-linear m-manifold which has nonempty, connected boundary and a p-connected q-spine K for some p and q. Let f:* Bd $W \longrightarrow S^{m-1}$ *be a surjective UV^p-map such that each nondegenerate point-inverse is a p-connected polyhedron of dimension $\leq q$. Then f extends to $F: W \longrightarrow B^m$ such that F is a UV^p-map, each nondegenerate point-inverse of F is a p-connected polyhedron of dimension $\leq q$, and F* (Int W) = Int B^m.

Proof. There is a piecewise linear homeomorphism

$$h: (W - K, \mathrm{Bd}\, W) \longrightarrow (\mathrm{Bd}\, W \times (0, 1], \mathrm{Bd}\, W \times \{1\}).$$

Let h_1 denote h followed by projection onto Bd W and h_2 denote h followed by projection onto $(0, 1]$. Consider $B^m - \{0\}$ to have spherical coordinates:

$$B^m - \{0\} = \{(s, t) | s \in S^{m-1}, t \in (0, 1]\}.$$

Define

$$F(x) = \begin{cases} 0, & \text{if } x \in K, \\ (fh_1(x), h_2(x)), & \text{if } x \in W - K. \end{cases}$$

Clearly this F satisfies the conclusion of the lemma.

Remark. If f has finitely many nondegenerate point-inverses, say k, then the image of the nondegenerate point-inverses of F is a k-frame, or k-od, with its branch point the image of K.

The following proof of Theorem 7 owes much to Bing's proof of Theorem 7.3 in [5] and Bean's proof of Theorem 2 in [4].

Proof. By Theorem 1(a), (b) and (d), X is the intersection of a sequence of compact, polyhedral manifolds M_i where $M_1 = M$, and for each i, $M_{i+1} \subset \operatorname{Int} M_i$ and each component M_i is a p-connected n-manifold that has an $(n-p-2)$-spine. If $2p+2 \geq n$, notice that for all i each component of M_i is an n-cell. This is given by Theorem 1(d) for $i \geq 2$ and by an argument using duality and the universal coefficient theorem for $M_1 = M$. In B^n choose a sequence of finite collections of polyhedral n-cells C_i such that there is a one-to-one correspondence between the components of M_i and the cells of C_i which commutes with the inclusion relations between components, and such that the intersection of the C_i's is a tame zero-dimensional set. Take $C_1 = B^n$.

The desired map f will be defined inductively. Specifically, we claim that for each k

$$f|M - \operatorname{Int} M_k : M - \operatorname{Int} M_k \to B^n - \operatorname{Int} C_k$$

can be defined such that there is at most one nondegenerate point-inverse on each component of $\operatorname{Bd} M_k$ in addition to the relevant properties listed in the conclusion. For $2p+2 \geq n$, and for all k, this map can be taken to be a homeomorphism. Hence, throughout the following induction argument we assume $2p+3 \leq n$.

For $k = 1$, we define $f|\operatorname{Bd} M$: $\operatorname{Bd} M \to S^{n-1}$ as follows. If $2p+4 \leq n$, define f first on an $(n-1)$-ball in $\operatorname{Bd} M$ to be a homeomorphism onto an $(n-1)$-ball in S^{n-1}. Then apply Lemma 8(i) with $q = p$ and $m = n-1$ to the closure of the complement of this ball in $\operatorname{Bd} M$ and define $f|\operatorname{Bd} M$ by Lemma 9. For $2p+3 = n$ use the same approach except to substitute Lemma 8(iii) with $q = p-1$ and $m = n-1$. Notice that if $2p+3 = n$ and $n \geq 6$, then $n \geq 7$ and $p \geq 2$.

Now assume $f|M - \operatorname{Int} M_k$ has been defined and $k \geq 1$. First choose a (minimal) collection of disjoint tubes (relative regular neighborhoods of arcs) joining $\operatorname{Bd} M_{k+1}$ to $\operatorname{Bd} M_k$ in $M_k - \operatorname{Int} M_{k+1}$ which miss the nondegenerate point-inverses in $\operatorname{Bd} M_k$. Let T_k denote the union of these tubes and define f on T_k to be a homeomorphism onto a union of corresponding tubes in $C_k - \operatorname{Int} C_{k+1}$.

