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AN APPROACH TO FIXED-POINT THEOREMS ON UNIFORM SPACES

BY

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ABSTRACT. Diaz and Metcalf [2] have some interesting results on the set of successive approximations of a self mapping which is either a nonexpansion or a contraction on a metric space with respect to the set of fixed points of the mapping. We have extended most of these results to a Hausdorff uniform space. We have also proved a Banach's contraction mapping principle on a complete Hausdorff uniform space and indicated some applications in locally convex linear topological spaces.

Introduction. Let (X, d) be a metric space. Then a mapping f of X into itself is called a contraction on X if there exists a real number r with $0 < r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all points x and y in X. Banach's contraction mapping principle states that if (X, d) is a complete metric space and f is a contraction on X, then f has a unique fixed point $a \in X$ such that $f^n(x) \longrightarrow a$ for each $x \in X$. This principle is well known for its wide scope of applications in analysis. It would, therefore, be of some interest to extend this principle in complete Hausdorff uniform spaces which are generalizations of complete metric spaces.

Let (X, h) be a uniform space, h being the uniformity, i.e., the family of entourages. Taylor [1] has introduced the following definitions:

Let $\mathfrak{B}$ be a base for h. If f maps X into itself; then

(a) f is said to be $\mathfrak{B}$-nonexpansion on X if $(x, y) \in H$ implies $(f(x), f(y)) \in H$ for each $H \in \mathfrak{B}$.

(b) f is said to be a $\mathfrak{B}$-contraction on X if, for each $H \in \mathfrak{B}$, there is a $K \in \mathfrak{B}$ such that $(x, y) \in H \circ K$ implies $(f(x), f(y)) \in H$.

(c) f is said to be asymptotically regular if for each $x \in X$ and entourage $H \in h$ there is a positive integer n_0 such that $(f^n(x), f^{n+1}(x)) \in H$ for $n \geq n_0$.

The following result is obtained in [1, Lemma 1.5].

Let (X, h) be a complete well-chained Hausdorff uniform space and $\mathfrak{B}$ a base for h. If f is a $\mathfrak{B}$-contraction on X, then f has a unique fixed point $a \in X$ such that $f^n(x) \longrightarrow a$ for each $x \in X$. (For definition of well-chained uniform space see [1, p. 166].)

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This result is not an exact analogue of Banach's contraction mapping principle in the sense that an additional condition of the space being well chained is imposed on X.

In the first section of this paper by giving a suitable definition of contraction mapping on a uniform space (which will reduce to the well-known definition of contraction mapping stated in the beginning when the uniform space is a metric space) we have obtained an exact analogue of Banach's contraction mapping principle on a complete Hausdorff uniform space.

In [2] Diaz and Metcalf have obtained a series of results on the cluster set of successive approximations in a metric space by using mainly the nonexpansion and contraction of a mapping with respect to the set of fixed points of the mapping. The main source of this work of Diaz and Metcalf is a paper of Tricomi [3] which is concerned with iteration of a real function. In §2 of our paper we have shown that most of the results of [2] can be extended to a uniform space.

In the concluding section we have indicated some applications of our results in locally convex linear topological spaces. Finally we should point out that the theory of nonexpansive mappings is growing very rapidly and a good account of the existing literature can be obtained in [4].

1. Let (X, b) be a uniform space, b being the uniformity. The uniform topology induced by b will be denoted by τ_b. A family $[\rho_\alpha: \alpha \in I]$ of pseudometrics on X is called an associated family for the uniformity b on X if the family $[H(\alpha, \epsilon): \alpha \in I, \epsilon > 0]$, where $H(\alpha, \epsilon) = [(x, y): \rho_\alpha(x, y) < \epsilon]$, is a subbase for b. (For definition of subbase and base for b see Kelly [5].) A family $[\rho_\alpha: \alpha \in I]$ of pseudometrics on X is called an augmented associated family for b if $[\rho_\alpha: \alpha \in I]$ is an associated family for b and has the additional property that, given $\alpha, \beta \in I$, there is a $\nu \in I$ such that $\rho_\nu(x, y) \geq \max(\rho_\alpha(x, y), \rho_\beta(x, y))$ for all $(x, y) \in X \times X$. An associated family and an augmented associated family for b will be respectively denoted by $A(b)$ and $A^*(b)$.

It is well known that if (X, b) is a uniform space and $[\rho_\alpha: \alpha \in I] = A^*(b)$, then the family $[H(\alpha, \epsilon): \alpha \in I, \epsilon > 0]$ is a base for b (see Thron [6, p. 179], or [5, pp. 188–189]). It is also well known that for each uniformity b on X, there exists a family $[\rho_\alpha: \alpha \in I]$ of pseudometrics on X such that $[\rho_\alpha: \alpha \in I] = A^*(b)$, and that every family $[\rho_\alpha: \alpha \in J]$ of pseudometrics on X determines a unique uniformity b on X such that $A(b) = [\rho_\alpha: \alpha \in J]$ and $A(b)$ can be enlarged to $A^*(b)$ by adjoining to $A(b)$ all the pseudometrics of the form $\max[\rho_{\alpha_k}: k = 1, 2, \cdots, n]$, where $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ is an arbitrary finite subset of the index set J (for details see [6, p. 177]).

We now make the following definitions. Let (X, b) be a uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. If f maps X into itself, then

(i) f will be said to be $A^*(b)$-nonexpansion on X, or simply nonexpansion on X if, for each $\alpha \in I$, $\rho_\alpha(f(x), f(y)) \leq \rho_\alpha(x, y)$ for all $(x, y) \in X \times X$.

(ii) f will be said to be $A^*(b)$-contraction on X, or simply contraction on X if for each $\alpha \in I$, there exists a real number $r(\alpha)$ with $0 < r(\alpha) < 1$ such that for all $(x, y) \in X \times X$ we have $\rho_\alpha(f(x), f(y)) \leq r(\alpha)\rho_\alpha(x, y)$ (we note that the above inequality implies $\rho_\alpha(f(x), f(y)) = 0$ if $\rho_\alpha(x, y) = 0$).

(iii) f will be said to be $A^*(b)$-asymptotically regular on X, or simply asymptotically regular on X if for each $x \in X$ and $\alpha \in I$,

$$\lim_{n\to\infty} \rho_\alpha(f^n(x), f^{n+1}(x)) = 0.$$

Remark 1. If f is $A^*(b)$-nonexpansion, -contraction, or -asymptotically regular on X, then it is trivial to see that f is also respectively $A(b)$-nonexpansion, -contraction, or -asymptotically regular on X. The converse of this is also true, i.e., if f is $A(b)$-nonexpansion, -contraction, or -asymptotically regular on X, then f is respectively $A^*(b)$-nonexpansion, -contraction, or -asymptotically regular on X. We prove it for the case of contraction. The case of nonexpansion and asymptotically regularity follows similarly.

