HEREDITARY QI-RINGS

BY

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ABSTRACT. We consider in this paper rings in which every quasi-injective right $R$-module is injective. These rings will be referred to as right QI-rings. For a hereditary ring, this is equivalent to the condition that $R$ be noetherian and a right $V$-ring. We also consider rings in which proper cyclic right $R$-modules are injective. These are right QI-rings which are either semisimple or right hereditary, right Ore domains in which indecomposable injective right $R$-modules are either simple or isomorphic to the injective hull of $R_R$.

Introduction. It has been shown that the following three conditions on a ring $R$ are equivalent: (1) each simple right $R$-module is injective; (2) each right ideal is the intersection of maximal right ideals; (3) $\text{Rad } M = 0$ for all $M \in \text{Mod-} R$. Rings satisfying any one of the above conditions are called right $V$-rings after Villamayor, who proved the equivalence of (1) and (2). In finding a counterexample to the conjecture that every $V$-ring is regular, Cozzens [3] produced an example of a noetherian $V$-ring in which every cyclic right $R$-module is either semisimple or free. This condition on the cyclics forces every quasi-injective right $R$-module to be injective. We shall call a ring in which every quasi-injective right $R$-module is injective a right QI-ring. In this paper we will consider the problem of characterizing hereditary QI-rings.

The class of finitely generated torsion right $R$-modules plays an important role in this consideration. We find that over a hereditary, noetherian, right $V$-ring, the class of finitely generated torsion right $R$-modules, the class of finitely generated semisimple right $R$-modules, and the class of finitely generated injective right $R$-modules are all synonymous. We use this equivalence to show that over a hereditary, noetherian ring $R$, $R$ is a right $V$-ring if and only if $R$ is a right QI-ring.

Following the example of Cozzens more closely, we also consider rings in which proper cyclic right $R$-modules are injective. We call these right PCI-rings. Every right PCI-ring is a right QI-ring. Further we show that a right PCI-ring is either semisimple or a right hereditary, right Ore domain in which every indecomposable injective right $R$-module is either simple or isomorphic to the injective hull of $R_R$.

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1. Preliminaries. Throughout this paper a ring $R$ is associated with an identity element. Each $R$-module is unitary. Unless right or left is specified, a condition will be assumed to hold on both sides of the ring. If $M \in \text{mod-}R$, then $E(M)$ denotes the injective hull of $M_R$.

An $R$-module $M_R$ is quasi-injective if each homomorphism of any submodule $N$ into $M$ can be extended to a homomorphism of $M$ into $M$. Semisimple right modules are clearly examples of quasi-injective modules. Johnson and Wong [8] have related injectives and quasi-injectives by proving that a module is quasi-injective if and only if it is closed under endomorphisms of its injective hull.

A submodule $J_R$ is uniform if for any two submodules $A_R$ and $B_R$, $A \cap B \neq 0$. An element $r$ of the ring $R$ is called regular if $sr \neq 0$ and $rs \neq 0$ for any nonzero element $s$ of $R$. For any right $R$-module $M$, an element $m \in M$ is a torsion element if $md = 0$ for some regular element $d \in R$. If every nonzero element of $M$ is a torsion element, then $M$ is said to be a torsion module; if no element of $M$ is torsion, $M$ is said to be torsion-free. Given a right $R$-module $M$, let $tM$ denote the collection of torsion elements of $M$. In the case where $R$ is a commutative ring, $tM$ always forms a submodule of $M$. In the noncommutative case however, we can conclude that $tM$ forms a submodule of $M$ for any $R$-module $M$ if and only if $R$ has a classical right quotient ring (see Levy [10] and Gentile [6]).

A ring $R$ is called a right QI-ring if every quasi-injective right $R$-module is injective.

A right $R$-module $C$ is a proper cyclic in case $R \rightarrow C \rightarrow 0$ is exact but $0 \rightarrow R \rightarrow C \rightarrow 0$ is not. A ring $R$ is called a right PCI-ring if every proper cyclic right $R$-module is injective.

