ABSTRACT. Let $G$ be a connected, simply connected Lie group and let $G_c$ be its complexification. Let $U$ be a unitary representation of $G$. The space of vectors $v$ at which $U$ is holomorphically extendible to $G_c$ is denoted $H^\omega(U)$. In [9] we characterized those $U$ for which $H^\omega(U)$ is dense. In the present work we study $H^\omega(U)$ as a topological vector space, proving e.g., that $H^\omega(U)$ is a Montel space if $U$ is irreducible and $G$ is nilpotent. We prove a representation theorem for $(H^\omega(U))'$ which yields a Bergman kernel type theorem for $G$. As an application we give a necessary and sufficient condition for the set of holomorphic functions on certain solvmanifolds to separate points.

Our main tool is a representation theorem for the dual space $(H^\omega(U))^\prime$, which states (in the unitary case) that every continuous linear functional is a finite
linear combination of functionals of the form \( v \rightarrow (U^\omega_v, w) \) (\( z \in G_c \), \( w \in B \) fixed).

In a large number of cases (e.g. when \( U \) is an induced representation) it can be shown that \( \mathcal{K}_\omega^\infty \) can be identified with a space of entire functions on \( G_c \) for which point evaluation is continuous. Our representation theorem yields a Bergman kernel type theorem for certain unbounded domains.

As an application of our results, we develop a necessary and sufficient condition for the entire functions on the complexification of a compact solvmanifold to separate points of the solvmanifold.

I. Throughout the sequel \( G \) will denote a connected, simply connected, solvable, type \( R \) Lie group and \( \mathfrak{g} \) will be its Lie algebra. Let \( U \) be a fixed unitary representation of \( G \) in a Hilbert space \( \mathcal{H} \). Let \( C^\infty(U) \) be the space of \( C^\infty \) vectors for \( U \) and, for \( X \in \mathfrak{g} \) and \( v \in C^\infty(U) \), let

\[
\partial U(X)v = \lim_{t \to 0} \frac{((U^{X^t}v - v)/t)}.
\]

The mapping \( X \rightarrow \partial U(X) \) defines a representation of \( \mathfrak{g} \) by skew-symmetric operators which we extend to a representation of the universal enveloping algebra \( \mathcal{U} \) of \( \mathfrak{g} \).

Let \( X = (X_1, \cdots, X_d) \) be a fixed, ordered, Jordan-Hölder basis for \( G \). Let \( N \) denote the positive integers. If \( n \in N^d \), let \( X^n = X_1^{n_1} \cdots X_d^{n_d} \). Also if \( n \in N \), let

\[
\{X^n = \{Y_1 \cdots Y_n | Y_i \in \{X_1, \cdots, X_d\}, 1 \leq i \leq n\}.
\]

For \( \nu \in C^\infty(U) \) and \( t > 0 \), let

\[
\sigma_t(\nu) = \sup \| \partial U(Y)\nu \|_{t^n/n!} | Y \in \{X^n, n \in N\}
\]

\[
r_t(\nu) = \sup \| \partial U(X^n)\|_{t^n/n!} | n \in N^d|.
\]

Recall ((1.1) and (1.3) of [9]) that \( \nu \in \mathcal{K}_\omega^\infty \) iff \( \sigma_t(\nu) \) (or equivalently \( r_t(\nu) \)) is finite for all \( t > 0 \).

**Proposition (1.1).** \( \mathcal{K}_\omega^\infty(U) \) is a Fréchet space and \( U^\omega \) is a holomorphic representation of \( G_c \) in \( \mathcal{K}_\omega^\infty \) by continuous operators. Furthermore, the topologies on \( \mathcal{K}_\omega^\infty \) defined via the families of seminorms \( \{\sigma_t\} \) and \( \{r_t\} \) both agree with the \( \mathcal{K}_\omega^\infty \) topology.

**Proof.** If \( K_n \) is any nested sequence of compact subsets of \( G_c \) for which \( K_n \subset K_{n+1} \) and \( \bigcup_{n=1}^{\infty} K_n = G_c \), then \( \rho_{K_n} \) form a basis of seminorms for \( \mathcal{K}_\omega^\infty \). Hence \( \mathcal{K}_\omega^\infty \) is countably normed and is a metric space.

If \( u_n \) is Cauchy in \( \mathcal{K}_\omega^\infty \) then \( z \rightarrow U(z)^n u_n \) converges (as a map of \( G_c \) into \( \mathcal{H} \)) uniformly on compacta to a holomorphic function \( \phi(z) \). It is easily seen that \( u_n \rightarrow \phi(e) \) in \( \mathcal{K}_\omega^\infty \). Thus \( \mathcal{K}_\omega^\infty \) is a Fréchet space.
It is easily verified that $H^\infty$ is also a Fréchet space in the topologies defined by the $\{\sigma_t\}$ and $\{\tau_t\}$. The equality of the various topologies then follows from the closed graph theorem and the fact that the injection $H^\infty \rightarrow H$ is continuous in each topology.