Let f be $A(b)$-contraction. Let $A(b) = [\rho_\alpha\colon \alpha \in J]$. Let $\rho \in A^*(b)$ be arbitrary. If $\rho \in A(b)$, then $\rho = \rho_\alpha$ for some $\alpha \in J$. Hence there will exist a real number $r(\alpha)$ with $0 < r(\alpha) < 1$ satisfying the condition of definition (ii) as f is $A(b)$-contraction. If $\rho \notin A(b)$, then $\rho = \max[\rho_{\alpha_k}\colon k = 1, 2, \cdots, n]$ for some finite subset $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ of J. Let $r = \max[r(\alpha_k)\colon k = 1, 2, \cdots, n]$, where $r(\alpha_k)$'s are obtained from the definition of $A(b)$-contraction of f. We will assert that r which clearly satisfies the relation $0 < r < 1$ is the required number for ρ. Let $(x, y) \in X \times X$. Then $\rho(x, y) = \rho_{\alpha_m}(x, y)$ for some $m = 1, 2, \cdots, n$ and $\rho(f(x), f(y)) = \rho_{\alpha_j}(f(x), f(y))$ for some $j = 1, 2, \cdots, n$.

Now noting that f is $A(b)$-contraction, and $\rho_{\alpha_j} \in A(b)$ and that $\rho_{\alpha_j}(x, y) \leq \rho_{\alpha_m}(x, y)$, we have

$$\begin{aligned}\rho(f(x), f(y)) &= \rho_{\alpha_j}(f(x), f(y)) \leq r(\alpha_j)\rho_{\alpha_j}(x, y) \leq r(\alpha_j)\rho_{\alpha_m}(x, y)\\ &= r(\alpha_j)\rho(x, y) \leq r\rho(x, y) \quad \text{as } r(\alpha_j) \leq r.\end{aligned}$$

Clearly r depends on $\alpha_1, \alpha_2, \cdots, \alpha_n$ and hence on β for which $\rho = \rho_\beta \in A^*(b) = [\rho_\alpha\colon \alpha \in I]$.

Thus we see that it does not matter whether we use $A(b)$ or $A^*(b)$ in the above definitions.

Remark 2. It is easy to see that the definition (c) of Taylor for asymptotic regularity stated in the introduction is equivalent to our definition (iii) above.

Also, if we take $\mathfrak{B} = [H(\alpha, \epsilon): \alpha \in I, \epsilon > 0]$, where $[\rho_\alpha: \alpha \in I] = A^*(b)$, then we see that definition (a) for $\mathfrak{B}$-nonexpansion coincides with definition (i) above for nonexpansion. However, definition (b) for contraction is not, in general, equivalent to the above definition (ii) for contraction. One can see this by comparing our Theorem 1.1 and the first lemma and the discussion following this lemma in [1].

Remark 3. If (X, b) is replaced by a metric space (X, d), then the above definitions (i), (ii) and (iii) reduce respectively to the well known definitions for nonexpansion, contraction and asymptotic regularity on the metric space (X, d).

Theorem 1.1 (Banach's Contraction Mapping Principle). *Let (X, b) be a Hausdorff complete uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let f be a contraction on X.*

Then f has a unique fixed point $a \in X$ such that $f^n(x) \to a$ in τ_b-topology for each $x \in X$.

Proof. Let $x_0 \in X$. Let $x_n = f(x_{n-1}) = f^n(x_0)$, $n = 1, 2, \cdots$. Let $\alpha \in I$ be arbitrary. If m and n are positive integers with $m < n$, then

$$\begin{aligned}
\rho_\alpha(x_m, x_n) &= \rho_\alpha(f^m(x_0), f^n(x_0)) = \rho_\alpha(f^m(x_0), f^m f^{n-m}(x_0)) \\
&\le \{r(\alpha)\}^m \rho_\alpha(x_0, f^{n-m}(x_0)) = \{r(\alpha)\}^m \rho_\alpha(x_0, x_{n-m}) \\
&\le \{r(\alpha)\}^m [\rho_\alpha(x_0, x_1) + \rho_\alpha(x_1, x_2) + \cdots + \rho_\alpha(x_{n-m-1}, x_{n-m})] \\
&\le \{r(\alpha)\}^m \rho_\alpha(x_0, x_1)[1 + r(\alpha) + \cdots + \{r(\alpha)\}^{n-m-1}] \\
&< \{r(\alpha)\}^m \rho_\alpha(x_0, x_1)/(1 - r(\alpha)) \to 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Hence $\{x_n\}_{n=1}^\infty$ is a ρ_α-Cauchy sequence (i.e. a Cauchy sequence in ρ_α-topology). Since $\alpha \in I$ is arbitrary, $\{x_n\}_{n=1}^\infty$ is a ρ_α-Cauchy sequence for each $\alpha \in I$. Let $S_p = \{x_n: n \ge p\}$ for all positive integers p and let $\mathfrak{B}$ be the filter basis $\{S_p: p = 1, 2, \cdots\}$. Then, since $\{x_n\}_{n=1}^\infty$ is a ρ_α-Cauchy sequence for each $\alpha \in I$, it is easy to see that the filter basis $\mathfrak{B}$ is Cauchy in the uniform space (X, b). To see this we first note that the family $[H(\alpha, \epsilon): \alpha \in I, \epsilon > 0]$ is a base for b as $A^*(b) = [\rho_\alpha: \alpha \in I]$. Now let $H \in b$ be an arbitrary entourage. Then there exist a $\nu \in I$ and $\epsilon > 0$ such that $H(\nu, \epsilon) \subset H$. Now since $\{x_n\}_{n=1}^\infty$ is a ρ_ν-Cauchy sequence in X, there exists a positive integer p such that $\rho_\nu(x_m, x_n) < \epsilon$ for $m \ge p$, $n \ge p$. This implies that $S_p \times S_p \subset H(\nu, \epsilon)$. Thus given any $H \in b$, we can find a $S_p \in \mathfrak{B}$ such that $S_p \times S_p \subset H$. Hence $\mathfrak{B}$ is a Cauchy filter in (X, b). Since (X, b) is complete and Hausdorff, the Cauchy filter $\mathfrak{B} = \{S_p\}$ converges to a unique point $a \in X$ in the τ_b-topology. Thus τ_b-$\lim S_p = a$. Now since f is ρ_α-continuous for each $\alpha \in I$, it follows that f is τ_b-continuous. Hence $f(a) = f(\tau_b\text{-}\lim S_p) = \tau_b\text{-}\lim f(S_p) = \tau_b\text{-}\lim S_{p+1} = a$. Thus a is a fixed point of f.

We now complete the proof of our theorem by showing that a is the only fixed point of f. We assume that f has another fixed point b such that $a \ne b$ and deduce an absurdity from this assumption. Since (X, b) is a Hausdorff space and $a \ne b$, there is an index $\beta \in I$ such that $\rho_\beta(a, b) \neq 0$. Since f is a contraction on X, $\rho_\beta(a, b) = \rho_\beta(f(a), f(b)) \leq r(\beta)\rho_\beta(a, b)$ which is absurd as $0 < r(\beta) < 1$ and $\rho_\beta(a, b) \neq 0$. This together with the fact that τ_b-lim $S_p = a$ implies τ_b-lim $x_n = a$ completes the proof.

2. In this section we extend some of the results of Diaz and Metcalf [2] to Hausdorff uniform spaces. To start with we write few facts in the form of lemmas.

Lemma 2.1. *Let* (X, b) *be a uniform space and let* $[\rho_\alpha: \alpha \in I] = A^*(b)$. *If* X *is* τ_b*-compact, then* X *is* ρ_α*-compact for each* $\alpha \in I$. *(A topological space* (X, τ) *is* τ*-compact if every* τ*-open covering of* X *has a finite subcovering.)*

Lemma 2.2. *Let* (X, b) *be a uniform space and let* $[\rho_\alpha: \alpha \in I] = A^*(b)$. *Let* $\{x_n\}_{n\in J}$ *be a net in* X.

