The Fredholm Spectrum of the Sum and Product of Two Operators

By

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Abstract. Let \( \mathcal{C}(X) \) denote the set of closed operators with dense domain on a Banach space \( X \), and \( L(X) \) the set of all bounded linear operators on \( X \). Let \( \Phi(X) \) denote the set of all Fredholm operators on \( X \), and \( \sigma_\Phi(A) \) the set of all complex numbers \( \lambda \) such that \( (\lambda - A) \notin \Phi(X) \). In this paper we establish conditions under which \( \sigma_\Phi(A + B) \subseteq \sigma_\Phi(A) + \sigma_\Phi(B) \), \( \sigma_\Phi(AB) \subseteq \sigma_\Phi(A) \cdot \sigma_\Phi(B) \), and \( \sigma_\Phi(AB) \subseteq \sigma_\Phi(A) \sigma_\Phi(B) \).

In this paper we will use the operational calculus developed in [3] to establish a property of the Fredholm spectrum of the sum and product of two operators.

Definition 1. A closed operator \( A \) from a Banach space \( X \) to a Banach space \( Y \) is called a Fredholm operator if:

1. the domain of \( A, D(A) \), is dense in \( X \).
2. \( \alpha(A) = \dim [N(A)] < \infty \).
3. \( R(A) \), the range of \( A \), is closed in \( Y \).
4. \( \beta(A) \), the codimension of \( R(A) \) in \( Y \), is finite.

It is shown in [1, Lemma 332] that condition (4) implies condition (3). A discussion of Fredholm operators can be found in [2].

We denote the set of Fredholm operators from \( X \) to \( Y \) by \( \Phi(X) \).

Definition 2. \( \lambda \in \Phi_A \) if and only if \( (\lambda - A) \notin \Phi(X) \).

Definition 3. \( \lambda \in \sigma_\Phi(A) \) if and only if \( \lambda \notin \Phi_A \).

Definition 4. A bounded operator \( B \) will be called a quasi-inverse of the closed operator \( A \) if:

1. \( R(B) \subseteq D(A) \) and \( AB = I + K_1 \), \( K_1 \in \mathcal{K}(X) \).
2. \( BA = I + K_2 \), \( K_2 \in \mathcal{K}(X) \).

\( \mathcal{K}(X) \) denotes the set of all compact operators on \( X \).


AMS (MOS) subject classifications (1970). Primary 47B30; Secondary 47A60, 47B05.

Key words and phrases. Fredholm operator, Fredholm spectrum, operational calculus.

(1) This research was partially funded by a Faculty Research Award Program grant, from the City University of New York.

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By [2, Theorem 2.9], $\Phi_A$ is open and is thus the union of a disjoint collection of connected open sets. Each such set, $\Phi_i(A)$, will be called a component of $\Phi_A$.

Let $C(X)$ denote the set of closed operators on $X$ with dense domain.

Suppose $A \in C(X)$ with $\Phi_A$ not empty, and let $\lambda \in \Phi_A$. In [3], a quasi-inverse of $\lambda - A$, $R_i^\prime(A)$, was constructed in the following way. In each $\Phi_i(A)$, a fixed point, $\lambda_i$, is chosen in a prescribed manner. There exist subspaces, $X_i$ and $Y_i$ such that $X = N(\lambda_i - A) \oplus X_i$, $X_i$ is closed, and $X = Y_i \oplus R(\lambda_i - A)$.

Let $F_{1i}$ be the projection of $X$ onto $N(\lambda_i - A)$ along $X_i$, and let $F_{2i}$ be the projection of $X$ onto $Y_i$ along $R(\lambda_i - A)$. $F_{1i}$ and $F_{2i}$ are bounded finite rank operators. $F_{1i}$ has a bounded inverse, $A_i F_{1i} = R(\lambda_i - A)$ along $X_i$.

Let the operator $T_i$ be defined by: $T_i x = A_i (I - F_{2i}) x$. $T_i$ is a quasi-inverse of $(\lambda_i - A)$.

$R_i^\prime(A)$ is then defined by $R_i^\prime(A) = T_i [(\lambda - \lambda_i) T_i + I]^{-1}$ when $\lambda \in \Phi_i(A)$ and $-1/(\lambda - \lambda_i) \in \rho(T_i)$. In [3, Theorems 2 and 5, §2], $R_i^\prime(A)$ is shown to be a quasi-inverse of $(\lambda - A)$ defined and analytic for all $\lambda \in \Phi_A$ except for at most an isolated set, $\Phi_0(A)$, having no accumulation point in $\Phi_A$.