That $U^\omega$ is a continuous operator follows easily from the definitions. To show that $U^\omega$ is holomorphic in $H^\infty$, it suffices, in virtue of (1.1) of [9], to show that $\{\partial U(Y)v_t^n/n! | Y \in \{X\}^n, n \in \mathbb{N}\}$ is bounded in $H^\infty$ for all $t > 0$ and all $v \in H^\infty$. This set, however, is clearly bounded in each $\sigma_t, s > 0$, and hence in $H^\omega$, thus proving the analyticity. Q.E.D.

Remarks. The equality of the topologies could also be shown directly from the estimates in §1 of [9]. Note also that we have so far used only the Banach property of $H$.

Yet another description of the $H^\omega$ topology is possible.

Definition (1.2). Let $s \in \mathbb{R}^d, s = (s_1, \ldots, s_d), s_i > 0$. Let $H^s(H) = H^s = \{v \in H | t \rightarrow U(t)X_jv \text{ is extendible to a continuous map of } |z| < s_i \text{ which is holomorphic on } |z| < s_i \text{ for } i = 1, \ldots, d\}$. Define

$$d(v, w)_s = \sum_{i=1}^d \left( (U_{exp is_jX_j}v, U_{exp is_jX_j}w) + (\frac{U_{exp-is_jX_j}v, U_{exp-is_jX_j}w}}{2} \right).$$

Let $\|v\|_s = (v, v)_s$.

Proposition (1.3). Under $\|\|_s, H^s$ is a Hilbert space. Also $H^\omega = \bigcap_{s_i > 0} H^s$ and the topology of $H^\omega$ is given by the family of restrictions of the norms $\|\|_s$.

Proof. From the maximum modulus principle and the unitarity of $U$, it follows that

$$\sup_{|z| < s_j} \|U_{exp zX_j}v\|^2 \leq \|U_{exp is_jX_j}v\|^2 + \|U_{exp -is_jX_j}v\|^2$$

for $v \in H^s$ and $j = 1, \ldots, d$. The completeness of $H^s$ follows from this.

That $\bigcap H^s = H^\infty$ follows from (1.5) of [9]. The topological part is another application of the closed graph theorem. Q.E.D.

The above proposition yields a characterization of $(H^\omega)'$. If $s = (s_1, \ldots, s_d) \in \mathbb{R}^d, s_i > 0$, let $U_s : H^{2s} \rightarrow H$ be given by

$$U_s = \sum_{j=1}^d \{U_{exp 2s_jX_j} + U_{exp -2is_jX_j}\}.$$

Also, if $t, r \in \mathbb{R}^d$, we will say $t > r$ iff $t_i > r_i$ for $i = 1, \ldots, d$.

Theorem (1.4). Given $\phi \in (H^\omega)'$, there is an $s_0 \in \mathbb{R}^d, s_0 > 0$, with the following property: For all $s > s_0$ there is a $u_s \in H^s$ which represents $\phi$ in the
sense that $\phi(v) = (U, v, w)$ for all $v \in H_\infty$.

**Proof.** Note that if $w, v \in H^s$, then, by uniqueness of analytic continuation, 
$(U_{\exp z} v, w) = (v, U_{w_{\exp z} v} w)$ for all $|z| \leq s_j$. It follows that, for $v \in H_\infty$ and $w \in H^s$, $(U, v, w) = (v, w)$. Thus, our assertion is that there is an $s_0$ for which $\phi$ is continuous in $||||$ for all $s > s_0$. This follows from (1.3) and its proof. Q.E.D.

**Corollary (1.5).** A subspace $M$ of $H_\infty$ is dense iff there is a sequence $s^p \in \mathbb{R}^d$ for which $s^p \to \infty$ as $p \to \infty$, $i = 1, \ldots, d$, and for which $U_{s^p}(M)$ is dense in $H_\infty$ in the topology of $H$ for all $p \in N$.

**Proof.** The usual annihilator argument.

**Corollary (1.6).** If $H$ is separable, so is $H_\infty$.

**Proof.** Let $v_n$, $n = 1, \ldots$, be a countable dense subset of $H$. For $n, m \in N$, $s \in \mathbb{R}^d$ if there is a point $w \in H_\infty$ for which $\|v_m - U, w\| < 1/n$, let $w_{s,n}$ be such a point. Otherwise let $w_{s,n} = 0$. Let $W = \{w_{s,n}\}$ be such that $W_{s^p}$ is dense in $H_{s^p}(H_\infty)$ for $s \in N^d$. Hence $M$ is dense in $H_\infty$ by (1.5). Finally, the set of linear combinations with complex rational coefficients of elements of $W_{s^p}$ is a countable dense subset of $H_\infty$. Q.E.D.

