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A SUMMATION FORMULA INVOLVING $\sigma(n)$

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ABSTRACT. The L^2 theories are known of the summation formula involving $\sigma_k(n)$, the sum of the kth power of divisors of n, as coefficients, for all k except $k = 1$. In this paper, techniques are used to overcome the extra convergence difficulty of the case $k = 1$, to establish a symmetric formula connecting the sums of the form $\sum \sigma_1(n)n^{-1/2}f(n)$ and $\sum \sigma_1(n)n^{-1/2}g(n)$, where $f(x)$ and $g(x)$ are Hankel transforms of each other.

1. **Introduction.** The summation formulae connecting the sums of the form

$$\sum \sigma_k(n)f(n) \qquad \text{and} \qquad \sum \sigma_k(n)g(n)$$

are known to exist for most values of k, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n and $f(x)$ and $g(x)$ are Watson or Fourier transforms of each other. For instance, if $k = 0$ there is the well-known Voronoi formula [10]; if $0 < |k| < 1$ a formula has been given by A. P. Guinand [5] and if $k > 1$ a formula has been given by the author [8]. The author and Guinand use the theory of transforms of functions of L^2-class, but this theory fails when $k = 1$. However, A. P. Guinand points out that if $L^p(0, \infty)$, $(1 < p \leq 2)$ is used, his result could be extended to include the case $|k| < 2$, the sum being Riesz summable $(R, n, 2)$. But he does not give any details, and the result then loses its symmetry. Moreover, higher order of summability has to be used.

In this paper we deal with the case $k = 1$ and establish a symmetric formula, using L^2 theory, connecting the sums

$$\sum \sigma(n)n^{-1/2}f(n) \qquad \text{and} \qquad \sum \sigma(n)n^{-1/2}g(n),$$

where $f(x)$ and $g(x)$ are Fourier transforms with respect to the kernel $-2\pi J_1(4\pi x^{1/2})$ and belong to a class of functions defined below. The series are summable $(R, n, 1)$ by Riesz means. We shall write $\sigma(n)$ for $\sigma_1(n)$, the sum of divisors of n.

Definition. A function $f(x) \in G_\lambda^p(0, \infty)$ if and only if for a fixed $\lambda > 1/p$ and $p > 1$ there exists a.e. a function $f^{(\lambda)}(x)$ such that

(i) $f(x) = (1/\Gamma(\lambda)) \int_x^\infty (t - x)^{\lambda-1} f^{(\lambda)}(t)\,dt$, $x > 0$,

(ii) $x^\lambda f^{(\lambda)}(x) \in L^p(0, \infty)$.

The function $f^{(\lambda)}(x)$ is the λth derivative of $f(x)$ where λ is an integer. Such a class

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of functions has been defined by A. P. Guinand [4] and J. B. Miller [7]. It can be shown that if $f(x) \in G_\lambda^p(0,\infty)$ then,

$$(1.1)\qquad \begin{aligned} &\text{(i)} \quad x^{r+1/2}f^{(r)}(x) \to 0 \qquad \text{as } x \to 0 \text{ or } \infty,\ 0 \le r < \lambda,\\ &\text{(ii)} \quad f(x) \in L^p(0,\infty). \end{aligned}$$

The class of functions G_λ^2 is a subclass of L^2. In this paper we shall use only the class $G_{\frac{1}{2}}^2(0,\infty)$.

2. **The kernel.** Consider the Hankel kernel

$$k(x) \equiv -2\pi J_1(4\pi x^{1/2}),$$

where $J_1(x)$ is the usual Bessel function of order 1. Its Mellin transform is

$$(2.1)\qquad \begin{aligned} K(s) &= \int_0^\infty k(x)x^{s-1}\,dx, \qquad -1/2 < R(s) < 3/4,\\ &= (1/\pi)(2\pi)^{1-2s}\Gamma(s-1/2)\Gamma(s+1/2)\cos s\pi,\\ &= \zeta(1/2-s)\zeta(3/2-s)/\zeta(s-1/2)\zeta(s+1/2), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Now define

$$m(x) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{(s+3/2)(s+1/2)}{(7/2-s)(5/2-s)(3/2-s)}K(s)x^{1-s}\,ds.$$