**Lemma 1.1.** Let $n$ be a positive integer and $A \in C(X)$ such that $\Phi_A$ is not empty. Then for each $\lambda \in \{\Phi_A \setminus \Phi_0(A)\}$, there exists a subspace $V_\lambda$, dense in $X$ and depending on $\lambda$, such that $\forall x \in V_\lambda$, $R_i^\prime(A) x \in D(A^n)$.

**Proof.** Let $\lambda \in \{\Phi_A \setminus \Phi_0(A)\}$. By [2, Theorem 2.5], $D(A^n) = D(\lambda - A)^n$ is dense in $X$ for all $n$. Therefore, $T_i^{-1}D(A^n) \cap R(A) = D(A^n - 1) \cap R(A)$ is dense in $R(A)$. By $T_i^{-1}D(A^n)$ we mean $\{x | T_i x \in D(A^n)\}$. Let $Y_i$ be the complement of $R(A)$ used in the construction of $T_i$. Since $T_i Y_i \rightarrow 0 \in D(A^n)$ and $X = R(A) + Y_i$, we have $T_i^{-1}D(A^n) = T_i^{-1}D(A^n) \cap R(A) \oplus Y_i$ is dense in $X$.

Therefore, $V_\lambda = [(\lambda - \lambda_i) T_i + I]T_i^{-1}D(A^n)$ is dense in $X$ because $[(\lambda - \lambda_i) T_i + I]$ is invertible. Q.E.D.

We denote the set of all bounded operators on $X$ by $L(X)$.

**Lemma 1.2.** Let $A \in \Phi(X)$, $B \in L(X)$, and $K \in K(X)$. Suppose $AB|_V = K|_V$ where $V$ is a dense subspace of $X$. Then $B \in K(X)$.

**Proof.** There exists $A_0 \in L(X)$ such that $A_0 A = I - K_1$, $K_1 \in K(X)$.

$$A_0 AB|_V = A_0 B|_V \quad (I - K_1) B|_V = K_2|_V \quad B|_V = (K_1 + K_2)|_V.$$ Since $B$ and $(K_1 B + K_2)$ are bounded, and $V$ is dense, we have $B = K_1 B + K_2$ by continuity. Q.E.D.

**Lemma 1.3.** Let $B \in L(X)$, $A \in C(X)$, $\mu \in \{\Phi_B \setminus \Phi_0(B)\}$ and $\lambda \in \{\Phi_A \setminus \Phi_0(A)\}$. Let there exist a positive integer $n$ and a compact operator $K_1$, such that
B: \( D(\mathbb{A}^n) \to D(\mathbb{A}) \) and \( ABx = BAx + Kx \), \( \forall x \in D(\mathbb{A}^n) \). Then there exists a compact operator \( K \), depending analytically on \( \lambda \) and \( \mu \), such that

\[
R^\lambda_\mu(A)R^\mu_\lambda(B) = R^\mu_\lambda(B)R^\lambda_\mu(A) + K.
\]

Proof. By Lemma 1.1 there exists a subspace \( V_\lambda \), dense in \( X \), such that \( \forall x \in V_\lambda, R^\lambda_\lambda(A)x \in D(\mathbb{A}^n) \). Let \( x \in V_\lambda \).

\[
(\lambda - A)BR^\lambda_\lambda(A)x = [B(\lambda - A) - K]R^\lambda_\lambda(A)x
\]

\[
= [B(I - K) + K_\lambda]x
\]

\[
= Bx + K_\lambda x.
\]

Therefore, \( (\lambda - A)[BR^\lambda_\lambda(A) - R^\lambda_\lambda(A)B]x = K_\lambda x \). Since this equality holds for all \( x \in V_\lambda \), we have by Lemma 1.2, that \( BR^\lambda_\lambda(A) - R^\lambda_\lambda(A)B = K \), and \( BR^\lambda_\lambda(A) = R^\lambda_\lambda(A)B + K \), \( K \in \mathbb{K}(X) \) for \( i = 1, 2, \ldots, 6 \).