Secondly, let

$$U_{k+1} = \operatorname{Cl}[\operatorname{Bd} M_{k+1} - (T_k \cap \operatorname{Bd} M_{k+1})].$$

Note that $T_k \cap \operatorname{Bd} M_{k+1}$ is a collection of $(m-1)$-cells. Hence, each component of $\operatorname{Bd} U_{k+1}$ is p-connected since $p \leq m-3$. Also, since each component of M_{k+1} is p-connected and has an $(n-p-2)$-spine, a general position argument allows us to conclude that each component of $\operatorname{Bd} M_{k+1}$ and, hence, of U_{k+1} is p-connected. Therefore, U_{k+1} has an $(n-p-2)$-spine each component of which is p-connected by Lemma 8(i) if $2p+4 \leq n$ and by Lemma 8(iii) if $2p+3 = n$. Now define f on U_{k+1} by Lemma 9.

Thirdly, let

$$N_k = \mathrm{Cl}\,[M_k - (T_k \cup M_{k+1})]$$

and

$$D_k = \mathrm{Cl}\,[C_k - (f(T_k) \cup C_{k+1})].$$

The same sort of general position argument as above allows us to conclude that each component of both N_k and $\mathrm{Bd}\, N_k$ is p-connected. Furthermore, $H_{p+1}(N_k, \mathrm{Bd}\, N_k) = 0$ by excision since $H_p(N_k) = 0$ and $H_{p+1}(M) = 0$. Therefore, by Lemma 8(ii), N_k has an $(n-p-2)$-spine each component of which is p-connected. Again define f on N_k by Lemma 9. Thus, we have defined f on all of $M_k - \mathrm{Int}\, M_{k+1}$, so $f\,|\,M - \mathrm{Int}\, M_{k+1}\colon M - \mathrm{Int}\, M_{k+1} \to B^n - \mathrm{Int}\, C_{k+1}$ is defined. We have completed the inductive step.

Therefore, we can define

$$f|M - \bigcap_{k=1}^{\infty} M_k : M - \bigcap_{k=1}^{\infty} M_k \to B^n - \bigcap_{k=1}^{\infty} C_k.$$

Each component Y of X is uniquely determined by a sequence of components one from each M_k and the corresponding cells of C_k intersect in a point which will be, by definition, the image of Y.

This completes the definition of the map f and it has the desired properties by construction so the proof is finished.

Remark. Because of the remark following the proof of Lemma 9, we can observe that the image of the nondegenerate point-inverses of f is an infinite tree which intersects each component of $\mathrm{Bd}\, C_k$ at most once and ends on $f(X)$.

Remark. Assuming that M is p-connected but not $(p+1)$-connected, the proof gives the same conclusion except that a countable number of $(n-p-1)$-dimensional polyhedra may occur as nondegenerate point-inverses. These are the spines of the manifolds N_k in the proof. This happens because the application of Lemma 8(ii) requires $H_{p+1}(N_k, \mathrm{Bd}\, N_k) = 0$ which results from $H_{p+1}(M) = 0$. No further improvements can be made in this direction since a UV^p-map induces isomorphisms of homotopy groups π_i for $i \le p$, [2, Theorem 5.1], [15, Theorem 3.1].

Similarly, in the case $n = 5$, the proof yields the same conclusion except that the dimension of all the added nondegenerate point-inverses may be $n - p - 1 = 3$. This occurs because neither 1-connected 4-manifolds nor 1-connected 5-manifolds necessarily have 2-spines, but they do have 3-spines trivially and by Lemma 8(i), respectively.

7. Extending monotone decompositions in high dimensions. In this section we prove the $p = 0$ analogue of Theorem 7. The proof is similar in some respects to

R. H. Bing's proof of this theorem in dimension 3, but the exact analogue of his proof fails.

Theorem 10. *Let* X *be a compact subset of* Int M *where* M *is a compact, piecewise-linear, connected n-manifold with nonempty connected boundary,* $n \geq 3$. *If* $\overline{H}^i(X) = 0$ *for* $i \geq n-1$, *then there is a surjective, monotone map* $f: M \to B^n$ *such that* $f(\text{Bd } M) = S^{n-1}$, $f(\text{Int } M) = \text{Int } B^n$, $f(X)$ *is a tame zero-dimensional set, and each point-inverse of* f *is a point, a component of* X, *or an* $(n-2)$*-dimensional polyhedron disjoint from* X.