(a) *If* $\{x_n\}_{n\in J}$ *is* τ_b*-convergent and converges to* x, *then for each* $\alpha \in I$, *it is* ρ_α*-convergent and converges to* x. *Conversely, if* $\{x_n\}_{n\in J}$ *is* ρ_α*-convergent and converges to* x *for each* $\alpha \in I$, *then it is* τ_b*-convergent and converges to* x.

(b) *If* y *is a* τ_b*-cluster point of the net* $\{x_n\}_{n\in J}$, *then* y *is a* ρ_α*-cluster point of the net* $\{x_n\}_{n\in J}$ *for each* $\alpha \in I$.

The proofs of the above two lemmas are trivial and hence omitted.

Lemma 2.3. *Let* (X, b) *be a Hausdorff uniform space and let* $[\rho_\alpha: \alpha \in I] = A^*(b)$. *If* A *and* B *are a disjoint pair of* τ_b*-compact subsets of* X, *then there exists at least one* $\beta \in I$ *such that* $\rho_\beta(A, B) > 0$.

Proof. Since X is τ_b-Hausdorff, it follows that A and B are disjoint τ_b-closed and τ_b-compact subsets of X. Hence we can find a symmetric entourage $W \in b$ such $W(A) \cap W(B) = \emptyset$ (e.g. see [7, Theorem 5, p. 117]) where for any subset C of X, $W(C) = \{y \in X: (x, y) \in W, x \in C\}$; i.e., $W(C) = \bigcup_{x\in C} W(x)$ where $W(x) = \{y \in x: (x, y) \in W\}$.

With this symmetric entourage W we can construct a sequence $\{W_n\}$ of symmetric entourages such that $W_n \circ W_n \circ W_n \subset W_{n-1}$, $W_1 = W \cap W^{-1} = W$ and $W_0 = X$ and then we can show that there exists a pseudometric $\rho \in A^*(b)$ such that for each positive integer n, $W_n \subset [(x, y): \rho(x, y) \leq 2^{-n}] \subset W_{n-1}$ (for details see Thron [6, pp. 178–179]). We first assert that for no pair (x, y) of points with $x \in A$, $y \in B$ we have $\rho(x, y) = 0$. We suppose contrary to what we wish to prove that for some pair (p, q) of points with $p \in A$ and $q \in B$, we have $\rho(p, q) = 0$. Then since $\rho(p, q) = 0$, $(p, q) \in W_n$ for each n and hence $(p, q) \in W_1 = W$ in particular. Then clearly $p \in W(A)$ and $p \in W(B)$. This contradicts that $W(A) \cap W(B) = \emptyset$. Thus we have proved our assertion. Next we prove that $\rho(A, B) > 0$. Since $\rho \in A^*(b)$ and A and B are τ_b-compact, we have by Lemma 2.1 that A and B are both ρ-compact.

Hence, if $\rho(A, B) = 0$, then there would exist a pair (x, y) of points with $x \in A$ and $y \in B$ such that $\rho(x, y) = 0$ which would contradict our established assertion that $\rho(x, y) \neq 0$ for each pair (x, y) of points with $x \in A$ and $y \in B$. Hence $\rho(A, B) > 0$. As $\rho \in A^*(b)$, $\rho = \rho_\beta$ for some $\beta \in I$. Thus we have proved the lemma.

Notation. Let (X, b) be a uniform space and let $f: X \to X$ be a mapping of X into X. Then $F(f)$ will be the set of fixed points of f, i.e., $F(f) = \{x \in X: f(x) = x\}$. For any $x \in X$, $L(x)$ will denote the set of all cluster points of the net (sequence) of iterates $\{f^n(x)\}_{n=1}^\infty$, i.e. $L(x)$ is the set of all $y \in X$ such that $\{f^{n_i}(x)\}_{i\in J} \to y$ in τ_b-topology for some subnet $\{f^{n_i}(x)\}_{i\in J}$ of the net $\{f^n(x)\}_{n=1}^\infty$.

The following is an extension of Theorem 6 in [2] to the uniform space.

Theorem 2.1 *Let (X, b) be a nonempty Hausdorff uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let $f: X \to X$ τ_b-continuous. Also let*

(a) *$f(X)$ be τ_b-compact; and*

(b) *f be asymptotically regular on X.*

*Then for each $x \in X$, the τ_b-cluster set $L(x)$ is a nonempty τ_b-closed and τ_b-connected subset of $F(f)$. In case $L(x)$ is just one point, then τ_b-$\lim f^m(x)$ exists and belongs to $F(f)$. In case $L(x)$ contains more than one point, then it is contained in the τ_b-boundary of $F(f)$. [The τ_b-boundary of a subset K of X = τ_b-closure of $K - \tau_b$-*Int *K where* Int *K stands for the interior of K.]*

Proof. The sequence $\{f^n(x)\}_{n=1}^\infty$ being also a net in the compact set $f(X)$ has a cluster point. Hence $L(x)$ is nonempty. We prove the rest of the theorem in few steps.

(i) $L(x)$ is a subset of $F(f)$. Let $y \in L(x)$. Then there is a subnet $\{f^{n_j}(x)\}_{j\in J}$ of the net $\{f^n(x)\}_{n=1}^\infty$ such that $f^{n_j}(x) \to y$ in τ_b-topology. Also since f is τ_b-continuous, the net $f^{n_j+1}(x) \to f(y)$ in τ_b-topology. Hence by Lemma 2.2, for each $\alpha \in I$, the net $f^{n_j}(x) \to y$ and the net $f^{n_j+1}(x) \to f(y)$ in the ρ_α-topology of X. *Let* $\alpha \in I$ be arbitrary. Then for each $j \in J$, we have

$$(1) \qquad \rho_\alpha(f(y), y) \leq \rho_\alpha(f(y), f^{n_j+1}(x)) + \rho_\alpha(f^{n_j+1}(x), f^{n_j}(x)) + \rho_\alpha(f^{n_j}(x), y).$$

Let $\epsilon > 0$ be arbitrarily chosen. Then since in ρ_α-topology of X, $f^{n_j+1}(x) \to f(y)$ and $f^{n_j}(x) \to y$ and since by asymptotic regularity we have $\rho_\alpha(f^{n_j+1}(x), f^{n_j}(x)) \to 0$, we can find a $p \in J$ such that for all $n_j \geq n_p$ we have simultaneously

$$(2) \quad \rho_\alpha(f^{n_j+1}(x), f(y)) < \epsilon/3, \qquad \rho_\alpha(f^{n_j+1}(x), f^{n_j}(x)) < \epsilon/3, \quad \text{and} \quad \rho_\alpha(f^{n_j}(x), y) < \epsilon/3.$$

Now from (1) and (2) we have $\rho_\alpha(f(y), y) < \epsilon$. Since ϵ is arbitrary, $\rho_\alpha(f(y), y) = 0$. Again since α is arbitrary, $\rho_\alpha(f(y), y) = 0$ for each $\alpha \in I$. Finally since X is Hausdorff, it follows that $f(y) = y$. (The uniform space X is Hausdorff iff, given two distinct points x and y, there is a $\beta \in I$ such that $\rho_\beta(x, y) \neq 0$.)