The integral converges in mean-square and consequently $x^{-1}m(x) \in L^2(0,\infty)$, and $m(x)$ is a generalized Hankel kernel in Watson's sense [11]. The integral above can easily be evaluated [3, p. 236] to give $m(x) = -x^{1/2}J_6(4\pi x^{1/2})$. To show the connection between $k(x)$ and $m(x)$, let us define

$$A(x) \equiv \lim_{T\to\infty}\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{K(s)}{3/2-s}x^{1-s}\,ds.$$

Then $k(x)$ is given by

$$A(x) = x^{-1/2}\int_0^x t^{1/2}k(t)\,dt.$$

Now define

$$B(x) = 3x^{-3/2}\int_0^x A(u)u^{1/2}\,du - A(x).$$

Then $m(x)$ is given by

$$m(x) = 5x^{-5/2}\int_0^x B(u)u^{3/2}\,du - B(x).$$

3. The preliminary results. We shall need the following results.

Lemma 3.1. *If $f(x) \in G_2^2(0,\infty)$, then $f(x)$ has a transform $g(x)$ with respect to the Hankel kernel $k(x)$, defined above. That is, both*

$$g(x) = \int_{\to 0}^{\to\infty} f(t)k(xt)\,dt; \qquad f(x) = \int_{\to 0}^{\to\infty} g(t)k(xt)\,dt,$$

where $k(x) = -2\pi J_1(4\pi x^{1/2})$.

Lemma 3.2. *If $f(x)$ and $g(x)$ are as defined in Lemma* 3.1, *then*

$$F(x) \equiv x^{5/2}(d/dx)^2\{x^{-1/2}f(x)\} \quad \textit{and} \quad G(x) \equiv x^{5/2}(d/dx)^2\{x^{-1/2}g(x)\}$$

are m-transforms [2] *of the class $L^2(0,\infty)$, where $m(x) = -x^{-1/2}J_6(4\pi x^{1/2})$. That is,*

$$\int_0^x t^{5/2}G(t)\,dt = x^{5/2}\int_0^\infty \frac{m(xt)}{t}F(t)\,dt$$

exists a.e. for $x > 0$. Also the reciprocal relation holds a.e. for $x > 0$.

Both of these are known results; they can be obtained as special cases of the results in Miller [7] and Guinand [5], respectively.

Next, consider the function

$$\begin{aligned}\phi(x) &= \left\{\sum_{n\le x}\sigma(n)(x-n) - \frac{\pi^2}{36}x^3 + \frac{1}{8\pi^2}(2\pi^2x^2 - 2\pi x + 1 - e^{-2\pi x})\right\}x^{-5/2} \\ &= h(x)x^{-5/2},\end{aligned} \tag{3.1}$$

say. It is known that [12, p. 415],

$$\sum_{n\le x}\sigma(n)(x-n) - \frac{\pi^2}{36}x^3 + \frac{1}{4}x^2 = O(x^{4/3+\varepsilon}) \quad \text{as } x\to\infty,\ \varepsilon > 0. \tag{3.2}$$

Then,

$$\begin{aligned}\phi(x) &= O(x^{-7/6+\varepsilon}) \qquad \text{as } x\to\infty, \\ &= O(x^{1/2}) \qquad \text{as } x\to 0.\end{aligned}$$

Therefore $\phi(x) \in L^2(0,\infty)$ and consequently has a Mellin transform

$$\Phi(s) \in L^2(1/2 - i\infty, 1/2 + i\infty),$$

where $\Phi(s) = \int_0^\infty \phi(x)x^{s-1}\,dx$. The integral is absolutely convergent for $-1/2 < R(s) < 7/6$. Now, for $-1/2 < R(s) < 7/6$

$$\Phi(s) = \int_0^\infty \left\{ \sum_{n\le x} \sigma(n)(x-n) - \frac{\pi^2}{36}x^3 \right.$$
$$\left. + \frac{1}{8\pi^2}(2\pi^2 x_2 - 2\pi x + 1 - e^{-2\pi x}) \right\} x^{s-7/2}\,dx$$
$$= \int_1^\infty \left\{ \sum_{n\le x} \sigma(n)(n-x) - \frac{\pi^2}{36}x^3 + \frac{1}{4}x^2 \right\} x^{s-7/2}\,dx$$
$$+ \frac{1}{8\pi^2}\int_1^\infty x^{s-7/2}\,dx - \frac{\pi^2}{36}\int_0^1 x^{s-1/2}\,dx - \frac{1}{4\pi}\int_1^\infty x^{s-5/2}\,dx$$
$$- \frac{1}{8\pi^2}\left\{ \int_0^1 \left(e^{-2\pi x} - 1 + 2\pi x - \frac{(2\pi x)^2}{2} \right) x^{s-7/2}\,dx + \int_1^\infty e^{-2\pi x} x^{s-7/2}\,dx \right\}$$
$$= I_1(s) + \frac{1}{8\pi^2}I_2(s) - \frac{\pi^2}{36}I_3(s) - \frac{1}{4\pi}I_4(s) - \frac{1}{8\pi^2}I_5(s),$$