The proof is similar to that of Theorem 7, but we do not get the $(n-2)$-spine from Lemma 8 whose complement is a product with $(0, 1]$. We can get a "layering" of sorts in the next lemma, however.

For that we need some more definitions. A *pinched manifold* is the identification space obtained from a manifold by identifying a single pair of points. The image of this pair is called the *pinch point.* A *p-handle* attached to a piecewise linear n-manifold W is a copy of $B^p \times B^{n-p}$ attached to Bd W by means of a homeomorphism on $S^{p-1} \times B^{n-p}$. Refer to [13] or [18] for additional definitions or background on handles.

Lemma 11. *Let* W *be a compact, connected, piecewise-linear n-manifold with connected, nonempty boundary and* $n \geq 3$. *Then there is a monotone, piecewise-linear map* $g: W \to [0, 1]$ *satisfying the following properties*:

(i) $g^{-1}(1) = \text{Bd } W$;

(ii) $g^{-1}(0)$ *is an* $(n-2)$*-dimensional polyhedron*;

(iii) *there is a finite collection* $1/2 < t_1 < t_2 < \cdots < t_k < 1$ *where* $g^{-1}(t_i)$ *is a pinched manifold*;

(iv) *if* $J = (0, t_1), (t_i, t_{i+1})$, *or* $(t_k, 1]$ *and* $t \in J$, *then* $g^{-1}(t)$ *is an* $(n-1)$*-manifold and* $g^{-1}(J) \cong g^{-1}(t) \times J$ *by a homeomorphism which takes* $g^{-1}(s)$ *to* $g^{-1}(t) \times \{s\}$ *for each* $s \in J$; *and*

(v) *there is an* $\epsilon > 0$ *and for each* i *a projection* $p_i: g^{-1}([t_i - \epsilon, t_i + \epsilon]) \to g^{-1}(t_i)$ *such that for* $t \in [t_i - \epsilon, t_i + \epsilon]$, $p_i | g^{-1}(t)$ *is one-to-one except on the pinch point and* $(p_i | g^{-1}(t))^{-1}$ *of the pinch point is homeomorphic to* S^{n-2} *for* $t < t_i$ *and is two points* $t > t_i$.

Proof. Since W and Bd W are connected, W can be obtained from a collar C of Bd W by attaching handles in increasing order none of which are 0-handles. Let U be C union all of the 1-handles and $V = \text{Cl}(W - U)$. Then V has a handle decomposition with handles of dimension $n-2$ at most which is obtained by "reversing the handles". Consider each p-handle $h^p = B^p \times B^{n-p}$ as $k^{n-p} = B^{n-p} \times B^p$. The order of attachment is also reversed and since Bd $W \cap V = \varnothing$, each k^{n-p} is attached as an $(n-p)$-handle. Then V has an $(n-2)$-spine K. Let $V-K$ be identified with $\mathrm{Bd}\, V \times (0, 1/2]$ and define g on $V-K$ by $g(b, t)=t$ for $(b, t) \in \mathrm{Bd}\, V \times (0, 1/2]$. Define g in K by $g(K)=0$.

Now let C be identified with $\mathrm{Bd}\, W \times [1/2, 1]$ such that $\mathrm{Bd}\, W$ is $\mathrm{Bd}\, W \times \{1\}$. The attaching tube of each 1-handle is a copy of $S^0 \times B^{n-1}$ lying in $\mathrm{Bd}\, W \times \{1/2\}$. For each 1-handle b_i in U, let D_i and E_i be caps in C over the two components of the attaching tube of b_i defined as the cone over each component to the point in $\mathrm{Bd}\, W \times \{3/4\}$ corresponding to the image of $0 \in B^{n-1}$. Define g on $C-(\bigcup(D_i \cup E_i))$ by $g(b, t)=t$ where $(b, t) \in \mathrm{Bd}\, W \times [1/2, 1]$. In each one-handle b_i let x_i be the image of $(0, 0) \in B^1 \times B^{n-1}$. Choose $t_i \in (1/2, 3/4)$, different for each i, and define $g(x_i)=t_i$. Extend g to $b_i \cup D_i \cup E_i$ by coning. This completes the definition of g. The properties listed in the conclusion can now be easily checked noting that each layer $g^{-1}(t)$ $(t \neq 0)$ is obtained from $\mathrm{Bd}\, W$ by removing pairs of $(n-1)$-cells and replacing each pair by another pair of $(n-1)$-cells, an annulus $B^1 \times S^{n-2}$, or (in finitely many cases) the cone over two copies of S^{n-2}.