(ii) $L(x)$ is a τ_b-closed subset of $F(f)$. It is well known that the cluster set of any net is always closed. Thus $L(x)$ is a τ_b-closed subset of $F(f)$ as we have proved in (i) that $L(x) \subset F(f)$.

(iii) We now prove that $L(x)$ is a τ_b-connected subset of $F(f)$. Although our proof is similar to the proof of the corresponding part of Theorem 6 in [2], our Lemma 2.3 will play the crucial role. If $L(x)$ consists of a single point, then there is nothing to prove. So we may suppose that $L(x)$ consists of more than one point. We assume that $L(x)$ is not a τ_b-connected subset of $F(f)$ and deduce a contradiction from this assumption. Since $L(x)$ is not a τ_b-connected subset of $F(f)$, $L(x) = S_1 \cup S_2$ where S_1 and S_2 are both nonempty and τ_b-closed and $S_1 \cap S_2 = \emptyset$. Also since $f(X)$ is compact, it follows that S_1 and S_2 are both τ_b-compact. Thus S_1 and S_2 are disjoint τ_b-closed and τ_b-compact subsets of X.

Now for each $\alpha \in I$, let

$$S_1^\alpha = \{y \in F(f)\colon \rho_\alpha(y, S_1) \leq \tfrac14 \rho_\alpha(S_1, S_2)$$

and

$$S_2^\alpha = \{y \in F(f)\colon \rho_\alpha(y, S_2) \leq \tfrac14 \rho_\alpha(S_1, S_2).$$

$F(f)$ being a τ_b-closed subset of τ_b-compact set $f(X)$ is τ_b-compact. Hence $F(f)$ is by Lemma 2.1 ρ_α-compact for each $\alpha \in I$. Hence S_1^α and S_2^α being ρ_α-closed in $F(f)$ are both ρ_α-compact subsets of $F(f)$ for each $\alpha \in I$. .

We first prove that $\lim_{m\to\infty} \rho_\alpha(f^m(x), S_1^\alpha \cup S_2^\alpha) = 0$ for each $\alpha \in I$. We suppose that this is not true for some $\beta \in I$ and obtain a contradiction. Then there would exist $\epsilon > 0$ and a subsequence $\{f^{n_i}(x)\}_{i=1}^\infty$ of the sequence $\{f^m(x)\}_{m=1}^\infty$ such that

$$\rho_\beta(f^{n_i}(x), S_1^\beta \cup S_2^\beta) \geq \epsilon > 0 \quad \text{for each } i = 1, 2, \cdots. \tag{3}$$

Now the subsequence $\{f^{n_i}(x)\}_{i=1}^\infty$ being a net in the τ_b-compact set $f(X)$ has a cluster point, say, z. Then obviously $z \in L(x)$ and z is a ρ_β-cluster point of the sequence $\{f^{n_i}(x)\}_{i=1}^\infty$ by Lemma 2.2. Hence there is a subsequence $\{f^{n_{p_j}}(x)\}_{j=1}^\infty$ of the sequence $\{f^{n_i}(x)\}_{i=1}^\infty$ such that $\lim_{j\to\infty} \rho_\beta(f^{n_{p_j}}(x), z) = 0$ because ρ_β-topology of X satisfies the first axiom of countability. Now since $z \in L(x) = S_1 \cup S_2 \subset S_1^\beta \cup S_2^\beta$, we have

$$\rho_\beta(f^{n_{p_j}}(x), S_1^\beta \cup S_2^\beta) \leq \rho_\beta(f^{n_{p_j}}(x), z), \qquad j = 1, 2, \cdots.$$

Hence

$$\lim_{j\to\infty} \rho_\beta(f^{n_{p_j}}(x), S_1^\beta \cup S_2^\beta) = 0$$

which contradicts (3).

Thus we have proved that

$$\lim_{m\to\infty} \rho_\alpha(f^m(x), S_1^\alpha \cup S_2^\alpha) = 0 \quad \text{for each } \alpha \in I.$$

Since S_1 and S_2 are disjoint, τ_b-closed and τ_b-compact subsets of X, there exists by Lemma 2.3 at least one $\nu \in I$ such that $\rho_\nu(S_1, S_2) > 0$. We now prove that S_1^ν and S_2^ν are disjoint. If $p \in S_1^\nu \cap S_2^\nu$ then, since S_1 and S_2 are both ρ_ν-compact subsets (by Lemma 2.1) of $F(f)$, there would exist points $a \in S_1$ and $b \in S_2$ such that $\rho_\nu(p, S_1) = \rho_\nu(a, p)$ and $\rho_\nu(p, S_2) = \rho_\nu(p, b)$.

Hence $0 < \rho_\nu(S_1, S_2) \leq \rho_\nu(a, b) \leq \rho_\nu(a, p) + \rho_\nu(p, b) = \rho_\nu(p, S_1) + \rho_\nu(p, S_2)$ $\leq ½\, \rho_\nu(S_1, S_2)$ which is absurd. Thus S_1^ν and S_2^ν are disjoint. Also we have noted earlier that both S_1^ν and S_2^ν are ρ_ν-closed and ρ_ν-compact subsets of $F(f)$. Hence $\rho_\nu(S_1^\nu, S_2^\nu) > 0$. In summary we have

$$\rho_\nu(S_1^\nu, S_2^\nu) > 0; \quad \lim_{m\to\infty} \rho_\nu(f^m(x), f^{m+1}(x)) = 0 \quad \text{and} \quad \lim_{m\to\infty} \rho_\nu(f^m(x), S_1^\nu \cup S_2^\nu) = 0.$$

In view of the last two limits we can find a positive integer M such that, for all $m \geq M$,

$$\rho_\nu(f^m(x), f^{m+1}(x)) < \rho_\nu(S_1^\nu, S_2^\nu)/3 \quad \text{and} \quad \rho_\nu(f^m(x), S_1^\nu \cup S_2^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3.$$

The rest of the proof is similar to that in [2] and we repeat this for the sake of clarity and completeness. It follows from the last inequality that for any $m \geq M$ we have either

$$(4) \qquad \rho_\nu(f^m(x), S_1^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3$$

or,

$$(5) \qquad \rho_\nu(f^m(x), S_2^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3.$$

The set of integers $m \geq M$ satisfying (4) is not empty as S_1 is not empty and the set of integers $m \leq M$ satisfying (5) is not empty as S_2 is not empty. Hence we can find a positive integer $n \geq M$ such that we have both

$$\rho_\nu(f^n(x), S_1^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3 \quad \text{and} \quad \rho_\nu(f^{n+1}(x), S_2^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3.$$

[For any $m_1 \geq M$ such that $\rho_\nu(f^{m_1}(x), S_1^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3$ there always exists a positive integer $m_2 > m_1$ such that $\rho_\nu(f^{m_2}(x), S_2^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3$. n can be chosen to be one less than smallest such m_2.]

By using ρ_ν-compactness of S_1^ν and S_2^ν we have

$$\rho_\nu(S_1^\nu, S_2^\nu) \leq \rho_\nu(f^n(x), S_1^\nu) + \rho_\nu(f^n(x), f^{n+1}(x)) + \rho_\nu(f^{n+1}(x), S_2^\nu).$$

But then by what we have proved above

$$\rho_\nu(S_1^\nu, S_2^\nu) < \rho_\nu(S_1^\nu, S_2^\nu)/3 + \rho_\nu(S_1^\nu, S_2^\nu)/3 + \rho_\nu(S_1^\nu, S_2^\nu) = \rho_\nu(S_1^\nu, S_2^\nu)$$

which is impossible. Hence our original assumption that $L(x)$ is not a τ_b-connected subset of $F(f)$ is wrong. Thus we have proved that $L(x)$ is a τ_b-connected subset of $F(f)$.