Before proceeding to the main body of the work, we shall consider several lemmas which will prove useful later.

Lemma 1 (Kurshan [9]). If $R$ is a ring in which every semisimple right $R$-module is injective, then $R$ is right noetherian.

Lemma 2. Let $M$ be a quasi-injective right $R$-module. If $M$ contains a copy of $R$, then $M$ is injective.

Proof. This is an obvious consequence of Baer’s criterion for injectivity [1].

Lemma 3. Let $R$ be a noetherian, hereditary, semiprime ring and let $\mathcal{F}_R$ denote the Serre class of finitely generated torsion right $R$-modules. Then $\text{Ext}_k(-, R) : \mathcal{F}_R \rightarrow R\mathcal{F}$ defines a duality.

Proof. Let $Q$ be the classical ring of right quotients of $R$. Then $\text{Hom}_R(-, Q/R)$ and $\text{Ext}_k(-, R)$ are naturally isomorphic on $\mathcal{F}_R$. Using this it follows readily that the functor takes $\mathcal{F}_R$ into $R\mathcal{F}$. 

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To complete the proof all we need show is that $\text{Hom}_R(\_, Q/R)$ is inverse to $\text{Ext}_R^1(\_, R)$ in $\mathcal{F}_R$. Let $A \in \mathcal{F}_R$. Then by Cartan and Eilenberg [2],

$$\text{Hom}_R(\text{Ext}_k^1(A, R), Q/R) \approx \text{Tor}_R^1(A, \text{Hom}_R(R, Q/R)) \approx \text{Tor}_R^1(A, Q/R)$$
and by Sandomierski [12], $\text{Tor}_R^1(A, Q/R) \approx A$, giving us the desired result.

2. Hereditary right $QI$-rings. In this section we will be concerned with characterizing hereditary right $QI$-rings.

**Theorem 4.** Let $R$ be a noetherian, hereditary right $V$-ring. Then finitely generated torsion right $R$-modules are semisimple and injective.

**Proof.** Since $\text{Rad } R = 0$, $R$ is semiprime and by Lemma 3 there exists a duality $\text{Ext}_k^1(\_, R) : \mathcal{F}_R \rightarrow \mathcal{F}_R$. Since $\mathcal{F}_R$ is a noetherian category, $\mathcal{F}_R$ is an artinian category, and hence a noetherian category, by this duality. Using the duality once more, since $\mathcal{F}_R$ is a noetherian category, $\mathcal{F}_R$ is an artinian category. Thus every finitely generated torsion right $R$-module contains a simple module which splits off since $R$ is a right $V$-ring. Hence every finitely generated torsion module is semisimple.

**Theorem 5.** Let $R$ be a hereditary, left noetherian ring. Then $R$ is a right $QI$-ring if and only if $R$ is a right noetherian, right $V$-ring.

**Proof.** Suppose $R$ is a right noetherian, right $V$-ring. By Faith [4], $R$ is a product of simple noetherian, right $V$-rings. Hence it suffices to consider a simple, noetherian hereditary right $V$-ring. Since $R$ is noetherian, it contains a uniform finitely generated, right ideal $U$. Since $R$ is hereditary and simple, $U$ is a projective generator. By Mitchell [11] there exists a category equivalence $\text{Mod-}R \rightarrow \text{Mod-End } U_R$ where $\text{End } U_R$ is an Ore domain by Goldie [7]. Since quasi-injectives and injectives are Morita invariants, it suffices to work in a hereditary, noetherian, right $V$-ring, right Ore domain.

Let $M$ be a quasi-injective right $R$-module. Since $R$ is noetherian, $E(M)$ is a direct sum of indecomposable, injective right submodules. By intersecting $M$ with each of these indecomposable injectives, we get a direct sum of indecomposable quasi-injectives, the direct sum of which is isomorphic to $M$. Hence it suffices to show that indecomposable quasi-injective right $R$-modules are injective.