\[
(\mu - B)[R^\mu_\lambda(B)R^\lambda_\lambda(A) - R^\lambda_\lambda(A)R^\mu_\lambda(B)] = (I - K_\lambda)R^\lambda_\lambda(A) - (\mu - B)R^\lambda_\lambda(A)R^\mu_\lambda(B)
\]

\[
= R^\lambda_\lambda(A) + K_\mu - R^\lambda_\lambda(A)(I - K_\lambda) + K_\mu = K_{10}.
\]

Therefore, \( R^\mu_\lambda(B)R^\lambda_\lambda(A) - R^\lambda_\lambda(A)R^\mu_\lambda(B) = K \), and \( R^\mu_\lambda(B)R^\lambda_\lambda(A) = R^\lambda_\lambda(A)R^\mu_\lambda(B) + K \), \( K \in \mathbb{K}(X) \) for \( i = 7, 8, 9, 10 \). The analyticity of \( K \) in \( \lambda \) and \( \mu \) follows from the analyticity of \( R^\mu_\lambda(B) \) and \( R^\lambda_\lambda(A) \).

Theorem 1. Let \( B \in L(X) \) and \( A \in C(X) \). Suppose there exist a positive integer \( n \) and a compact operator \( K \) such that \( B: D(\mathbb{A}^n) \to D(\mathbb{A}) \) and \( ABx = Bx + Kx \) for all \( x \in D(\mathbb{A}^n) \). Then \( \sigma_\alpha(A + B) \subseteq \sigma_\alpha(A) + \sigma_\alpha(B) \). If \( \sigma_\alpha(A) \) is empty, we interpret \( \sigma_\alpha(A) + \sigma_\alpha(B) \) to be the empty set.

Proof. If \( \sigma_\alpha(A) + \sigma_\alpha(B) \) is the entire complex plane, then the theorem is trivially true. We therefore assume that \( \sigma_\alpha(A) + \sigma_\alpha(B) \) is not the entire plane.

Let \( y \) be a fixed point not contained in \( \sigma_\alpha(A) + \sigma_\alpha(B) \). We shall show that \( y \in \Phi(A + B) \). If \( \lambda \in \sigma_\alpha(B) \), then \( (y - \lambda) \in \Phi_\lambda \). Since \( \sigma_\alpha(A) \) is closed and \( \sigma_\alpha(B) \) is compact, there exists an open set \( U \supset \sigma_\alpha(B) \) such that \( \Phi(U) \), the boundary of \( U \), is bounded, and when \( \lambda \in U \), \( (y - \lambda) \in \Phi_\lambda \). Let \( A_1 = y - A \). \( \Phi_\lambda \) if and only if \( \lambda \in \Phi_{A_1} \). Therefore, \( \sigma_\alpha(B) \subseteq U \subseteq \Phi_{A_1} \). There exists a bounded Cauchy domain \( D \) such that \( \sigma_\alpha(B) \subseteq D \subseteq U \). See [4, Theorem 3.3]. Since \( \Phi_{A_1} \) does not accumulate in \( \Phi_{A_1} \) and \( \Phi_0(B) \) does not accumulate in \( \Phi_B \), \( D \) can be chosen so that \( R^\lambda_\mu(A) \) and \( R^\mu_\lambda(B) \) are analytic on \( B(D) \), the boundary of \( D \).

Define the operators \( S_1 \) and \( S_2 \) by

\[
S_1 = -\frac{1}{2\pi i} \int_{B(D)} R^\mu_\lambda(A)R^\mu_\lambda(B) d\lambda,
\]
and

\[ S_2 = -\frac{1}{2\pi i} \int_{+B(D)} R_{\lambda}'(B) R_{\lambda}'(A_1) \, d\lambda. \]

\( R_{\lambda}'(A_1) \) is of the form \( TC(\lambda) \), where \( C(\lambda) \) is bounded operator valued analytic function of \( \lambda \) and \( T \) is a fixed bounded operator such that \( T: X \to D(A_1) = D(A) \).

Therefore, \( S_1: X \to D(A) \).