(iv) If $L(x)$ consists of a single point, then we prove that τ_b-lim $f^m(x)$ exists. Let us denote the only cluster point of $\{f^m(x)\}_{m=1}^{\infty}$ by p. We prove that τ_b-lim $f^m(x) = p$. Let us assume that τ_b-lim $f^m(x) \neq p$ and deduce a contradiction from this assumption. Then there is a subnet $\{f^{m_k}(x)\}_{k\in K}$ of the net $\{f^m(x)\}_{m=1}^{\infty}$ such that $\{f^{m_k}(x)\}_{k\in K}$ has no subnet converging to p in the τ_b-topology. But since $\{f^{m_k}(x)\}_{k\in K}$ is a net in the τ_b-compact set $f(X)$, it has a subnet converging to a point, say, q in the τ_b-topology, i.e., it has a τ_b-cluster point q. Clearly $p \neq q$ and q is also a τ_b-cluster point of the net $\{f^m(x)\}_{m=1}^{\infty}$. Hence $L(x)$ consists of at least two distinct points p and q. This contradiction proves that τ_b-lim $f^m(x) = p$.

(v) Finally we prove that if $L(x)$ consists of more than one point, then it is contained in the τ_b-boundary of $F(f)$. Since $L(x)$ consists of more than one point, it is clear that $f^m(x) \notin F(f)$ for any $m = 0, 1, 2, \cdots$, where $f^0(x) = x$. Let $y \in L(x)$ be arbitrary. If y belonged to Int $F(f)$, then it would follow that Int $F(f)$ would contain $f^k(x)$ for some positive integer k which would contradict the assumption that $L(x)$ contains more than one point. Hence $y \in \tau_b$-boundary of $F(f)$. Thus $L(x) \subset \tau_b$-boundary of $F(f)$.

The following theorem is the generalization of some parts of the main result (Theorem 2) of [2] to uniform space.

Theorem 2.2. *Let (X, b) be a Hausdorff uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let $f: X \to X$ be τ_b-continuous. Also suppose that*

(i) *$F(f)$ is nonempty and compact;*

(ii) *for each $x \in X$, with $x \notin F(f)$ we have for each $\alpha \in I$, $\rho_\alpha(f(x), F(f)) < \rho_\alpha(x, F(f))$ if $\rho_\alpha(x, F(f)) \neq 0$ and $\rho_\alpha(f(x), F(f)) = 0$ if $\rho_\alpha(x, F(f)) = 0$.*

Then for each $x \in X$ the set $L(x)$ is a closed subset of $F(f)$. If $L(x)$ consists of more than one point, then $L(x)$ is contained in the τ_b-boundary of $F(f)$.

Proof. We have nothing to prove if $L(x)$ is empty. So we may suppose that $L(x)$ is nonempty. Again, if $x \in F(f)$ or $f^k(x) \in F(f)$ for some integer $k \geq 1$, then obviously τ_b-lim $f^m(x)$ exists and belongs to $F(f)$ and thus the eorem is proved in this case. So we prove the theorem in the remaining case, i.e. we assume that $f^m(x) \notin F(f)$ for each $m = 0, 1, 2, \cdots$.

(1) First we prove that for each $\alpha \in I$, $\lim_{m\to\infty} \rho_\alpha(f^m(x), F(f))$ exists and nonnegative. Let α be arbitrary. Now if $\rho_\alpha(f^k(x), F(f)) = 0$ for some integer $k = 0, 1, 2, \cdots$, then by the second part of condition (ii) of the theorem every element of the sequence $\{\rho_\alpha(f^m(x), F(f))\}_{m=1}^{\infty}$ starting from the kth element (from the first

element if $k = 0$) is zero. Consequently $\lim_{m\to\infty} \rho_\alpha(f^m(x), F(f)) = 0$. In the remaining case, i.e., when $\rho_\alpha(f^m(x), F(f) \neq 0$ for each $m = 0, 1, 2, \cdots$, the sequence of positive numbers $\{\rho_\alpha(f^m(x), F(f)\}_{m=1}^\infty$ is a decreasing sequence by virtue of the first part of condition (ii) of the theorem and, therefore, $\lim_{m\to\infty} \rho_\alpha(f^m(x), F(f))$ exists and is nonnegative.

(2) Next we prove that $L(x)$ is a subset of $F(f)$. Let $y \in L(x)$. It suffices to prove that $y \in F(f)$. We assume that $y \notin F(f)$ and arrive at a contradiction from this assumption. Since $y \in L(x)$, there is a subnet $\{f^{n_j}(x)\}_{j\in J}$ of the net $\{f^m(x)\}_{m=1}^\infty$ such that $f^{n_j}(x) \to y$ in the τ_b-topology of X. Also since f is continous, the net $f^{n_j+1}(x) \to f(y)$ in the τ_b-topology of X. Now by our Lemma 2.2, for each $\alpha \in I$, $f^{n_j}(x) \to y$ and $f^{n_j+1}(x) \to f(y)$ in ρ_α-topology of X. The rest of the proof depends on the simple fact that if $\{f^{n_j}(x)\}_{j\in J}$ is a subnet of $\{f^m(x)\}_{m=1}^\infty$ then $\{f^{n_j+1}(x)\}_{j\in J}$ is also a subnet of $\{f^m(x)\}_{m=1}^\infty$.(1)

Let $\alpha \in I$ be arbitrary. Then noting that in ρ_α-topology of X, $f^{n_j}(x) \to y$ and $f^{n_j+1}(x) \to f(y)$ and using the continuity of the real valued function $\rho_\alpha(x, F(f))$, $x \in X$, we have

$$\rho_\alpha(f^{n_j}(x), F(f)) \to \rho_\alpha(y, F(f)) \quad \text{and} \quad \rho_\alpha(f^{n_j+1}(x), F(f)) \to \rho_\alpha(f(y), F(f)).$$

Hence in view of the fact that $\lim_{m\to\infty} \rho_\alpha(f^m(x), F(f))$ exists and that $\{\rho_\alpha(f^{n_j}(x), F(f))\}_{j\in J}$ and $\{\rho_\alpha(f^{n_j+1}(x), F(f))\}_{j\in J}$ are both subnets of the net $\{\rho_\alpha(f^m(x), F(f))\}_{j\in J}$ of real numbers, we have $\rho_\alpha(f(y), F(f)) = \rho_\alpha(y, F(f))$. Now since by assumption $y \notin F(f)$, the above equality together with condition (ii) of the theorem implies that $\rho_\alpha(y, F(f)) = 0$. Since α is arbitrary, $\rho_\alpha(y, F(f)) = 0$ for each $\alpha \in I$. But this contradicts our Lemma 2.3 as $\{y\}$ and $F(f)$ are disjoint pair of τ_b-compact subsets of Hausdorff uniform space X. Hence $y \in F(f)$.

(3) That $L(x)$ is closed is well known and the proof that $L(x)$ is in the τ_b-boundary of $F(f)$ when it consists of more than one point is exactly the same as given in part (v) of the proof of Theorem 2.1.