**Remarks.** In view of the proof of Theorem (II.2) of [9], we might expect that one could produce a countable dense set of entire vectors via regularization — i.e. vectors of the form $\int_G \phi(g) U, v d\gamma$ where $\phi(g)$ is an $L^1$ entire vector. Although this definitely seems possible, we are unable to prove that the set of such vectors is dense in $H_\infty$. The difficulty is that the integral does not seem to converge in $H_\infty$. It is for this reason, also, that we are unable to prove that $U$ is topologically irreducible if $U$ is (cf. Example (8.20) of [10]).

(1.4) also provides, in the case that $U$ is given as a direct integral of other representations, another description of $(H_\infty^*)$.

This description is based on the following fundamental theorem of Goodman's [4, Lemma (3.1)].

**Theorem (A.1).** If $U = \int_M U^\alpha d\alpha$ where $M$ is an analytic Borel space and $U^\alpha$ is an integrable family of unitary representations of $G$ in $H^\alpha$, then $v \in H$, $v = |v^\alpha|$ is in $H_\infty(U)$ if $v^\alpha \in H_\infty(U^\alpha)$ for a.e. $\alpha$ and, for all compact sets $\Omega \subset G_c$, $a \to \sup_{z \in \Omega} \|U^\alpha v^\alpha\|^2$ is in $L^1(M)$.

In this case, $(U^\alpha z) v^\alpha = U^\alpha z v^\alpha$ for all $z \in G_c$ and a.e. $\alpha$.

If the representation space of $U$ is separable then the same statement, more or less, is true for $(H_\infty^*)^*$. 


Proposition (1.7). In the notation of (A.1), if $\phi \in H_\infty^\omega(U)'$ then for a.e. $\alpha$ there is a uniquely determined functional $\phi^\alpha \in H_\infty^\omega(U^\alpha)'$ for which $\phi(v) = \int_M \phi^\alpha(v^\alpha) \, d\alpha$ for all $v = \{v^\alpha\} \in H_\infty^\omega(U)$. The integral is absolutely convergent.

Conversely, if $\phi^\alpha \in H_\infty^\omega(U^\alpha)'$ are such that $\alpha \rightarrow \phi^\alpha(v^\alpha)$ is in $L^1(M)$ for all $v = \{v^\alpha\} \in H_\infty^\omega(U)$, then $v \rightarrow \int_M \phi^\alpha(v^\alpha) \, d\alpha$ defines an element of $H_\infty^\omega(U)'$.

Proof. The first part, except for uniqueness, follows from (1.4) and (A.1). The uniqueness follows as in (C.1) of [8] except that we now obtain the required countable dense subset of $H_\infty^\omega(U^\alpha)$ via the following lemma.

Lemma. If $\{v^\alpha\}_{n=1}^\infty$ is dense in $H_\infty^\omega(U)$, then for a.e. $\alpha \in M$, $\{v^\alpha\}_{n=1}^\infty$ is dense in $H_\infty^\omega(U^\alpha)$.

Proof. Let $s^p \in \mathbb{R}^d$, $p \in \mathbb{N}$, be a sequence satisfying the hypothesis of (1.5). Let $K^p = U_{s^p}$ and $K^p_\alpha = U_{s^p}(H_\infty^\omega(U^\alpha))$. If $E \subset M$ is measurable, let $\Pi_E : H \rightarrow K$ be the map that takes $\{v^\alpha\}$ onto $\{w^\alpha\}$ where $w^\alpha = v^\alpha$ if $\alpha \in E$ and is zero otherwise. It follows from (A.1) that $\Pi_E$ leaves $K^p$ invariant. Hence $\Pi_E$ commutes with the projection $\Pi^p$ onto $K^p$. It follows from Theorem (P. 6) of [6, p. 92] that $\Pi^p$ is a direct integral of projections $\Pi^p_\alpha$. $\Pi^p_\alpha$ is the projection onto $K^p_\alpha$ and, hence $K^p$ is the direct integral of the $K^p_\alpha$.

But $U_{s^p}(\{v^\alpha\})$ is dense in $K^p$. Thus $U_{s^p}(\{v^\alpha\})$ is, for a.e. $\alpha$, dense in $K^p_\alpha$. Upon choosing a set of $\alpha$ for which this is true for all $p \in \mathbb{N}$, the lemma follows from (1.5).

The converse statement of (1.7) follows from the closed graph theorem as in [8]. Q.E.D.

Corollary (1.8). If in (1.7) each $U^\alpha$ is finite dimensional, then $\phi$ is given via a function $\alpha \rightarrow w^\alpha \in H^\alpha$ in the sense that $\phi(v) = \int_M (v^\alpha, w^\alpha) \, d\alpha$ for all $v \in H_\infty^\omega(U)$.

Furthermore $\alpha \rightarrow w^\alpha$ represents an element of $H_\infty^\omega(U)'$ iff $\alpha \rightarrow (v^\alpha, w^\alpha)$ is integrable for all $\{v^\alpha\} \in H_\infty^\omega(U)$.