say. Now in the convergence strip $-1/2 < R(s) < 7/6$

$$I_2(s) = -\frac{1}{s-5/2}, \qquad I_3(s) = \frac{1}{s+1/2}, \qquad I_4(s) = -\frac{1}{s-3/2},$$

and these expressions give the analytic continuations of these functions for $s \neq 5/2, -1/2, 3/2$ respectively. Now consider

$$I_5(s) = \int_0^1 \left\{ e^{-2\pi x} - 1 + 2\pi x - \frac{(2\pi x)^2}{2} \right\} x^{s-7/2}\,dx + \int_1^\infty e^{-2\pi x} x^{s-7/2}\,dx.$$

This defines an analytic function within the strip, and gives the continuation for all $R(s) > -1/2$. For the part of the latter region where $R(s) > 5/2$, this can be rearranged as

$$I_5(s) = \int_0^\infty e^{-2\pi x} x^{s-7/2}\,dx - \int_0^1 x^{s-7/2}\,dx + 2\pi\int_0^1 x^{s-5/2}\,dx - 2\pi^2\int_0^1 x^{s-3/2}\,dx$$
$$= (2\pi)^{5/2-s}\Gamma(s-5/2) - \frac{1}{s-5/2} + \frac{2\pi}{s-3/2} - \frac{2\pi^2}{s-1/2}.$$

That is, $I_5(s)$ is equal to $(2\pi)^{5/2-s}\Gamma(s-5/2)$ less the principal parts at $s = 1/2$, $3/2$, $5/2$. Hence the latter expression gives the analytic continuation of $I_5(s)$ for all s, except $s = -1/2, -3/2, -5/2, \ldots$. The analytic continuation of the contribution of $I_2(s)$, $I_3(s)$, $I_4(s)$, $I_5(s)$ to $\Phi(s)$ is therefore given by

$$-\frac{1}{8\pi^2}\frac{1}{s-5/2} - \frac{\pi^2}{36}\frac{1}{s+1/2} + \frac{1}{4\pi}\frac{1}{s-3/2} - \frac{(2\pi)^{5/2-s}}{8\pi^2}\Gamma(s-5/2)$$
$$+ \frac{1}{8\pi^2}\frac{1}{s-5/2} - \frac{1}{4\pi}\frac{1}{s-3/2} + \frac{1}{4}\frac{1}{s-1/2}$$
$$= \frac{1}{4}\frac{1}{s-1/2} - \frac{\pi^2}{36(s+1/2)} - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2).$$

That is, for $R(s) < 7/6$, $s \neq -1/2, -3/2, -5/2, \ldots$, $\Phi(s)$ has an analytic continuation given by

$$\Phi(s) = \int_1^\infty \left\{ \sum_{n \le x} \sigma(n)(x-n) - \frac{\pi^2}{36}x^3 + \frac{1}{4}x^2 \right\} x^{s-7/2}\,dx$$
$$+\frac{1}{4}\frac{1}{s-1/2} - \frac{\pi^2}{36(s+1/2)} - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2).$$

Now for $R(s) < -1/2$,

$$\Phi(s) = \int_1^\infty \sum_{n \le x} \sigma(n)(x-n)x^{s-7/2}\,dx - \frac{\pi^2}{36}\int_1^\infty x^{s-1/2}\,dx + \frac{1}{4}\int_1^\infty x^{s-3/2}\,dx$$
$$+\frac{1}{4}\frac{1}{s-1/2} - \frac{\pi^2}{36(s+1/2)} - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2)$$
$$= \int_1^\infty \sum_{n \le x} \sigma(n)(x-n)x^{s-7/2}\,dx - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2).$$