We will now show that there exist compact operators \( K_1 \) and \( K_2 \) such that

\[ (y - B - A)S_1 = I + K_1 \quad \text{and} \quad S_2(y - B - A) = (I + K_2)|_{D(A)}. \]

\[ y - B - A = (y - \lambda - A) + (\lambda - B) = -(\lambda - A_1) + (\lambda - B). \]

\[ (y - B - A)S_1 = -\frac{1}{2\pi i} \int_{+B(D)} (\lambda - B) R_{\lambda}'(A_1) R_{\lambda}'(B) \, d\lambda. \]

Since \( (\lambda - A_1)R_{\lambda}'(A_1) \) is of the form \( I + F(\lambda) \) where \( F(\lambda) \) is a bounded finite rank operator depending analytically on \( \lambda \) and \( (2\pi i)^{-1} \int_{+B(D)} R_{\lambda}'(B) \, d\lambda \) is of the form \( I + K_3 \), see \([3, \text{Theorem 13, \S 2}] \), the first integral is of the form \( I + K_4 \).

By Lemma 1.3, there exists a compact operator \( K(\lambda) \), depending analytically on \( \lambda \), such that \( R_{\lambda}'(A_1)R_{\lambda}'(B) = R_{\lambda}(B)R_{\lambda}'(A_1) + K(\lambda) \). Therefore, the second integral is equal to

\[ \frac{-1}{2\pi i} \int_{+B(D)} (\lambda - B) R_{\lambda}'(B) R_{\lambda}'(A_1) \, d\lambda = \frac{1}{2\pi i} \int_{+B(D)} (\lambda - B) K(\lambda) \, d\lambda. \]

The first of these two integrals equals \( \int_{+B(D)} R_{\lambda}'(A_1) \, d\lambda + K_5^* \). \( K_5 \in \mathbb{K}(X) \). As in \([3, \text{proof of Theorem 10, \S 2}] \), \( \int_{+B(D)} R_{\lambda}'(A) \, d\lambda \) is compact. Since \((2\pi i)^{-1} \int_{+B(D)} (\lambda - B) K(\lambda) \, d\lambda \) is also compact, we have that \((y - B - A)S_1 = I + K_1^*, K_1 \in \mathbb{K}(X) \).

By a similar argument we have that \( S_2(y - B - A) = I + K_2 \). \( K_2 \in \mathbb{K}(X) \). Therefore, by \([2, \text{Lemma 2.4}] \), \( (y - B - A) \in \Phi(X) \) and, thus, \( y \in \Phi(A + B) \). Therefore, \( \sigma(\Phi(A + B)) \subset \sigma(\Phi(A)) + \sigma(\Phi(B)) \). This completes the proof of the theorem.

**Theorem 2.** Let \( A \in C(X) \) and \( B \in L(X) \cap \Phi(X) \). Let \( B: D(A) \to D(A) \) and let there exist a compact operator \( K \) such that \( BAx = ABx + Kx \), \( \forall x \in D(A) \). Then \( BA \) is preclosed and

1. \( \sigma^b(AB) \subset \sigma^b(A) \sigma^b(B) \), and
2. \( \sigma^b(AB) \subset \sigma^b(A) \sigma^b(B) \).

**Proof.** Since \( (AB)|_{D(A)} \) is preclosed and \( K \) is bounded, we have that \( BA \) is preclosed.

Let \( \mathbb{C} \) denote the set of all complex numbers. Since \( 0 \notin \sigma^b(B) \) and \( \sigma^b(B) \)
is not empty, we have that if \( \sigma_0(A) = C \) then \( \sigma_0(B)\sigma_0(A) = C \), and the lemma is established. We therefore assume that \( \sigma_0(A) \neq C \).

Suppose \( \gamma \) is a fixed point not in \( \sigma_0(B)\sigma_0(A) \). We proceed to show that \( \gamma \in \Phi(BA) \).

Since \( \sigma_0(A) \) is closed, \( \sigma_0(B) \) is compact, and \( 0 \notin \sigma_0(B) \), there exists an open set \( U \supset \sigma_0(B) \) such that \( 0 \notin U, B(U) \), the boundary of \( U \), is bounded, and \( (y - \mu A) \in \Phi(X) \) \( \forall \mu \in U \). Let \( D \) be a bounded Cauchy domain such that \( \sigma_0(B) \subset D \subset U \). \( (y - \mu A) = \mu y (1/\mu - (1/\gamma)A) = (y/\lambda) (\lambda - (1/\gamma)A) \), where \( \lambda = 1/\mu \).