Example (1.9). Let $G = \mathbb{R}$ and let $U$ be the regular representation. $\mathbb{R}$ is $C$ and $H_\infty^\omega(U)$ is the space of entire functions $f$ on $C$ for which

$$\sup_{|y| \leq \delta} \int_{-\infty}^{\infty} |f(x + iy)|^2 \, dx < \infty$$

for all $\delta > 0$ (cf. [3, p. 64]).

By the Paley-Wiener theorem this is the space of functions which, when restricted to the real line, satisfy $\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \exp \{2\delta |\lambda| \} \, d\lambda < \infty$ for all
defines a direct integral decomposition of $U$ into one-dimensional representations. (1.8) applies and shows that, to each element $\phi$ of $(H(\omega))'$, there is a unique function $\phi: \mathbb{R} \to \mathbb{C}$ for which $\phi(f) = \int_{-\infty}^{\infty} \hat{\phi}(\lambda) \hat{f}(\lambda) \, d\lambda$. The space of such $\phi$ can be characterized as the set of functions for which $|\hat{\phi}(\lambda)| \leq e^{\delta|\lambda|}$ for some $\delta > 0$ (depending on $\phi$).

One of our main uses of (1.3) will be in the study of the duality theory of $H(\omega)$.

Definition (1.10). For $\omega \in H(\omega)$ and $z \in G$, define the seminorm $\|\cdot\|_{z, \omega}$ by $\|v\|_{z, \omega} = \sup_{z \in \Omega} \|v\|_{z, \omega}$ for $v \in H(\omega)$. Let $\overline{H(\omega)} = \overline{H(\omega)}(\Omega)$ be $H(\omega)$ with the topology defined by the family $\|\cdot\|_{z, \omega}$.

Proposition (1.11). $\overline{H(\omega)}$ is $H(\omega)$ with its weak topology.

Proof. (1.4).

We shall also need an intermediary topology.

Definition (1.12). If $\Omega \subset G$, $\Omega$ compact, let $\|v\|_{\Omega, \omega} = \sup_{\Omega} \|v\|_{z, \omega}$ for $v, w \in H(\omega)$. Let $H(\omega)$ be $H(\omega)$ with the topology defined via the family $\|\cdot\|_{\Omega, \omega}$.

Recall that a locally convex topological vector space $F$ is said to be semi-Montel if all closed and bounded sets are compact. If $F$ is also barrelled, $F$ is said to be Montel.

Proposition (1.13). $H(\omega)$ and $\overline{H(\omega)}$ are semi-Montel spaces.

Proof. By the uniform boundedness principle in $H(\omega)$, $H(\omega)$, $H(\omega)$, and $\overline{H(\omega)}$ all have the same bounded sets. Since the injection $H(\omega) \to H(\omega)$ is continuous, it suffices to prove the assertion for $H(\omega)$.

Let $\{v^a\}_{a \in A}$ be a bounded net in $H(\omega)$. Then $\{v^a\}_{a \in A}$ is bounded in $H$. Hence we may assume $v^a$ converges weakly to some element $v \in H$. Now, let $w \in H$ and set $\phi_w(z) = \sup_{a \in A} \|v^a\|_{z, \omega}$ for $z \in G$. Let $\Omega_w$ be the space of entire functions on $G$ which are dominated by $\phi_w$. From the boundedness of $\phi_w$, $\phi_w$ is bounded on compact subsets. Hence $\Omega_w$ is a closed bounded subset of the space of entire functions with the compact open topology. By Montel's theorem, $\Omega_w$ is compact. Let $\Omega = \bigcap_{w \in H} \Omega_w$. By the Tychnoff theorem $\Omega$ is compact. Let $\{X^a\}_{a \in A} \in \Omega$ be the net $X^a = z \to (U_z v^a, w)$ and let $\{X^a_{n\beta}\}_{\beta \in A}$ be a convergent subnet, say $X^a_{n\beta} \to X$. Then $X^a_{n\beta} \to X^w$ uniformly on compact subsets of $G$ for all $w \in H$. $X^w$ is an entire function and, for $g \in G$, $X^w(g) = \lim_{\beta \in A} \|U_g v^a\|_{\Omega_w} = \lim_{\beta \in A} \|U_g v^a\|_{\Omega_w}$. Thus, by (1.2) of [9], $v \in H(\omega)$ and $v_{n\beta} \to v$ in $H(\omega)$. Q.E.D.

Corollary (1.14). $H(\omega)$ is reflexive.
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Proof. Since $\mathcal{H}_\infty^{\omega}$ is semi-Montel, $\mathcal{H}_\infty^{\omega}$ is weakly reflexive by Proposition 1 of [5, p. 227]. $\mathcal{H}_\infty^{\omega}$ is barrelled since it is Fréchet. Our conclusion then follows from Proposition 6 of [5, p. 229]. Q.E.D.

Remark. The above proposition is actually true for any representation of $G$ in a reflexive Fréchet space. The same proof carries over with only minor changes.