The integral in the right-hand side is

$$\int_1^\infty \sum_{n \le t} \sigma(n)\,dt \int_t^\infty x^{s-7/2}\,dx = (5/2-s)^{-1}\int_1^\infty \sum_{n \le t} \sigma(n)t^{s-5/2}\,dt$$
$$= (5/2-s)^{-1}\sum_1^\infty \{\sigma(1) + \cdots + \sigma(n)\}\int_n^{n+1} t^{s-5/2}\,dt$$
$$= [(s-5/2)(s-3/2)]^{-1}\sum_{n=1}^\infty \sigma(n)n^{s-3/2}.$$

It is well known that

$$\sum_{n=1}^\infty \sigma_k(n)n^{-(s+(1/2)k)} = \zeta(s-\tfrac{1}{2}k)\zeta(s+\tfrac{1}{2}k), \qquad R(s) > \max(1+\tfrac{1}{2}k, 1-\tfrac{1}{2}k).$$

And if we put $k = 1$ and replace s by $1-s$, we obtain

$$\sum_{n=1}^\infty \sigma(n)n^{s-3/2} = \zeta(1/2-s)\zeta(3/2-s), \qquad R(s) < -\tfrac{1}{2}.$$

Thus, by analytic continuation, we have for all s

$$\Phi(s) = [(s-5/2)(s-3/2)]^{-1}\zeta(1/2-s)\zeta(3/2-s) - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2). \tag{3.3}$$

Lemma 3.3. *Let $\phi(x)$ be a function defined by* (3.1). *Then*

$$\int_0^x t^{5/2}\phi(t)\,dt = x^{5/2}\int_0^\infty \phi(t)\frac{m(xt)}{t}\,dt,$$

where $m(x) = -x^{1/2}J_6(4\pi x^{1/2})$.

Proof. Since $\phi(x)$ and $m(x)/t$ both belong to $L^2(0,\infty)$, by Parseval's theorem for Mellin transforms

$$(3.4)\qquad \begin{aligned} &x^{5/2}\int_0^\infty \phi(t)\frac{m(xt)}{t}\,dt \\ &\quad = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Phi(1-s)\frac{(s+3/2)(s+1/2)}{(7/2-s)(5/2-s)(3/2-s)}K(s)x^{7/2-s}\,ds. \end{aligned}$$

By (2.1) and (3.3)

$$\begin{aligned} K(s)\Phi(1-s) &= \frac{\zeta(3/2-s)\zeta(1/2-s)}{(s+3/2)(s+1/2)} \\ &\quad - (2\pi)^{-1/2-s}\Gamma(s+1/2)\Gamma(s-1/2)\Gamma(-s-3/2)\cos s\pi \\ &= \frac{\zeta(3/2-s)\zeta(1/2-s)}{(s+3/2)(s+1/2)} - \frac{\pi(2\pi)^{-1/2-s}}{(s+3/2)(s+1/2)}\Gamma(s-1/2). \end{aligned}$$

Now, the right-hand side of (3.4) yields

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left\{\frac{\zeta(3/2-s)\zeta(1/2-s)}{(s-5/2)(s-3/2)} - \frac{1}{2}(2\pi)^{1/2-s}\Gamma(s-5/2)\right\}\frac{x^{7/2-s}}{7/2-s}\,ds.$$

The bracketed expression in the integrand above is the Mellin transform of $\phi(x)$, and the other factor is $l(1-s)$ where $l(s)$ is the Mellin transform of the function $t^{5/2}$ if $0<t<x$; 0 if $t>x$. Therefore, by Parseval's theorem for Mellin transform the above integral is equal to $\int_0^x \phi(t)t^{5/2}\,dt$, which proves the lemma. Thus $\phi(x)$ is self-reciprocal with respect to the generalized Hankel kernel $m(x)$.

4. **The summation formula.** If the assumptions of Lemmas 3.2 and 3.3 are satisfied, then by Parseval's theorem for the pairs $\phi(x)$, $\phi(x)$ and $F(x)$, $G(x)$, we have $\int_0^\infty \phi(x)F(x)\,dx = \int_0^\infty \phi(x)G(x)\,dx$, or,

$$(4.1)\qquad \lim_{N\to\infty}\int_0^N h(x)\left(\frac{d}{dx}\right)^2\{x^{-1/2}f(x)\}\,dx = \lim_{N\to\infty}\int_0^N h(x)\left(\frac{d}{dx}\right)^2\{x^{-1/2}g(x)\}\,dx.$$