Next we need a lemma corresponding to Lemma 9 for the type of layering just obtained.

Lemma 12. *Let W be a compact, connected, piecewise-linear n-manifold, $n \geq 3$, with connected, nonempty boundary. If f: $\mathrm{Bd}\, W \to S^{n-1}$ is a monotone map such that the nondegenerate point-inverses of f form a finite collection of $(n-2)$-dimensional polyhedra, then f extends to a monotone map F: $W \to B^n$ such that each nondegenerate point-inverse of F is an $(n-2)$-dimensional polyhedron.*

Proof. Let g: $W \to [0, 1]$ be the map given by Lemma 11. Assume the intervals $[t_i - \epsilon, t_i + \epsilon]$ are mutually disjoint. There is a map

$$P_k : g^{-1}(1) \times [t_k, 1] \to g^{-1}([t_k, 1])$$

such that $P_k \mid g^{-1}(1) \times \{s\}$ is the homeomorphism given in (v) if $s \neq t_k$, and $P_k \mid g^{-1}(1) \times \{t_k\} = p_k \mid g^{-1}(1)$. Choose a tame arc A in $\mathrm{Bd}\, W$ such that A contains the pair of points $(p_k \mid g^{-1}(1))^{-1}$ of the pinch point of $g^{-1}(t_k)$, and A intersects the nondegenerate point-inverses in exactly an endpoint. (Assume this pair of points is disjoint from the nondegenerate point-inverses to avoid trivial case-taking.) Then $f(A)$ is an arc in S^{n-1}, and there is a pseudo-isotopy b: $S^{n-1} \times I \to S^{n-1} \times I$ shrinking $f(A)$ to a point. Now define $F \mid g^{-1}([t_k, 1])$ by $F(x) = b(f \times \mathrm{id}(P_i^{-1}(x)))$. This is well defined since P_i is one-to-one everywhere except onto the pinch point of $g^{-1}(t_k)$, and b was chosen to correct that. Then define F on $g^{-1}([t_k - \epsilon, t_k])$ by $F(x) = F(p_k(x))$. Next define F successively on each $g^{-1}([t_i - \epsilon, t_{i+1} - \epsilon])$ in the same way as above. Finally define F on $g^{-1}([0, t_1 - \epsilon])$ by Lemma 9, and the proof is complete.

Proof of Theorem 10. Follow the same steps as in the proof of Theorem 7 except to use Lemma 12 instead of Lemmas 8 and 9 when it comes to defining the map f on the manifold $N_k = \mathrm{Cl}[M_k - (T_k \cup M_{k+1})]$. Notice that on Bd M and on $U_{k+1} = \mathrm{Cl}[\mathrm{Bd}\ M_{k+1} - (T_k \cap \mathrm{Bd}\ M_{k+1})]$ there is no problem since an $n-1$ manifold with boundary has an $(n-2)$-spine trivially.

Remark. Theorems similar to Theorems 7 and 10 can be proven in the case Bd $M = \emptyset$. Simply map a ball disjoint from X homeomorphically onto $\mathrm{Cl}(S^n - B^n)$ as the first step in the induction; the rest is the same. Of course, here the range is S^n, not B^n; and one should add $X \neq M$ to the hypothesis. We state this version in terms of decompositions to illustrate the relationship between the two points-of-view.

Corollary 13. *Let M be a compact, $(p+1)$-connected, piecewise-linear n-manifold with* Bd M *empty, $n \geq 3$, $p \geq 0$. Let G be an upper semicontinuous UV^p decomposition of M whose nondegenerate elements are the components of a compact set X. Assume $\overline{H}^i(X) = 0$ for $i \geq n-p-1$; and if $p \geq 1$ assume X has CC, and $n \geq 6$. Then there is an upper semicontinuous UV^p decomposition H of M which extends G (in the sense that each nondegenerate element of G is an element of H), such that $M/H \cong S^n$, and the nondegenerate elements of H which are not in G are $(n-p-2)$-dimensional polyhedra.*

This is Bing's theorem [5, Theorem 7.3] in the case that $p = 0$ and $n = 3$ since the condition that $\overline{H}^i(X) = 0$ for $i \geq n-1$ implies the non-locally-separating condition of Bing.