Corollary (1.15). The relative topologies gotten by restricting $\mathcal{M}_\infty^{\omega}$ and $\mathcal{L}_\infty^{\omega}$ to a fixed bounded set agree. In particular a sequence $x_n$ converges in $\mathcal{M}_\infty^{\omega}$ iff it converges in $\mathcal{L}_\infty^{\omega}$.

Proof. It suffices to consider only the $\mathcal{M}_\infty^{\omega}$ bounded sets, in which case the proposition follows from compactness. Q.E.D.

In general, the topology of $\mathcal{H}_\infty^{\omega}$ cannot be any nicer than that of $\mathcal{H}$ since if $U$ is uniformly continuous, for example, $\mathcal{H}_\infty^{\omega} = \mathcal{H}$. However, if $G$ is nilpotent and $U$ is irreducible, the topology is, in some sense, significantly nicer.

Theorem (1.16). If $G$ is nilpotent and $U$ is irreducible, then $\mathcal{H}_\infty^{\omega}$ is a Montel space.

Before proving (1.16), we comment that we do not know the most general type of group $G$ for which the theorem is true. However (1.16) implies that $G$ is C.C.R. for, if $f$ is any entire vector for the $L^1$ left regular representation of $G$, then $U_f = \int_G f(g) U g \, dg$ maps $\mathcal{H}$ into $\mathcal{H}_\infty^{\omega}$. By the closed graph theorem $U_f$ is continuous. From the continuity of the injection $\mathcal{M}_\infty^{\omega} \rightarrow \mathcal{H}$, it follows that $U_f$, as a map of $\mathcal{H}$ into $\mathcal{H}$, is a compact operator if $\mathcal{H}_\infty^{\omega}$ is a Montel space. Finally, every operator $U_b$, $b \in L^1(G)$, is a uniform limit of such $U_f$. Hence every such $U_b$ is compact and $G$ is C.C.R.

Auslander-Moore [1, Chapter V] have shown that, for type R groups, C.C.R. is equivalent to type I. We suspect that (1.16) is also.

Proof of (1.16). Let $B$ be closed and bounded in $\mathcal{H}_\infty^{\omega}$. To show that $B$ is compact it suffices, we claim, to show that $U_z B$ is compact in $\mathcal{H}$ for all $z \in G$, for in this case $B = \bigcap_{z \in G} U_z B$ is compact. Hence every net $\{x_\alpha\}_{\alpha \in A}$ in $B$ has a subnet $\{y_\beta\}_{\beta \in A}$, for which $U_z y_\beta$ converges in $\mathcal{H}$ for all $z \in G$. By (1.3) $y_\beta$ then converges in $\mathcal{H}_\infty^{\omega}$. Thus, it suffices to show that every $\mathcal{H}_\infty^{\omega}$ bounded set $B$ is compact in $\mathcal{H}$.

Now in [12], Pukánszky showed (Part II, Chapter II, Theorem 2 and its proof) that since $U$ is irreducible, there is an element $X \in \mathcal{U}(G)$ for which $\partial U(X)$ has a bounded inverse $T$. From the eigenvalues for $T$ as computed in [12], $T$ is a compact operator. Hence $B = T \partial U(X) B$ is a compact subset of $\mathcal{H}$. Q.E.D.
II. Function spaces. We are, in this section, interested in representations realized in spaces of functions on $G$ as defined below.

Definition (II.1). Let $H_0$ be a Hilbert space and let $H$ be a space of $H_0$ valued locally integrable functions (with a.e. equal functions identified) with respect to Haar measure. Suppose $H$ is topologized in such a manner that:

1. $H$ is a Banach space.
2. The injection of $H$ into $L^1_{loc}(G, H_0)$ is continuous.
3. For all $f \in H$ and $g \in G$, the function $R_g f : x \mapsto f(gx)$ is in $H$ and $g \mapsto R_g f$ defines a continuous representation of $G$ in $H$.

Then $H$ is said to be a regular Banach space of $H_0$-valued functions and $R$ is said to be the regular representation of $G$ in $H$.

If $F$ is any Fréchet space and $F : G \rightarrow F$ is a $C^\infty$ function then define, for $X \in \mathcal{L}$,

$$
\mathcal{X}_F(g) = \lim_{t \to 0} \frac{F(g \exp tX) - F(g)}{t}.
$$

In this terminology, we have the following generalization of results of Goodman [4] and Poulsen [11].

Proposition (II.2). Let $f \in H$, $f \in C^\infty(H_0)$ iff $f$ is a $C^\infty$ $H_0$-valued function and $X_1, \ldots, X_n \in \mathcal{L}$.

$f \in C^\infty_0(R)$ iff $f$ is extendible to an entire $H_0$-valued map of $G_c$ for which the maps $R_z f : x \mapsto f(gz)$ are in $H$ for all $z \in G_c$ and $z \mapsto R_z f$ is continuous in $H$.