Let us consider first the left-hand side of (4.1). Integrating by parts twice, we obtain

$$(4.2)\qquad \lim_{N\to\infty}\left\{h(x)\frac{d}{dx}\{x^{-1/2}f(x)\} - [h'(x)x^{-1/2}f(x)]_0^N + \int_0^N x^{-1/2}f(x)h''(x)\,dx\right\},$$

where h' and h'' denote the first derivative and second derivative of h. By (1.1), the integrated terms are $O(x)$ as $x\to 0$ and consequently vanish. At $x=N$, the first integrated term, in (4.2) by (3.2) and (1.1), gives $\lim_{N\to\infty} O(N^{-2/3+\varepsilon}) = 0$. The second term at $x=N$, however, will be shown limitable to zero by Riesz means $(R,N,1)$ as $N\to\infty$. It is to be noted that [10, p. 413]

$$
\begin{aligned}
h'(x) &= \sum_{n\le x} \sigma(n) - \frac{\pi^2}{12}x^2 + \frac{1}{4\pi}(2\pi x - 1 + e^{-2\pi x}) \\
&= O(x \log x), \qquad x \to \infty, \\
&= O(x^2), \qquad x \to 0.
\end{aligned} \tag{4.3}
$$

Also, $f(x) = O(x^{-1/2})$, $x \to 0$ or ∞.

Thus the second integrated term in (4.2) is $O(\log N) \to \infty$ as $N \to \infty$. The expression (4.2) can now be written as

$$
\begin{aligned}
&\lim_{N\to\infty} -h'(N)N^{-1/2}f(N) \\
&\quad + \lim_{N\to\infty} \int_0^N x^{-1/2}f(x)\left\{D\left[\sum_{n\le x}\sigma(n)\right] - \frac{\pi^2}{6}x + \frac{1}{2}(1 - e^{-2\pi x})\right\}dx, \\
&\qquad\qquad = \lim_{N\to\infty}\{S_1(N) + S_2(N)\},
\end{aligned}
$$

say. We shall show that $\int_0^N S_1(t)\,dt = O(N)$, as $N \to \infty$. Now,

$$
\lim_{N\to\infty} \frac{1}{N}\int_0^N S_1(t)\,dt = -\lim_{N\to\infty}\frac{1}{N}\int_0^N h'(t)t^{-1/2}f(t)\,dt.
$$

In the above integral, split the range of integration $(0,N)$, into $(0,1)$ and $(1,N)$. The integral with the range $(0,1)$, contributes a term which is $\lim_{N\to\infty} O\{(1/N)\int_0^1 t\,dt\} = 0$. By integrating by parts the integral with the range $(1,N)$, we get

$$
\lim_{N\to\infty}\frac{1}{N}\left\{[h(t)t^{-1/2}f(t)]_1^N - \int_1^N h(t)D[t^{-1/2}f(t)]\,dt\right\}. \tag{4.4}
$$

By using the results (1.1) and (3.2), it can be shown that the integrated term vanishes as $N \to \infty$.

Therefore, the expression (4.4) reduces to

$$
\begin{aligned}
-\lim_{N\to\infty}\frac{1}{N}\int_1^N h(t)D[t^{-1/2}f(t)]\,dt &= \lim_{N\to\infty}\frac{1}{N}O\left(\int_1^N h(t)t^{-3/2}\,dt\right) \\
&= \lim_{N\to\infty}\frac{1}{N}O(N^{5/6+\varepsilon}) = \lim_{N\to\infty} O(N^{-1/6+\varepsilon}) = 0,
\end{aligned}
$$

by (1.1) and (3.2) again, and consequently

$$
\lim_{N\to\infty}\frac{1}{N}\int_0^N S_1(t)\,dt = 0,
$$

that is, $S_1(N)$ is limitable $(R,N,1)$ by Riesz means to zero as $N \to \infty$. Next, consider

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N S_2(t)\,dt = \lim_{N\to\infty}\frac{1}{N}\Big\{\int_0^N dt\int_0^t x^{-1/2}f(x)\,d\Big(\sum_{n\le x}\sigma(n)\Big)$$

$$-\int_0^N dt\int_0^t x^{-1/2}f(x)\Big(\frac{\pi^2}{6}x - 1/2 + \frac{1}{2}e^{-2\pi x}\Big)dx\Big\}$$

$$= \lim_{N\to\infty}\Big\{\int_0^N x^{-1/2}f(x)\Big(1-\frac{x}{N}\Big)d\Big(\sum_{n\le x}\sigma(n)\Big)$$