Let $M$ be an indecomposable quasi-injective right $R$-module. Let $m \in M$. If $mR \approx R$ then $M$ contains a copy of the ring and, by Lemma 2, $M$ is injective. If $\text{ann}_R(m) \neq 0$, then $mR \approx R/I$ for some right ideal $I$. Since $R$ is a domain, $I$ contains a regular element and hence $R/I$ is a finitely generated torsion module. By Theorem 4, $mR$ is injective and thus $mR = E(M)$ forcing $M$ to be injective.

Conversely, suppose $R$ is a right $QI$-ring. Since simple right $R$-modules are quasi-injective, $R$ is a right $V$-ring. Since semisimples are also quasi-injective, $R$ is right noetherian by Lemma 1.

We can take our knowledge of the structure of these hereditary, noetherian, right $V$-rings a step further by characterizing the class of finitely generated injectives.
Corollary 6. Let \( R \) be a noetherian, hereditary, right \( V \)-ring. Then a finitely generated right \( R \)-module is injective if and only if it is semisimple.

Proof. Since \( R \) is right noetherian and a right \( V \)-ring, every semisimple right \( R \)-module is injective.

Conversely, assume that \( M \) is a finitely generated injective right \( R \)-module. By Theorem 4, \( tM \), the torsion submodule of \( M \), is semisimple and injective. Thus \( M \cong tM \oplus K \) where \( K \) is torsion-free. Suppose \( K \neq 0 \). As in the proof of Theorem 5, we have a category equivalence \( T: \text{Mod-}R \to \text{Mod-}S \) where \( S \) is a hereditary, noetherian, right \( V \)-ring, right Ore domain. By Theorem 4, all finitely generated torsion modules are semisimple both in \( \text{Mod-}R \) and \( \text{Mod-}S \). If \( T(K) \) is not a torsion module, then \( T(K) \) must contain a copy of \( S \) since this is the only cyclic in \( \text{Mod-}S \) which is not a torsion module. Since \( T(K) \) is injective, it contains \( E(S_2) \) which implies that \( E(S_2) \) is finitely generated. Then by Faith and Walker [5], \( S \) is right artinian. However, every right \( V \)-ring is semiprime. So \( S \) must be semisimple by the Wedderburn-Artin theorem. \( R \) must also be semisimple in which case every right \( R \)-module is semisimple. Hence we may suppose that \( K = 0 \). Then \( M = tM \) is semisimple.

3. Right PCI-rings. As was mentioned in the introduction, one of the motivating factors for this topic was the example of Cozzens which provided a nontrivial right QI-ring. Cozzens' example has the property that proper cyclic right \( R \)-modules are injective. We shall call rings with this property right PCI-rings.

Theorem 7. If \( R \) is a right noetherian, right PCI-ring, then \( R \) is a right hereditary, right QI-ring.

Proof. Let \( M \) be any right \( R \)-module with injective hull \( E(M) \). Consider \( E(M)/M \). If \( \overline{m} \in E(M)/M \), then \( \text{ann}_R(\overline{m}) \neq 0 \) and is in fact a large right ideal. Thus \( \overline{m}R \) is a proper cyclic and is injective by hypothesis. So \( \overline{m}R \) splits off as a summand of \( E(M)/M \). Take a maximal collection \( \{\overline{m}_iR\}_{i \in I} \) of linearly independent cyclic summands of \( E(M)/M \). Suppose \( E(M) \neq \sum_{i \in I} \overline{m}_iR \). Since \( R \) is right noetherian, every direct sum of injectives is injective. Thus \( \sum_{i \in I} \overline{m}_iR \) is injective and \( E(M)/M \cong \sum_{i \in I} \overline{m}_iR \oplus A \) for some submodule \( A \) of \( E(M)/M \). Every cyclic submodule \( C \) of \( A \) is proper and thus injective. So \( C \) splits off contradicting the maximality of \( \{\overline{m}_iR\}_{i \in I} \). Hence \( E(M)/M = \sum_{i \in I} \overline{m}_iR \cong \sum_{i \in I} \oplus \overline{m}_iR \) is a direct sum of proper cyclics and is therefore injective. Thus \( 0 \to M \to E(M) \to E(M)/M \to 0 \) is an injective resolution for \( M \). Thus inj dim \( R \leq 1 \) implying that \( R \) is right hereditary.