8. Semicellularity criteria for maps. We now give some criteria for the point-inverses of a map between manifolds to be (p, q)-semicellular. There are fewer conditions needed than for an arbitrary set. Lacher first noticed this in relation to cellularity. The following theorem includes his result [16, Theorem 5.4] for completeness.

Theorem 14. *Let $f: M \to N$ be a proper, surjective, UV^p-map between manifolds without boundary where M has dimension $m \geq 5$, N has dimension $n \geq 3$, and $p \geq 1$. Then the following may be concluded:*

(i) *If $m \leq 2p+1$, then each point-inverse is cellular.*

(ii) *If $m = 2p+2$ and $H_{p+1}(M) = 0$, then each point-inverse is cellular.*

(iii) *If $m \geq 2p+2$, then $f^{-1}(y)$ has a compact, p-connected, polyhedral manifold neighborhood which has an $(m-p-1)$-spine for all $y \in N$.*

(iv) *If $m \geq 2p+2$, then $f^{-1}(y)$ has a compact, p-connected, polyhedral manifold neighborhood which has an $(m-p-2)$-spine for all but a locally finite subset of points $y \in N$.*

(v) *If* $m \geq 2p+3$ *and* $H_{p+1}(M) = 0$, *then* $f^{-1}(y)$ *has a compact, p-connected, polyhedral manifold neighborhood which has an* $(m-p-2)$*-spine for every* $y \in N$.

Before we prove Theorem 14, we need the following lemmas.

Lemma 15. *Let* $f: M \to N$ *be a proper, surjective,* UV^1*-map between manifolds without boundary where* N *has dimension* $n \geq 3$. *If* Y *is a compact, zero-dimensional set in* N *which is contained in an n-ball and is tame there, then* $f^{-1}(Y)$ *has* CC.

Proof. Let $X = f^{-1}(Y)$. Given an open set U containing X choose a finite collection of open balls B in $f(U)$ containing Y, and let $V = f^{-1}(B)$. On each component of $V - X$, f is a UV^1-map so f induces isomorphisms on the fundamental groups by [2, Theorem 5.1], [15, Theorem 3.1]. Since $n \geq 3$, $B - Y$ is simply connected. Hence, X has CC.

Lemma 16. *Let* $f: M \to N$ *be a proper, surjective,* UV^p*-map between manifolds without boundary wehre* M *has dimension* m, N *has dimension* n *and* $1 \leq p \leq n-2$. *Then*:

(i) $\overline{H}^i(f^{-1}(y)) = 0$ *for* $i \geq m-p$ *for every* $y \in N$;

(ii) $\overline{H}^{m-p-1}(f^{-1}(y)) = 0$ *for all but a locally finite set of points* $y \in N$; *and*

(iii) *if* $H_{p+1}(M) = 0$, *then* $\overline{H}^{m-p-1}(f^{-1}(y)) = 0$ *for every* $y \in N$.

Proof. Statements (i) and (ii) are proven in [17]; see Theorems 1.3, 2.3 and §4. To prove (iii) consider the commutative diagram:

$$\begin{tikzcd}
H_{p+1}(M) \arrow[r] & H_{p+1}(M, M-X) \arrow[r] & H_p(M-X) \arrow[r, "\alpha"] \arrow[d, "(f|M-X)_*"] & H_p(M) \arrow[d, "f_*"] \\
H_p(B-\{y\}) \arrow[r] & H_p(B) \oplus & H_p(N-\{y\}) \arrow[r, "\beta"] & H_p(N)
\end{tikzcd}$$

In this diagram B denotes a ball containing y and $X = f^{-1}(y)$. The top line is part of the exact sequence of the pair $(M, M-X)$ and the bottom line is part of the Mayer-Vietoris sequence for $N = B \cup (N - \{y\})$. Since f is a UV^p-map $(f|M-X)_*$ and f_* are isomorphisms [16, Lemma 5.1]. Therefore, α is an injection since $H_p(B-\{y\}) = H_p(B) = 0$. By hypothesis $H_{p+1}(M) = 0$, so by exactness $H_{p+1}(M, M-X) = 0$. This completes the proof since by duality [23, Theorem 6.2.17] $H_{p+1}(M, M-X) \cong \overline{H}^{m-p-1}(X)$. (Note: An orientable neighborhood can be obtained since X has UV^1 as in §2. Use this if M is nonorientable.)