$$-1/2\int_0^N x^{-1/2}f(x)\Big(\frac{\pi^2}{3}x - 1 + e^{-2\pi x}\Big)\Big(1-\frac{x}{N}\Big)dx\Big\}$$

$$= \lim_{N\to\infty}\Big\{\sum_{n=1}^N \sigma(n)n^{-1/2}f(n)\Big(1-\frac{n}{N}\Big)$$

$$-1/2\int_0^N x^{-1/2}f(x)\Big(\frac{\pi^2}{3}x - 1 + e^{-2\pi x}\Big)\Big(1-\frac{x}{N}\Big)dx\Big\},$$

by Stieltjes integration since $\sum_{n\le x}\sigma(n)$ is a steadily increasing step function. Thus the left-hand side of (4.1) is reduced to the above expression. On treating the right-hand side of the equation (4.1) in the same manner, we obtain an expression similar to the last one in $g(x)$. Hence our main theorem:

Theorem. *Let $f(x) \in G_2^2(0,\infty)$, and define a function $g(x)$ by*

$$g(x) = \int_{\to 0}^{\to\infty} f(t)k(xt)\,dt, \qquad x > 0,$$

where $k(x) = -2J_1(4\pi x^{1/2})$. Then $g(x) \in G_2^2(0,\infty)$, and

$$\lim_{N\to\infty}\Big\{\sum_{n=1}^N \sigma(n)n^{-1/2}f(n)\Big(1-\frac{n}{N}\Big)$$

$$-1/2\int_0^N x^{-1/2}f(x)\Big(\frac{\pi^2}{3}x - 1 + e^{-2\pi x}\Big)\Big(1-\frac{x}{N}\Big)\,dx\Big\}$$

$$= \lim_{N\to\infty}\Big\{\sum_{n=1}^N \sigma(n)n^{-1/2}g(n)\Big(1-\frac{n}{N}\Big)$$

$$-1/2\int_0^N x^{-1/2}g(x)\Big(\frac{\pi^2}{3}x - 1 + e^{-2\pi x}\Big)\Big(1-\frac{x}{N}\Big)dx\Big\}.$$

Also, $f(x) = \int_{\to 0}^{\to\infty} g(t)k(xt)\,dt$, $x > 0$.

5. **Examples.** 1. Let $f(x) = (e^{-2\pi ax} - e^{-2\pi bx})x^{-1/2}$, $a, b > 0$. Then

$$g(x) = (e^{-2\pi x/a} - e^{-2\pi x/b})x^{-1/2}.$$

It can be verified that $f(x)$ and $g(x)$ satisfy the conditions of the main theorem. Thus

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{\sigma(n)}{n}(e^{-2\pi an}-e^{-2\pi bn})\left(1-\frac{n}{N}\right)\right.$$
$$\left.-1/2\int_0^N\frac{1}{x}(e^{-2\pi ax}-e^{-2\pi bx})\ \left(\frac{\pi^2}{3}x-1+e^{-2\pi x}\right)\left(1-\frac{x}{N}\right)dx\right\}$$
$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{\sigma(n)}{n}(e^{-2\pi n/a}-e^{-2\pi n/b})\left(1-\frac{n}{N}\right)\right.$$
$$\left.-1/2\int_0^N\frac{1}{x}(e^{-2\pi x/a}-e^{-2\pi x/b})\ \left(\frac{\pi^2}{3}x-1+e^{-2\pi x}\right)\left(1-\frac{x}{N}\right)dx\right\},$$

provided one of the limits exists. It is obvious that the above identity exists and without the convergence factors, therefore, it can be written as

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}(e^{-2\pi an}-e^{-2\pi bn})-\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}(e^{-2\pi n/a}-e^{-2\pi n/b})$$
$$=1/2\int_0^\infty\frac{1}{x}[(e^{-2\pi ax}-e^{-2\pi bx})-(e^{-2\pi x/a}-e^{-2\pi x/b})]\left(\frac{\pi^2}{3}x-1+e^{-2\pi x}\right)dx.$$