Let \( D' \) be the image of \( D \) under the map \( \lambda = 1/\mu \). Let \( A_1 = (1/\gamma)A \). Since \( \forall \mu \in D, 1/\mu \in \Phi_{A_1} \), and since \( R_\lambda(A_1) \) is analytic in \( \lambda \) throughout \( \Phi_{A_1} \), except for at most an isolated set having no accumulation point in \( \Phi_{A_1} \), we can assume that \( R_\lambda(A_1) \) is analytic in \( \lambda \) on \( B(D') \). Note. \( R_\lambda(A_1) \) is of the form \( TC(\lambda) \) where \( C(\lambda) \) is an analytic operator valued function of \( \lambda \) and \( T \) is a bounded operator such that \( R(T) \subset D(A_1) = D(A) \).

Let

\[
S_1 = \frac{1}{2\pi i} \int_{\partial B(D')} \frac{-1}{\gamma \lambda} R_\lambda'(A_1) R_\lambda'(B) d\lambda
\]

and

\[
S_2 = \frac{1}{2\pi i} \int_{\partial B(D')} \frac{-1}{\gamma \lambda} R_\lambda'(B) R_\lambda'(A_1) d\lambda.
\]

Since \( R(S_1) \subset D(A) \), \( (y - BA) S_1 \) is defined, and \( (y - BA) S_1 = (y - BA) S_1 \).

\[
y - BA = y - BYA_1 = y(I - BA_1)
\]

\[
= yB(\lambda - A_1) - y\lambda B + yI = yB(\lambda - A_1) + y(I - \lambda B).
\]

Therefore,

\[
(y - BA) S_1 = \frac{1}{2\pi i} \int_{\partial B(D')} \frac{-1}{\gamma \lambda} B(\lambda - A_1) R_\lambda'(A_1) R_\lambda'(B) d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{\partial B(D')} (1/\gamma - B) R_\lambda'(A_1) R_\lambda'(B) d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\partial B(D')} \frac{-1}{\gamma \lambda} B[I + K_1(\lambda)] R_\lambda'(B) d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{\partial B(D')} (1/\gamma - B)[R_\lambda'(B) R_\lambda'(A_1) + K_2(\lambda)] d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\partial B(D')} \frac{-1}{\gamma \lambda} B R_\lambda'(B) d\lambda + K_3
\]

\[
- \frac{1}{2\pi i} \int_{\partial B(D')} [I + K_3(\lambda)] R_\lambda'(A_1) d\lambda + K_5
\]

\[
= \frac{1}{2\pi i} \int_{\partial B(D')} \frac{1}{\mu} B R_\lambda'(B) d\mu + K_3 - \frac{1}{2\pi i} \int_{\partial B(D')} R_\lambda'(A_1) d\lambda + K_6.
\]
Since \( 0 \notin D \), we have
\[
\frac{1}{2\pi i} \int_{D'} \frac{1}{\mu} \mathcal{R}'(B) \, d\mu = B \frac{1}{2\pi i} \int_{D} \frac{1}{\mu} \mathcal{R}'(B) \, d\mu = I + K_7
\]
by [3, Theorems 14, 9 and 13, \S 2].

Since \( \mathcal{R}'(A_1) \) is analytic in \( D' \) except for at most a finite number of points, we have that \( (2\pi i)^{-1} \int_{D'} \mathcal{R}'(A_1) \, d\lambda = K_8 \) by [3, Lemma 7.4, \S 2]. Therefore, \( (y - B\overline{A})S_1 = I + K', \ K', \ K_i (\lambda) \in \mathcal{K}(x) \) for \( i = 1, 2, \ldots, 8 \).

Claim. \( D(\overline{B}A) \subseteq D(AB) \) and \( \overline{BA}x = ABx + Kx \ \forall x \in D(\overline{B}A) \).