Proof of Theorem 14. Lacher [16, Corollary 4.4(2)] has shown that if a compact set X has property UV^p and $\overline{H}^i(X) = 0$ for $i \geq p+1$, then X has property UV^∞. Thus, the cellularity criterion of McMillan [19] yields (i) and (ii), and the semicellularity criterion, Theorem 1, yields (iii), (iv) and (v).

Remark. Lemmas 15 and 16 also prove a converse of Theorem 7. For this we need the full strength of Lemma 15, rather than the version where $Y = \{y\}$ used in the proof of Theorem 14. The corresponding strengthening of Lemma 16 is not needed since $\overline{H}^i(f^{-1}(Y)) = 0$ if and only if $\overline{H}^i(f^{-1}(y)) = 0$ for every $y \in Y$ where Y is a compact, zero-dimensional set.

Lacher [16] has proven that Theorem 14 (i) and (ii) are true assuming only that f is a UV^{p-1}-map and a $p - uv(Z)$-map. The latter means that for each $y \in N$, given an open set U containing $f^{-1}(y)$, there is an open set V with $f^{-1}(y) \subset V \subset U$ such that i: $V \subset U$ induces the zero homomorphism $H_p(V; Z) \to H_p(U; Z)$. Improving this result slightly, we have the following.

Theorem 17. *Let f: $M \to N$ be a proper, surjective, UV^{p-1} and $p - uv(Z_q)$-map where M has dimension $m \geq 5$, N has dimension $n \geq 3$,* Bd $M =$ Bd $N = \emptyset$ *and q is a prime integer. If $m = 2p$, or if $m = 2p+1$ and $H_p(M) = 0$, then $f^{-1}(y)$ is cellular for every $y \in N$.*

The following lemma, needed to prove Theorem 17, is a generalization of Theorem 3 of [20], and is proven in the same way. The terminology is due to Stallings [26] and is also defined in [20]. Here we use it only for the statement of Lemma 18 and the proof of Theorem 17.

Lemma 18. *Let M be a manifold, Y be a Hausdorff space, and f: $M \to Y$ be a UV^{p-1} and $p - uv(Z_q)$-map where $p \geq 1$ and q is a prime integer. If there is a $y_0 \in Y$ such that given an open set U containing $f^{-1}(y_0)$, there is an open set V with $f^{-1}(y_0) \subset V \subset U$ such that each singular p-sphere in V projects under f to a null-homotopic p-sphere in $f(U)$, then $f^{-1}(y_0)$ has an (ω, q)-trivial π_p-shape.*

Proof of Theorem 17. Let $y \in N$. Given any neighborhood U of $f^{-1}(y)$ in M, choose an open n-ball B in $f(U)$ containing y. Let $V = f^{-1}(B)$. Then any singular p-sphere in V projects under f to a null-homotopic singular p-sphere in $f(U)$. By Lemma 18, $f^{-1}(y)$ has an (ω, q)-trivial π_p-shape. To prove that $f^{-1}(y)$ has property UV^p, let U be any open set containing $f^{-1}(y)$ and choose open sets V_1 and V_2 such that $f^{-1}(y) \subset V_1 \subset V_2 \subset U$; V_2 is a compact, polyhedral $(p-1)$-connected m-manifold that has a p-spine; and each singular p-sphere in V_1 represents an element of F_ω of the q-central series of $F = \pi_p(V_2)$. To obtain V_2 use Theorem 14(iii) when $m = 2p$ and Theorem 14(v) when $m = 2p+1$. The definition of (ω, q)-trivial π_p-shape yields V_1. Since V_2 is $(p-1)$-connected, $F = \pi_p(V_2) \cong H_p(V_2)$. But $H_p(V_2)$ is a free abelian group since V_2 has a p-spine. Thus, F is a free abelian group and $F_\omega = 0$ [26]. Therefore, each singular p-sphere in V_1 is null-homotopic in $V_2 \subset U$, and f has property UV^p by definition. Application of Theorem 14(i) completes the proof.

An example constructed like the dyadic solenoid using regular neighborhoods of p-spheres rather than solid tori illustrates that an arbitrary compactum in a manifold can have properties UV^{p-1} and $p-uv(Z_2)$, but not property UV^p. Theorem 17 says that this cannot happen for point-inverses of a map.

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