Since \( R \) is right noetherian, for any quasi-injective right \( R \)-module \( M \) we can get a decomposition of \( M \) into a direct sum of indecomposable quasi-injective right \( R \)-modules. Thus it suffices to show that indecomposable quasi-injectives are injective. Let \( M \) be an indecomposable quasi-injective right \( R \)-module and let \( m \in M \). If \( mR \approx R \), then \( M \) is injective by Lemma 2. Otherwise \( mR \) is a proper
cyclic and is injective. Since \( M \) is indecomposable, \( mR = M \) and \( M \) is injective. Thus \( R \) is a right QI-ring.

It can be observed from the above proof that every indecomposable injective right \( R \)-module is either cyclic or isomorphic to \( E(R) \). Using the following lemma due to Faith, we can say even more about the structure of these rings.

**Lemma 8.** Let \( R \) be a noetherian right PCI-ring. Then the endomorphism ring of an indecomposable injective right \( R \)-module is a field.

**Proof.** Let \( M \) be an indecomposable injective right \( R \)-module. If \( M \) has a cyclic submodule \( C \) such that \( C \cong R \), then \( M \cong E(R) \). Thus \( E(R) \) is indecomposable. \( R \) has a semisimple classical quotient ring \( Q \) and since the singular right ideal is zero, \( Q \cong E(R) \). Thus \( E(R) \) is a right and left semisimple artinian ring. Since it is indecomposable over \( R \), \( E(R) \) is a field. End \( M \cong \text{End } E(R) \cong E(R) \). Thus End \( M \) is a field.

Otherwise any nonzero submodule \( N \) of \( M \) contains a proper cyclic \( C \). Since \( C \) is injective, \( C = N = M \). Thus \( M \) has no proper submodules and is therefore simple. So End \( M \) is a field.

**Corollary 9.** If \( R \) is a right noetherian, right PCI-ring, then either \( R \) is a semisimple ring or else \( R \) is a right hereditary, right Ore domain in which every indecomposable injective right \( R \)-module is either simple or isomorphic to the right quotient field \( E(R) \).

**Proof.** If \( R \) has no indecomposable injective right \( R \)-module isomorphic to \( E(R) \), then as in the proof of Lemma 8, every indecomposable injective is simple, and hence given a right \( R \)-module \( M \) it is contained in a semisimple right \( R \)-module \( E(M) \) and thus is semisimple.

If \( R \) is not semisimple, then there exists an indecomposable injective right \( R \)-module \( M \) such that \( M \cong E(R) \). This implies that \( E(R) \) is indecomposable. Since \( E(R) \) is indecomposable as a ring, it is a field. Thus \( R \) is a right Ore domain and every indecomposable injective right \( R \)-module is either simple or isomorphic to \( E(R) \).

**Corollary 10.** Suppose \( R \) is not semisimple. Then the following conditions on a left noetherian, left hereditary ring \( R \) are equivalent:

1. \( R \) is a right noetherian, right PCI-ring.
2. \( R \) is a right hereditary, right QI-ring, right Ore domain.
3. \( R \) is a right hereditary, right hereditary \( V \)-ring which is a right Ore domain.

**Proof.** (1) implies (2). This follows from Theorem 7 and Corollary 10.
(2) implies (3). This follows from Theorem 5.
(3) implies (1). We want to show that proper cyclics are injective. By Theorem 4, finitely generated torsion right \( R \)-modules are injective. Let \( C \) be a proper cyclic, \( C \cong R/I \) for some right ideal \( I \). Since \( R \) is an Ore domain, \( r \in I \) implies that \( r \) is regular and \( cr = 0, c \in C \). Thus \( C \) is a finitely generated torsion module and is injective.
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