Proof. Suppose $f$ satisfies the hypothesis of the “only if” part of the above. For each $t \in R$ and $X \in \mathcal{L}(G)$ let $g_t = \int_0^1 R(\exp tX) \tilde{X}_f dt$. (This $H$ valued integral exists by Lemma 2, p. 12 of [7].) Let $w \in (H_0)'$ and let $\phi \in \mathcal{D}(G)$. By evaluating with functionals of the form

$$
g \mapsto \int_G \phi(x) (g(x), w) \, dx \quad (g \in H)
$$

it is easily seen that $g_t = R(\exp tX)f$. It follows that $f$ is a $C^1$-vector and hence, by induction, that $f$ is a $C^\infty$-vector. Conversely, if $f$ is a $C^\infty$-vector for $R$ and $w \in (H_0)'$, then it follows by evaluating $\partial R(X)f$ with functionals of the above form that $x \mapsto \langle \partial R(X)f(x), w \rangle$ is the distributional derivative (along $X$) of $x \mapsto \langle f(x), w \rangle$. It follows by induction and the Sobolev theorem that $x \mapsto \langle f(x), w \rangle$ is a $C^\infty$-function. Since weak $C^\infty$ implies strong $C^\infty$, we get that $f$ is a $C^\infty$ $H_0$ valued map (see [7, Lemma 1, p. 47]).

To prove the entirety part, let $f \in C^\infty_0(R)$. Then $f$ is, in particular, a $C^\infty$-vector and hence is continuous. Let $K$ be a compact neighborhood of $e$ in $G$ and let $C(K, H_0)$ be the Banach space of continuous $H_0$ valued functions on
$K$ given the sup-norm topology.

The restriction map $f \mapsto f|_K$ of $H^\omega(R)$ into $C(K, H_0)$ is continuous by the closed graph theorem and the regularity of $H$. It follows that point evaluation is a continuous linear transform on $H^\omega(R)$. We conclude from (1.1) above that $z \mapsto (U^\omega_z f)(e)$ is a holomorphic function on $G_c$. This is the desired extension. Clearly $U^\omega$ acts as claimed.

Finally, if $f$ has an extension satisfying the above hypothesis, let $\gamma$ be a closed curve in $G_c$. By evaluating with the appropriate functionals (as in the $C^\infty$ part), it is easily shown that $\int_\gamma R f \, ds \in H$ is zero. Hence Morera's theorem finishes the proof.

**Corollary (II.3).** Let $R$ be the unitary representation of $G$ induced from a unitary representation $L$ of a closed subgroup of $G$. If $U$ is realized as in Blattner [2], then $R$ is a regular representation and the characterization of (II.2) applies.

**Corollary (II.4).** Let $H_0 = C$, $H = L^p(G)$ (1 < $p$ < $\infty$). Let $R$ be the right regular representation of $G$ in $H$. Then (II.2) applies and we obtain the expected characterization of $H^\omega(X^R)$.

Remark (II.2) is a generalization of results from [11]. (II.4) in the nilpotent case is due to Goodman [4].

**Remark (II.5).** Note that in the above proof we showed that $f \mapsto f(e)$ is continuous from $H^\omega(R)$ into $H_0$. It follows that $f \mapsto f(z)$ is continuous for all $z \in G_c$. If $H_0 = C$ we obtain from (1.4) a Cauchy-like representation theorem for $H^\omega(G)$. In fact, if $R$ is the right regular representation of $G$, our representation theorem takes the following form:

**Corollary.** Let $R$ be the $L^2$ right regular representation. Then, for all $z \in G_c$, there is an $s_0 \in R^d$, $s_0 > 0$, with the following property: For all $s > s_0$ there is a $w_s^z \in H^s(R)$ such that

$$f(z) = \sum_j \left\{ \int_G f(g \exp is_j X_j) \overline{w}_s^z(g) \, dg + \int_G f(g \exp -is_j X_j) \overline{w}_s^z(g) \, dg \right\}$$

for all $f \in H^\omega(R)$.

Note, incidentally, that the modular function does not appear in the above formula. The reason is that for type $R$ groups, the trace of the adjoint representation is unity and hence $G$ is unimodular. Type $R$ groups are the only ones whose regular representation has nonzero entire vectors.

In general, the kernels $w_s^z$ seem to be difficult to compute. For $R$ the right regular representation and $z$ real, it suffices to compute $w_s^z$, for let $L$ be the $L^2$ left regular representation of $G$. Since $L$ and $R$ commute, $L$ leaves $H^s(R)$ invariant and $L$ is unitary in $\|\|_s$. Thus, for $f \in H^\omega(R)$ and $g \in G_c$, the
Thus \( w^g_s = L_g w^e_s \).