The next theorem is an extension of Theorem 3 in [2] to uniform space.

Theorem 2.3. *Let (X, b) be a Hausdorff uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let $f: X \to X$ be τ_b-continuous. Further suppose that*

(i) *$F(f)$ is nonempty;*

(ii) *for each $x \in X$, with $x \notin F(f)$ and each $p \in F(f)$, we have for each $\alpha \in I$, $\rho_\alpha(f(x), p) < \rho_\alpha(x, p)$ if $\rho_\alpha(x, p) \neq 0$, and $\rho_\alpha(f(x), p) = 0$ if $\rho_\alpha(x, p) = 0$.*

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(1) Note that if n is the corresponding function for the first subnet, i.e. $n(j) = n_j$ for each $j \in J$, then the function n^1 defined by $n^1(j) = n(j) + 1 = n_j + 1$ is the required function for the latter subnet.

Then for each $x \in X$, either $\{f^m(x)\}_{m=1}^{\infty}$ has no τ_b-convergent subnet, or τ_b-$\lim f^m(x)$ exists and belongs to $F(f)$.

Proof. We have nothing to prove if $L(x)$ is empty. So we may assume that $L(x)$ is nonempty. If $x \in F(f)$, or $f^k(x) \in F(f)$ for some integer $k \geq 1$, then obviously τ_b-$\lim f^m(x)$ exists and belongs to $F(f)$ and, therefore, the theorem is proved in this case. Hence we assume that $f^m(x) \notin F(f)$ for each $m = 0, 1, 2, \cdots$ and prove our theorem in the following steps.

(a) In this step we prove that for each $\alpha \in I$, $\lim_{m\to\infty} \rho_\alpha(f^m(x), p)$ exists and nonnegative, where p is any point belonging to $F(f)$. Such a p exists by condition (i) of the theorem. The proof is similar to the proof of the part 1 of our previous theorem. Let $\alpha \in I$ be arbitrary. Now if $\rho_\alpha(f^k(x), p) = 0$ for some integer $k = 0, 1, 2, \cdots$, then by the second part of condition (ii) of our theorem each element of the sequence $\{\rho_\alpha(f^m(x), p)\}_{m=1}^{\infty}$ starting from the kth element (from the first element if $k = 0$) is zero and hence $\lim_{m\to\infty} \rho_\alpha(f^m(x), p) = 0$ in this case.

If $\rho_\alpha(f^m(x), p) \neq 0$ for each $m = 0, 1, 2, \cdots$, then by the first part of condition (ii) of the theorem the sequence of positive numbers $\{\rho_\alpha(f^m(x), p)\}_{m=1}^{\infty}$ is decreasing sequence and, therefore, $\lim_{m\to\infty} \rho_\alpha(f^m(x), p)$ exists and is nonnegative.

(b) We next prove that $L(x)$ is a subset of $F(f)$. Here again the proof is very much similar to the proof of part 2 of the previous theorem. Let $y \in L(x)$. Then there is a subnet $\{f^{n_j}(x)\}_{j\in J}$ of the net $\{f^m(x)\}_{m=1}^{\infty}$ such that $f^{n_j}(x) \to y$ in the τ_b-topology of X. Also by the τ_b-continuity of f, $f^{n_j+1}(x) \to f(y)$. Hence by our Lemma 2.2, for each $\alpha \in I$ we have that $f^{n_j}(x) \to y$ and $f^{n_j+1}(x) \to f(y)$ in the ρ_α-topology of X. Then by using the above two limits and the continuity of the real valued function $\rho_\alpha(x, p)$, $x \in X$, we have

$$\rho_\alpha(f^{n_j}(x), p) \to \rho_\alpha(y, p) \quad \text{and} \quad \rho_\alpha(f^{n_j+1}(x), p) \to \rho_\alpha(f(y), p).$$

Now, since we have proved in part (a) that $\lim_{m\to\infty} \rho_\alpha(f^m(x), p)$ exists, we have $\rho_\alpha(f(y), p) = \rho_\alpha(y, p)$ because of the same reason given in part 2 of the previous theorem, i.e. because $\{f^{n_j}(x)\}_{j\in J}$ and $\{f^{n_j+1}(x)\}_{j\in J}$ are both subnets of the net $\{f^m(x)\}_{m=1}^{\infty}$. We now assume that $y \notin F(f)$ and readily deduce a contradiction from this assumption. Since $y \notin F(f)$ and $\rho_\alpha(f(y), p) = \rho_\alpha(y, p)$, it follows from condition (ii) of our theorem that $\rho_\alpha(y, p) = 0$. Since α is arbitrary, we have $\rho_\alpha(y, p) = 0$ for each $\alpha \in I$. This implies that $y = p$ as X is τ_b-Hausdorff. This is a contradiction because of the fact that $p \in F(f)$. Thus we have proved that $y \in L(x)$.

(c) We now prove that $L(x)$ contains at most one point. We suppose that $L(x)$ contains two points, p and q. By Lemma 2.2, p and q are also ρ_α-cluster points of the net $\{f^m(x)\}_{m=1}^{\infty}$ for each $\alpha \in I$. Let α be arbitrary. Then there are two subsequences $\{f^{m_i}(x)\}_{i=1}^{\infty}$ and $\{f^{n_i}(x)\}_{i=1}^{\infty}$ of the sequence $\{f^m(x)\}_{m=1}^{\infty}$ such that $f^{m_i}(x) \to p$ and $f^{n_i}(x) \to q$ in the ρ_α-topology of X (as the ρ_α-topology of X satisfies the first axiom of countability) i.e.,

$$\lim_{i\to\infty} \rho_\alpha(f^{m_i}(x), p) = 0 \quad \text{and} \quad \lim_{i\to\infty} \rho_\alpha(f^{n_i}(x), q) = 0.$$

We can select a subsequence $\{m_i'\}_{i=1}^\infty$ of $\{m_i\}_{i=1}^\infty$ such that $m_i' > n_i$ for $i = 1, 2, \cdots$. Then $f^{m_i'}(x) \to p$ in the ρ_α-topology of X, i.e., $\lim_{i\to\infty} \rho_\alpha(f^{m_i'}(x), p) = 0$. Also we have $f^m(x) \notin F(f)$ for each $m = 0, 1, 2, \cdots$ and $q \in F(f)$ by part (b). We show that $f^{m_i'}(x) \to q$ in the ρ_α-topology of X, i.e., $\lim_{i\to\infty} \rho_\alpha(f^{m_i'}(x), q) = 0$.

We prove this by considering two cases (A) and (B).

(A) $\rho_\alpha(f^{m_k'}(x), q) = 0$ for some $k = 1, 2, \cdots$.

Then $\rho_\alpha(f^{m_{k+r}'}(x), q) = 0$ for each $r = 1, 2, \cdots$. For, let $m_{k+r}' - m_k' = t$. Then by the second part of condition (ii) of the theorem

$$0 = \rho_\alpha(f^{m_k'}(x), q) = \rho_\alpha(f^{m_k'+1}(x), q) = \rho_\alpha(f^{m_k'+t}(x), q) = \rho_\alpha(f^{m_{k+r}'}(x), q).$$

Hence it follows that in this case $\lim_{i\to\infty} \rho_\alpha(f^{m_i'}(x), q) = 0$.