The right-hand side yields

$$\frac{\pi^2}{6}\int_0^\infty(e^{-2\pi ax}-e^{-2\pi bx})\,dx-\frac{\pi^2}{6}\int_0^\infty(e^{-2\pi x/a}-e^{-2\pi x/b})\,dx$$
$$-1/2\int_0^\infty(e^{-2\pi ax}-e^{-2\pi bx})\frac{1}{x}dx$$
$$+1/2\int_0^\infty(e^{-2\pi x/a}-e^{-2\pi x/b})\frac{1}{x}dx+1/2\int_0^\infty(e^{-2\pi x(a+1)}-e^{2\pi x(b+1)})\frac{1}{x}dx$$
$$-1/2\int_0^\infty(e^{-2\pi x(1+1/a)}-e^{-2\pi x(1+1/b)})\frac{1}{x}dx=\frac{\pi}{12}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{\pi}{12}(a-b)$$
$$-\frac{1}{2}\log\frac{b}{a}+\frac{1}{2}\log\frac{a}{b}+\frac{1}{2}\log\frac{b+1}{a+1}-\frac{1}{2}\log\frac{a(b+1)}{b(a+1)}$$

$$=\frac{\pi}{12}\left\{\left(\frac{1}{a}-a\right)-\left(\frac{1}{b}-b\right)\right\}-\frac{1}{2}\log\frac{b}{a}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}(e^{-2\pi an}-e^{-2\pi bn})-\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}(e^{-2\pi n/a}-e^{-2\pi n/b})$$
$$=\frac{\pi}{12}\left\{\left(\frac{1}{a}-a\right)-\left(\frac{1}{b}-b\right)\right\}-\frac{1}{2}\log\frac{b}{a}. \tag{5.1}$$

A special case of this formula is a known result [6], and is derived below. Let $b = 1$; then

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{-2\pi an} - \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{-2\pi n/a} = \frac{\pi}{12}\left(\frac{1}{a} - a\right) + \frac{1}{2}\log a. \tag{5.2}$$

Now,

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{-2\pi an} = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{m}{lm} e^{-2\pi alm} = -\sum_{m=1}^{\infty} \log(1 - e^{-2\pi am})$$
$$= -\log \prod_{m=1}^{\infty} (1 - e^{-2\pi am}).$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{-2\pi n/a} = -\log \prod_{m=1}^{\infty} (1 - e^{-2\pi m/a}).$$

Therefore, (5.2) becomes

$$\log \prod_{m=1}^{\infty} (1 - e^{-2\pi m/a}) = \frac{\pi}{12}\left(\frac{1}{a} - a\right) + \frac{1}{2}\log a + \log \prod_{m=1}^{\infty} (1 - e^{-2\pi ma}),$$

or,

$$\prod_{m=1}^{\infty} (1 - e^{-2\pi m/a}) = a^{1/2} \exp \frac{\pi}{12}\left(\frac{1}{a} - a\right) \prod_{m=1}^{\infty} (1 - e^{-2\pi ma}).$$

This is a well-known result in the theory of elliptic modular functions [6]. Also the above formula can be considered as a transformation formula for $\eta(ia)$ where $\eta(z)$ is Dedekind's eta-function.

2. Let $f(x) = x^{1/2}\exp(-2\pi ax)$, $a > 0$. Then

$$g(x) = -2\pi \int_0^{\infty} t^{1/2} \exp(-2\pi at) J_1(4\pi x^{1/2} t^{1/2})\, dt$$
$$= -a^{-2} x^{1/2} \exp(-2\pi x/a).$$

On substituting the values for $f(x)$ and $g(x)$ in the main theorem, and evaluating the integrals, we get the identity

$$\sum_{n=1}^{\infty} \sigma(n)\exp(-2\pi an) + a^{-2} \sum_{n=1}^{\infty} \sigma(n)\exp(-2\pi n/a) = \frac{1}{24}\left(1 + \frac{1}{a^2}\right) - \frac{1}{4\pi a}.$$

Now let $a = 1$ and obtain

$$2 \sum_{n=1}^{\infty} \sigma(n)\exp(-2\pi n) = \frac{1}{12} - \frac{1}{4\pi}.$$

If we write

$$\sum_{n=1}^{\infty} \sigma(n)\exp(-2\pi n) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} me^{-2\pi lm}$$
$$= \sum_{m=1}^{\infty} m \sum_{l=1}^{\infty} e^{-2\pi lm} = \sum_{m=1}^{\infty} \frac{m}{e^{2\pi m}-1},$$

the above identity becomes

$$2\sum_{m=1}^{\infty} \frac{m}{e^{2\pi m}-1} = \frac{1}{12} - \frac{1}{4\pi}.$$

We note that many identities of the type given in [1] can be established as special cases of the main theorem but in most cases the integrals involved cannot be evaluated explicitly.

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