If \( G = \mathbb{R} \), we may explicitly compute \( w^e_{s/2} = w \) as follows: From the Paley-Wiener theorem and (II.2), it follows that the Fourier transform \( \hat{f} \) of a function \( f \in L^2(G) \) is in \( \mathbb{H}^\omega(R) \) iff \( \int_0^\infty e^{\alpha t} |\hat{f}(t)|^2 \, dt < \infty \) for all \( \alpha > 0 \). (Cf. Goodman [3, p. 64], for details.) The analytic extension of \( \hat{f} \) is given by

\[
\hat{f}(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-izx} f(x) \, dx.
\]

Hence, letting \( \hat{w} \) denote the conjugate of the inverse Fourier transform

\[
(2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \, dx = \hat{f}(0) = \int_{-\infty}^{\infty} \hat{f}(x+is) \hat{w}(x) \, dx + \int_{-\infty}^{\infty} \hat{f}(x-is) \hat{w}(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} 2(\cosh sx) f(x) \, dx.
\]

Thus \( \hat{w}(x) = 2(2\pi)^{-1/2} \cosh sx \). The Fourier transform can be explicitly computed via a contour integral to be \( w(x) = \left[ 4s \cosh\left(\frac{nx}{2s}\right) \right]^{-1} \).

III. Applications to complex solvmanifolds. Let \( S \) be a solvable, connected, simply connected Lie group (not necessarily type \( R \)) and let \( \Gamma \) be a closed but not necessarily connected subgroup. Then the homogeneous space \( M = S/\Gamma \) is a solvmanifold. If \( S \) is a complex Lie group and \( \Gamma \) is a complex Lie subgroup (i.e. the component of the identity of \( \Gamma \) is a complex analytic subgroup of \( S \)), then \( S/\Gamma \) is a complex manifold and will be called a complex solvmanifold.

Lemma (III.1). Let \( \Gamma_0 \) be the component of the identity of \( \Gamma \) and let \( (\Gamma_0')_c \subset S_c \) be its complexification. Let \( \Gamma_c = \Gamma \cdot (\Gamma_0')_c \). Then \( \Gamma_c \) is a closed, complex Lie subgroup of \( S_c \) and \( M \) is canonically imbedded in \( M_c = S_c / \Gamma_c \).

Proof. There is a Jordan-Hölder basis \( B \) of \( \mathfrak{L}(S) \) which contains a Jordan-Hölder basis \( B_0 \) of \( \mathfrak{L}(\Gamma_0) \). Letting \( B \) define holomorphic coordinates for \( S_c \) as in the proof of (1.1) of [9], one sees that \( B_0 \) defines holomorphic coordinates for \((\Gamma_0')_c \) and hence \((\Gamma_0')_c \cap S = \Gamma_0 \). Let \( \Gamma_c = \Gamma \cdot (\Gamma_0')_c \).

We claim that \((\Gamma_0')_c \) is invariant under \( \Gamma \) and hence that \( \Gamma_c \) is a subgroup. To see this let \( \| \| \) be a complex norm on \( \mathfrak{L}_c(S) \) and let \( \mathcal{U} \) be a closed ball in \( \mathfrak{L}_c(S) \) such that

(i) \( \exp \) is a homeomorphism onto the image \( U \) of \( \mathcal{U} \) in \( S_c \), and

(ii) \( \mathcal{U} \) is sufficiently small in the sense defined below.

For \( z \in S_c \) and \( \alpha \in \mathbb{C} \), \( |\alpha| \leq 1 \), define \( \alpha z = \exp(\alpha(\log z)) \) wherever \( \log z \) is defined and single valued. If \( z \in U \) and \( U \) is sufficiently small, the elements \( y = (-i/2) (\overline{z}^{-1} z) \) and \( x = z(iz)^{-1} \) are defined and satisfy \( z = x(iz) \) (\( \overline{\cdot} \) denotes the canonical conjugation on \( S_c \)). Furthermore, it is easily seen that \( y = \overline{x} \), \( x = \overline{x}^{-1} \) and, hence, that \( y \) and \( x \) are in \((\Gamma_0')_c \cap S = \Gamma_0 \). It follows that if \( g \) is
a fixed element of $\Gamma$ and $U$ is small enough (relative to $g$), then

$$gzg^{-1} = (g^{-1})(giy)g^{-1}) = (g^{-1}(i(gyg^{-1}))) \in (\Gamma_0)^c.$$  

Since $U \cap (\Gamma_0)^c$ generates $(\Gamma_0)^c$, $g$ leaves $(\Gamma_0)^c$ invariant. Hence $\Gamma \cdot (\Gamma_0)^c$ is a subgroup.

It follows similarly that $\Gamma_c$ is closed since to show closure it suffices to show that there is a closed neighborhood $U$ of $e$ in $S_c$ such that $U \cap \Gamma$ is closed. If $U$ is sufficiently small we can work with real and imaginary parts as above.

The map $S/\Gamma \to S \cdot \Gamma_c/\Gamma_c \subset M_c$ is easily seen to be an imbedding (since $\Gamma_c \cap S = \Gamma$) and the lemma follows. Q.E.D.