(B) $\rho_\alpha(f^{m_i'}(x), q) \neq 0$ for each $i = 1, 2, \cdots$.

Then by the first part of condition (ii) of the theorem

$$\rho_\alpha(f^{m_i'}(x), q) < \rho_\alpha(f^{m_i'-1}(x), q) < \cdots < \rho_\alpha(f^{n_i}(x), q).$$

Hence $\lim_{i\to\infty} \rho\rho_\alpha(f^{m_i'}(x), q) = 0$ in this case either as $\lim_{i\to\infty} \rho_\alpha(f^{n_i}(x), q) = 0$. Thus we have $\lim_{i\to\infty} \rho_\alpha(f^{m_i'}(x), q) = 0$. Now by the triangle inequality,

$$\rho_\alpha(p, q) \le \rho_\alpha(p, f^{m_i'}(x)) + \rho_\alpha(f^{m_i'}(x), q) \quad \text{for } i = 1, 2, \cdots.$$

Hence $\rho_\alpha(p, q) = 0$ as $\lim_{i\to\infty} \rho_\alpha(p, f^{m_i'}(x)) = 0$ and $\rho_\alpha(f^{m_i'}(x), q) = 0$. Since α is arbitrary, $\rho_\alpha(p, q) = 0$ for each $\alpha \in I$. Since X is τ_b-Hausdorff, this implies that $p = q$.

(d) Finally we prove that if $L(x)$ consists of just one point, then τ_b-lim $f^m(x)$ exists. Let $\{y\} = L(x)$. Then there is a subnet $\{f^{n_j}(x)\}_{j\in J}$ of the net $\{f^m(x)\}_{m=1}^\infty$ such that τ_b-lim $f^{n_j}(x) = y$. Now by Lemma 2.2, $f^{n_j}(x) \to y$ in the ρ_α-topology of X for each $\alpha \in I$. Let $\alpha \in I$ be arbitrarily chosen. Let $\epsilon > 0$. Then since $f^{n_j}(x) \to y$ in the ρ_α-topology of X, there is a $s \in J$ such that $\rho_\alpha(f^{n_j}(x), y) < \epsilon$ for all $j \succ s$ and hence for all $n_j \geq n_s$ where $\succ$ is the relation in J. We now show that for all positive integers $m \geq n_s$, $\rho_\alpha(f^m(x), y) < \epsilon$. We have at most two cases:

Case 1. $0 = \rho_\alpha(f^{n_s}(x), y) < \epsilon$.

Case 2. $0 < \rho_\alpha(f^{n_s}(x), y) < \epsilon$.

In Case 1, $0 = \rho_\alpha(f^{n_s}(x), y) = \rho_\alpha(f^{n_s+1}(x), y) = \cdots = \rho_\alpha(f^m(x), y) < \epsilon$.

In Case 2, $\rho_\alpha(f^m(x), y) < \rho_\alpha(f^{m-1}(x), y) < \cdots < \rho_\alpha(f^{n_s}(x), y) < \epsilon$.

Thus, in all cases, $\rho_\alpha(f^m(x), y) < \epsilon$ whenever $m \geq n_s$. Hence $f^m(x) \to y$ in the ρ_α-topology of X. Now since α is arbitrary, $f^m(x) \to y$ in the ρ_α-topology of X for each $\alpha \in I$. Hence by Lemma 2.2, τ_b-lim-$f^m(x) = y$.

Remark 4. The Remark 10 in [2] concerning the work of Edelstein [8, Theorem 1 and 3.2] applies equally here.

Corollary 2.1. *Suppose in addition to the hypotheses of Theorem 2.3 that for each $x \in X$, $L(x) \neq \emptyset$, then for each $x \in X$, $\{f^n(x)\}_{n=1}^{\infty}$ converges in τ_b-topology to a fixed point of f.*

Proof. This follows immediately from the above Theorem 2.3.

Remark 5. If we assume in the above Theorem 2.3 that $f(X)$ is compact, then this will insure the additional condition assumed in the above corollary, i.e., for each $x \in X$, $L(x) \neq \emptyset$.

The next theorem is patterned after the Theorem 3.1 of [2].

Theorem 2.4. *Let (X, b) be a Hausdorff uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let $f: X \to X$ be τ_b-continuous. Also let*

(i) *$F(f)$ be nonempty;*

(ii) *for each $x \in X$ with $x \notin F(f)$ and each $p \in F(f)$, we have for each $\alpha \in I$, $\rho_\alpha(f(x), p) \leq \rho_\alpha(x, p)$;*

(iii) *f be asymptotically regular on X.*

Then for each $x \in X$, either $\{f^m(x)\}_{n=1}^{\infty}$ contains no τ_b-convergent subnet, or τ_b-lim $f^m(x)$ exists and belongs to $F(f)$.

Proof. The proof that $L(x) \subset F(f)$ is exactly the same as given in the part (i) of the proof of Theorem 2.1. The proof that $L(x)$ contains at most one point can be obtained from the part (c) of the proof of the above Theorem 2.3 by ignoring the case A and replacing all the strict inequality signs appearing in case B by $\leq$ signs. The rest of the proof of this theorem can be obtained from part (d) of the Theorem 2.3 by ignoring Case 1 and replacing all the strict inequality signs by $\leq$ in the proof of Case 2.

Corollary 2.2. *Let (X, b) be a Hausdorff uniform space and let $[\rho_\alpha: \alpha \in I] = A^*(b)$. Let f be an asymptotically regular τ_b-continuous mapping of a subset $Y \subset X$ into Y. Also suppose that*

(i) *$f(Y)$ is τ_b-compact;*

(ii) *for each $x \in Y$ with $x \notin F(f)$, we have for each $\alpha \in I$, $\rho_\alpha(f(x), p) \leq \rho_\alpha(x, p)$.*

Then for each $x \in Y$, the sequence $\{f^m(x)\}_{m=1}^{\infty}$ converges in the τ_b-topology to a fixed point of f.

Proof. Since f is asymptotically regular on Y and $f(Y)$ is compact, we have by Theorem 2.1 that $L(x) \neq \emptyset$ for each $x \in Y$ and $F(f) \neq \emptyset$. Hence the corollary follows from the above Theorem 2.4.

3. **An extended remark.** Since our results will concern only with locally convex linear Hausdorff topological spaces, (E, τ) will denote a locally convex linear Hausdorff topological space throughout the rest of the paper.

It is well known that given (E, τ), there always exists a family $[p_\alpha: \alpha \in I]$ of seminorms on E which generates the topology τ in E. More specifically, there always exists a family $[p_\alpha: \alpha \in I]$ of seminorms on E such that the family of scalar multiple rU, $r > 0$, of finite intersections $U = \bigcap_{k=1}^{n} U_{\alpha_k}$, where $U_{\alpha_k} = \{x: p_{\alpha_k}(x) \leq 1\}$, forms a neighbourhood base at 0 for the topology τ (see [9, p. 48], or [10, p. 203]).

In the sequel this zero neighbourhood base will be denoted by $\mathfrak{B}$.

Now for each $\alpha \in I$, the function $\rho_\alpha: E \times E \to R$ (the real line) defined by $\rho_\alpha(x, y) = p_\alpha(x - y)$ for each pair $(x, y) \in E \times E$ is a pseudometric on E. Thus by what we have noted in the beginning of §1 the family $[\rho_\alpha: \alpha \in I]$ of pseudometrics on E (obtained from the family $[p_\alpha: \alpha \in I]$ as above) determines a unique uniformity b on E such that $A^*(b) = [\rho_\alpha: \alpha \in I]$. It is well known that $\tau_b = \tau$ (e.g. see [9, p. 16]).