Proof. Let \( x \in D(\overline{B}A) \). Then there exists a sequence, \( \{x_n\}_{n=1}^\infty \), such that \( x_n \in D(A) \), \( x_n \to x \) as \( n \to \infty \), and \( ABx_n \to \overline{BA}x \) as \( n \to \infty \). Therefore
\[
\lim_{n \to \infty} ABx_n = \lim_{n \to \infty} BAX_n - \lim_{n \to \infty} Kx_n = \overline{BA}x - Kx.
\]
Therefore, \( Bx \in D(A) \) and \( ABx = \lim_{n \to \infty} ABx_n = \overline{BA}x - Kx \).

\[
y - BA = B(\lambda - A_1) + y(I - \lambda B) = (y - A_1)B + y(I - \lambda B) + K_9.
\]

\[
S_2(y - BA) = \frac{1}{2\pi i} \int_{D'} \frac{-1}{\gamma^2} \mathcal{R}'(B) \frac{1}{\mathcal{R}_\lambda'(A_1)} \, d\lambda \left[ (y - A_1)B + y(I - \lambda B) + K_9 \right]
\]

\[
= \frac{1}{2\pi i} \int_{D'} \frac{-1}{\gamma^2} \mathcal{R}'(B)[I + K_10(\lambda)]B \, d\lambda
\]

\[
+ \frac{1}{2\pi i} \int_{D'} \frac{-1}{\gamma^2} \mathcal{R}'(A_1) \mathcal{R}_\lambda'(B) - K_2(\lambda)[y(I - \lambda B) + K_9] \, d\lambda
\]

\[
= \left[ \frac{1}{2\pi i} \int_{D} \frac{1}{\mu} \mathcal{R}'(B) \, d\mu \right] B + K_{11}
\]

\[
+ \frac{1}{2\pi i} \int_{D'} (-1) \mathcal{R}_\lambda'(A_1) \mathcal{R}_\lambda'(B)(1/\lambda - B) \, d\lambda + K_{12}
\]

\[
= I + K_{13} + \frac{1}{2\pi i} \int_{D'} (-1) \mathcal{R}_\lambda'(A_1) [I + K_{14}(\lambda)] \, d\lambda
\]

\[
= I + K_{15},
\]

\( K_i, K_\lambda(\lambda) \in \mathcal{K}(x) \) for \( i = 9, 10, \ldots, 14 \).

Therefore \( (y - BA) \in \Phi(x) \) by [2, Lemma 2.4]. This completes the proof that \( \sigma_\Phi(\overline{B}A) \subseteq \sigma_\Phi(B) \sigma_\Phi(A) \).

To show that \( \sigma_\Phi(AB) \subseteq \sigma_\Phi(B) \sigma_\Phi(A) \) we proceed to show that \( (y - AB) \in \Phi(x) \).

Let \( S_1 \) and \( S_2 \) be as above. Since \( R(S_1) \subseteq D(A) \) and \( ABx = BAX - Kx, \ \forall x \in D(A) \), we have that \( (y - AB)S_1 = (y - BA)S_1 + K_{15} = I + K' + K_{15} = I + K_{16} \).

\[
S_2(y - AB) = S_2[y(\lambda - A_1)B + y(I - \lambda B)]
\]

\[
= I + K_{17}, \quad K_{15}, K_{16}, K_{17} \in \mathcal{K}(x).
\]

Therefore, by [2, Lemma 2.4], \( (y - AB) \in \Phi(x) \). This completes the proof that \( \sigma_\Phi(AB) \subseteq \sigma_\Phi(B) \sigma_\Phi(A) \).

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Corollary 2.1. Let $A \in C(X)$ and $B \in L(X)$. Assume $0 \notin \sigma(B)$. Let $B: D(A) \to D(A)$ and let there exist a compact operator, $K$, such that $BAx = ABx + Kx$, $\forall x \in D(A)$. Then $BA$ and $AB|_{D(A)}$ are closed and

1. $\sigma(AB|_{D(A)}) = \sigma(BA) \subseteq \sigma(B)\sigma(A)$, and
2. $\sigma(AB) \subseteq \sigma(B)\sigma(A)$.

Proof. Since $B$ is invertible and $A$ is closed, $BA$ is closed. We proceed to show that $AB|_{D(A)}$ is closed.

Let $x_n \in D(A)$, $x_n \to x$ and $ABx_n \to y$. Since $B$ is bounded and $A$ is closed, we have $y = ABx$. We have only to show that $x \in D(A)$, $BAx_n = (ABx_n + Kx_n)$ converges to some vector, $z$. Therefore, $Ax_n \to B^{-1}z$, and, since $A$ is closed, $x \in D(A)$. This shows that $AB|_{D(A)}$ is closed.

Since $BA = AB|_{D(A)} + K$, we have by [2, Theorem 2.8] that $\sigma(AB|_{D(A)}) = \sigma(BA)$. The remainder of the corollary follows from Theorem 2.

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