We are interested in the following questions about $M_c$.

(i) Under what conditions does the set of entire functions $\Omega(M_c)$ separate points of $M$?

(ii) When does $\Omega(M_c)$ separate points of $M$?

Our answers are as follows:

**Theorem (III.2).** Suppose $\Gamma$ contains no nontrivial normal analytic subgroups. Then:

(i) If $S$ is type $R$, then $\Omega(M_c)$ separates points of $M$. If $S/\Gamma$ is compact, it is also necessary that $S$ be type $R$ for point separating to hold.

(ii) If $S$ is type $R$, then there is a closed complex subgroup $\Gamma' \subset S_c$ with the following properties:

(a) $\Gamma' \supset \Gamma_c$ and $(\Gamma')_0 = (\Gamma_c)_0 = (\Gamma_0)^c$.

(b) $\Gamma' \cap S = \Gamma$ and there is a natural imbedding of $M$ into $S_c/\Gamma'$.

(c) $\Omega(S_c/\Gamma')$ separates points of $S_c/\Gamma'$.

I.e., if we change our notion of complexification slightly, $M$ has a complexification for which the holomorphic functions separate points.

**Proof.** (i) Let $S$ be type $R$. Let $C_0(M)$ be the Banach space all continuous functions vanishing at infinity given the sup norm. We may identify $C_0(M)$ with a space of functions $C$ on $G$ which are invariant under right translation by elements of $\Gamma$. $S$ acts on $C$ via left translation and this action defines a representation $R$ of $S$ in $C$. $C$ is a regular Banach space of complex functions (in the sense of II.1) and $R$ is the left regular representation of $S$ in $C$. By (II.3) of [5], $R$ has a dense set of entire vectors and by (II.2) above each entire vector $f$ extends to an entire function $f_c$ on $S_c$. Since $f$ is invariant under $\Gamma$, $f_c$ is invariant under $(\Gamma_0)^c$ (by uniqueness of analytic extensions in $(\Gamma_0)^c$) and hence under $\Gamma_c$. Thus, upon projection $f_c$ defines an element $\tilde{f_c}$ of $\Omega(M_c)$. The set of such $\tilde{f_c}$ separate points of $M$ since they are dense in $C_0(M)$. 

If $S/\Gamma$ is compact, but not necessarily type $R$, let $L$ denote the left regular representation of $S$ in $L^2(S/\Gamma)$.

It follows from (II.2) and the above reasoning that $\mathcal{H}_\infty^0(L)$ can be identified with a subspace of $\Omega(\mathcal{M})$. In fact, since $M$ is compact, it is easily seen that $\Omega(\mathcal{M}) = \mathcal{H}_\infty^0(L)$. It follows from (II.4) of [9] that every element of $\Omega(\mathcal{M})$ is left fixed by left translation by elements of the Green kernel $K$ of $S$ (recall that the Green kernel is the smallest analytic normal subgroup $K$ for which $S/K$ is type $R$). In particular, every element of $\Omega(\mathcal{M})$ is constant on $K$. If $\Omega(\mathcal{M})$ separates points of $M$, then $K \subseteq \Gamma$. Hence $K = \{e\}$ and $S$ is type $R$.

(ii) Let $\pi: \mathcal{S} \to \mathcal{M}$ be the projection map and let $\Gamma' = \{z \in \mathcal{S} \mid f(\pi(z)) = f(\pi(e)) \vee f \in \Omega(\mathcal{M})\}$. $\Gamma'$ is a closed subgroup of $\mathcal{S}$ which contains $\Gamma$. We claim that $\Gamma'$ is a complex subgroup. To see this, let $Z \in \mathcal{L}(\mathcal{S})$ be such that $\exp tZ \in \Gamma'$ for all $t \in \mathbb{R}$.

If $f \in \Omega(\mathcal{M})$, then the map $w \to f(\pi(\exp wZ))$ of $\mathbb{C}$ into $\mathbb{C}$ is holomorphic and is constant on $\mathbb{R}$. Hence it is constant and, in particular $f(\pi(\exp iz)) = f(\pi(e))$ for all $t \in \mathbb{R}$. Thus $iz \in \mathcal{L}(\Gamma')$, as claimed.

Similarly, it follows that $\Gamma'$ is invariant under the canonical conjugation in $\mathcal{S}$. Therefore $\mathcal{L}(\Gamma')$ is the complexification of a real Lie subalgebra of $\mathcal{S}$. Since $\Gamma' \cap S = \Gamma$ (since $\Omega(\mathcal{M})$ separates points of $S/\Gamma$), it follows that $\mathcal{L}(\Gamma')$ is the complexification of $\mathcal{L}(\Gamma)$ and hence that $(\Gamma')_0 = (\Gamma_0)'$, as claimed. The rest of the proposition follows. Q.E.D.

Remarks. We know of no examples where $\Gamma' \neq \Gamma_c$. It would be interesting to know if such examples exist.

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