Now it is straightforward to see that the following definitions (i$'$), (ii$'$) and (iii$'$) are equivalent respectively to definitions (i), (ii) and (iii) given in §1.

Let $\mathfrak{U}$ be the family of zero neighbourhoods in E.

Then we have seen that $\mathfrak{B}$ defined above is a base for $\mathfrak{U}$. Let f maps $X \subset E$ into X, then

(i$'$) f is said to be nonexpansion on X if $x - y \in U$ implies $f(x) - f(y) \in U$ for each $U \in \mathfrak{B}$ and $(x, y) \in X \times X$.

(ii$'$) f is said to be a contraction on X if for each $U \in \mathfrak{B}$, there is a real number r_U, $0 < r_U < 1$, such that $x - y \in U$ implies $f(x) - f(y) \in r_U U$ for each $U \in \mathfrak{B}$ and $(x, y) \in X \times X$.

(iii$'$) f is asymptotically regular on X if for each $x \in X$ and $U \in \mathfrak{U}$, there is a positive integer n_0 such that $f^n(x) - f^{n+1}(x) \in U$ for $n \geq n_0$.

[For (i) $\Leftrightarrow$ (i$'$) and (iii) $\Leftrightarrow$ (iii$'$) see Remark 2 and [1]. We prove that (ii$'$) $\Leftrightarrow$ (ii). To prove this we first show that f is a contraction with respect to $A^*(b)$. Let $\rho_\alpha \in A(b)$ and $(x, y) \in X \times X$. Also let $\rho_\alpha(x, y) = r$. Then $x - y \in rU_\alpha = U \in \mathfrak{B}$ where $U_\alpha = \{x: p_\alpha(x) \leq 1\}$. Hence by (ii$'$) there is a real number r_U, $0 < r_U < 1$, such that $f(x) - f(y) \in r_U U$. This implies that $\rho_\alpha(f(x), f(y)) \leq r_U \rho_\alpha(x, y)$. Clearly r_U depends on α. Hence we can write $r_U = r(\alpha)$.

Similarly we prove that (ii) $\Rightarrow$ (ii$'$). This can be done as follows. Let $U \in \mathfrak{B}$. Then $U = r \bigcap_{k=1}^{n} U_{\alpha_k}$, $r > 0$ and $U_{\alpha_k} = \{x: p_{\alpha_k}(x) \leq 1\}$. Let ρ_{α_k} be the corresponding pseudometrics. Choose $r_U = \max (r(\alpha_k): k = 1, 2, \cdots, n)$ where α_k's are obtained from the definitions (ii).]

Since now we will be concerned with only the locally convex topology τ in E, unless otherwise stated, all topological concepts such as continuity, convergence, closedness, etc. will hereafter be meant with respect to the topology τ in E.

A subset X of E is called starshaped if there is a point $p \in X$ such that for each each $x \in X$ and real t with $0 < t < 1$, $tx + (1-t)p \in X$. p is called the star centre of X. Every convex subset of E is thus starshaped.

The following result which we write as a lemma is due to Taylor [1]. However our concept of a contraction mapping and Theorem 1.1 provides the following simpler proof.

Lemma 3.1. *Let X be a complete bounded starshaped subset of E and let f be nonexpansion on X. Then 0 lies in the closure of $(I-f)X$, i.e., in $\overline{(I-f)X}$, where I is the identity map on X.*

Proof. For each t, $0 < t < 1$, we define $f_t(x) = tf(x) + (1-t)p$, $x \in X$, p being the star centre of X, f_t is a self mapping on X as X is starshaped. Let $U \in \mathfrak{B}$ be arbitrary. Let $x - y \in U$. Then $f_t(x) - f_t(y) = t(f(x) - f(y)) \in tU$ as f is nonexpansion on X. Thus f_t is a contraction on X. Now since X is complete and by our Theorem 1.1 f_t has a unique fixed point x_t, say, in X. Now

$$\begin{aligned}(I-f)(x_t) &= x_t - f(x_t)\\ &= x_t - (f_t(x_t) - (1-t)p)/t, \quad \text{from the definition of } f_t,\\ &= (1 - 1/t)(x_t - p) \longrightarrow 0 \quad \text{as } t \longrightarrow 1, \text{ because } X \text{ is bounded.}\end{aligned}$$

Remark 6. Note that in the proof of the above lemma we have not used the fact that X is connected.

Let us express the condition (ii) of Theorem 2.4 and condition (ii) of Theorem 2.3 by saying respectively that f is nonexpansion on X with respect to $F(f)$ and f is contractive (in the terminology of [8]) on X with respect to $F(f)$. Thus in (E, τ) f is nonexpansion on $X \subset E$ with respect to $F(f)$ if $x - p \in U$ implies $f(x) - p \in U$ for each $U \in \mathfrak{B}$ and $x \in X$ with $x \notin F(f)$ and $p \in F(f)$. f is contractive on X with respect to $F(f)$ if the condition (ii) of Theorem 2.3 holds where ρ_α's are obtained from the corresponding p_α's.

Theorem 3.1. *Let f be a nonexpansion on a bounded complete starshaped subset X of E. Also let $(I-f)X$ be closed. Then f has a fixed point.*

Proof. By Lemma 3.1, $0 \in \overline{(I-f)X} = (I-f)X$. Hence there is a point $p \in X$ such that $(I-f)(p) = 0$, i.e. $f(p) = p$.

Corollary 3.1. *Let f be a nonexpansion on a compact starshaped subset X of E. Also suppose that f is contractive on X with respect to $F(f)$. Then for each $x \in X$, the sequence $\{f^m(x)\}_{m=1}^{\infty}$ converges to a fixed point of f.*

Proof. By continuity of $(I - f)$, $(I - f)X$ is compact and hence closed. Now by Theorem 3.1 $F(f)$ is nonempty. Also since $f(X)$ is compact, $L(x) \neq 0$ for each $x \in X$. Hence the corollary follows from Theorem 2.3.

Theorem 3.2. *Let f be a continuous asymptotically regular mapping on a closed bounded subset X of E. Also suppose that $(I - f)$ maps closed and bounded subsets of X into closed subset of E. Then for each $x \in X$, $L(x)$ is nonempty and a closed subset of $F(f)$. If in addition f is nonexpansion on X with respect to $F(f)$, then for each $x \in X$, the sequence $\{f^m(x)\}_{m=1}^{\infty}$ converges to a fixed point of f.*

Proof. The proof is identical to the proof of Theorem 3.3 in [1] because the nonexpansion of f with respect to $F(f)$ is only used there. Alternatively, we prove in the same way as in Theorem 3.3 of [1] that $F(f) \neq \emptyset$ and $L(x) \neq \emptyset$ for each $x \in X$ and then we refer to our Theorem 2.4.

Remark 7. Clearly this theorem includes the Theorem 3.3 of [1] which includes the Theorem 6 of Browder and Petryshyn [11]. Also we note that the present theorems weaken the conditions of Theorem 3.3 of [1] in exactly the same way as the Theorem 3.4 of Diaz and Metcalf [2] does to the conditions of Theorem 6 of [11].

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