COUNTABLE BOX PRODUCTS OF ORDINALS

BY

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ABSTRACT. The countable box product of ordinals is examined in the paper for normality and paracompactness. The continuum hypothesis is used to prove that the box product of countably many $\alpha$-compact ordinals is paracompact and that the box product of another class of ordinals is normal. A third class trivially has a nonnormal product.

Because I have found a countable box product of ordinals useful in the past [1], this class of spaces particularly interests me. The purpose of this paper is to tell what I know about which of these spaces is paracompact or normal.

In [2] I prove that the continuum hypothesis implies the box product of countably many $\sigma$-compact, locally compact, metric spaces is paracompact. I prove here that the continuum hypothesis implies the box product of countably many $\sigma$-compact ordinals is paracompact (Theorem 1) and the box product of another class of ordinals is normal (Theorem 2). The proof of Theorems 1 and 2 is a quite messy join of the techniques of [1] and [2] which raises some doubt in my mind as to whether these theorems are worth proving. Because I care, because I think these spaces are set theoretically interesting and topologically useful, because I think these theorems are best possible, the theorems are worth the mess to me.

A. If $(X_\lambda)_{\lambda \in \Lambda}$ is a family of topological spaces, a box in $\prod_{\lambda \in \Lambda} X_\lambda$ is a set $\prod_{\lambda \in \Lambda} U_\lambda$ where each $U_\lambda$ is open in $X_\lambda$. The box product of $(X_\lambda)_{\lambda \in \Lambda}$ is $\prod_{\lambda \in \Lambda} X_\lambda$ topologized by using the set of all boxes in it as a basis.

Throughout the paper the following notation is used.

An ordinal $\alpha$ is the set of all ordinals less than $\alpha$ and $\alpha$ is topologized by the interval topology. The statement that $\alpha$ is a cardinal means that $\alpha$ is an ordinal and no smaller ordinal has the same cardinality as $\alpha$.

The notation $\prod_{\lambda \in \Lambda} \beta_\lambda$ is used to mean the ordinary Cartesian product of the $\beta_\lambda$'s and never the cardinal or ordinal arithmetic product. Similarly $\alpha^\delta$ means the set of all functions from $\beta$ into $\alpha$ rather than an arithmetic operation.

If $\alpha$ is an ordinal, let $\text{cf}(\alpha)$ denote the cofinality of $\alpha$; that is $\text{cf}(\alpha)$ is the smallest ordinal $\delta$ such that there is a subset $\Delta$ of $\alpha$, order isomorphic with $\delta$, such that $\beta < \alpha$ implies there is a $\gamma \in \Delta$ with $\beta \leq \gamma$. Observe that $\alpha$ is a $\sigma$-compact ordinal if and only if $\alpha$ is compact or $\text{cf}(\alpha) = \omega_0$.

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Let \([\text{CH}]\) and \([\text{GCH}]\) denote the continuum hypothesis is true and the generalized continuum hypothesis is true, respectively.

To avoid repetition assume that for each \(n \in \omega_0\), \(\alpha_n\) is a positive ordinal, and let \(X\) be the box product of \(\{\alpha_n\}_n \in \omega_0\). If \(x \in X\) or \(U \subseteq X\), let \(x(n)\) and \(U(n)\) denote the projection of \(x\) and \(U\), respectively, on \(\alpha_n\).

B. Let \(S = \{n \in \omega_0 \mid \alpha_n\) is not \(\sigma\)-compact\}.

Case 0. The trivial case. There are \(n \in S\) and \(m \in \omega_0 - \{n\}\) with \(\alpha_m > \text{cf}(\alpha_n)\).

Theorem 0 yields \(X\) is not normal.

Case 1. The basic case. \(S = \emptyset\). Theorem 1 yields \([\text{CH}]\) \(X\) is paracompact.

Case 2. The other case. Not Cases 0 or 1.

Since not Case 1, \(S \neq \emptyset\). And \(s \in S\) imply \(\alpha_s\) is not paracompact; so \(X\) is not paracompact. Define \(\kappa(X) = \sup\{\text{cardinality of } \prod_{\alpha \in \omega_0 - \{s\}} \beta_n \mid s \in S, \beta_n \leq \alpha_n\) and \(\beta_s\) is compact\}. Since \(S \neq \emptyset\) and not Case 0, there is a unique uncountable \(\delta\) such that \(\text{cf}(\alpha_s) = \delta\) for all \(s \in S\). Since not Case 0 and \(\delta\) is uncountable, \(\kappa(X) \leq \sup\{\text{cardinality of } \beta_m \mid 0 < \beta\) is a cardinal less than \(\delta\}\}. Thus \([\text{GCH}]\) implies \(\kappa(X) < \delta\) unless \(\delta\) is the cardinal successor of an infinite \(\sigma\)-compact cardinal.

Theorem 2 proves \(\kappa(X) < \delta\) and \([\text{CH}]\) implies \(X\) is normal.

Thus in Case 2, if \(\delta\) is not the cardinal successor of an infinite \(\sigma\)-compact cardinal, then \([\text{GCH}]\) \(X\) is normal but not paracompact.

I give some examples to show the flavor of the results.

(1) \(\alpha_0 = \omega_2\) and \(\alpha_n = \omega_0 + 1\) for \(n > 0\); \([\text{CH}]\) \(X\) is normal; Case 2.

(2) \(\alpha_n = \omega_k\) for a finite \(k > 1\) and all \(n\); \([\text{CH}]\) \(X\) is normal; Case 2.

(3) \(\alpha_n = \omega_{\omega_0 + 2}\) for \(n\) even and \(\alpha_n = \omega_0 + 1\) for \(n\) odd; \([\text{GCH}]\) \(X\) is normal; Case 2.

(4) \(\alpha_n = \omega_n\) for all \(n\); \(X\) is not normal; Case 0.

(5) \(\alpha_n = \omega_{\omega_0}\) for all \(n\); \([\text{CH}]\) \(X\) is paracompact; Case 1.

(6) \(\alpha_n = \omega_0 + 1\) for all \(n\); \([\text{CH}]\) \(X\) is paracompact; Case 1.

But I conjecture there is a model of set theory in which \(X\) is not normal.

(7) \(\alpha_0 = \omega_0\) and \(\alpha_n = \omega_0 + 1\) for all \(n > 0\); Case 2 but none of the present results apply. I conjecture \(X\) is not normal in a model of set theory including \([\text{CH}]\).

I also conjecture \(X\) is normal in a different model of set theory.

C. Let \(T = \{t \in \omega_0 \mid \alpha_t\) is \(\sigma\)-compact but not compact\}.

Lemma 0. Without loss of generality, \(T = \emptyset\).

Proof. If \(t \in T\), choose nonlimit ordinals \(\alpha_{q_t} < \alpha_{t_t} < \cdots\) having \(\alpha_t\) as a limit. Define \(\mathcal{A} = \{\prod_{\alpha \in \omega_0} A_{\alpha} \mid A_{\alpha} = \alpha_{\alpha}\) if \(n \in \omega_0 - T\), and either \(A_{\alpha} = \alpha_{q_t}\) or \(A_{\alpha} = \alpha_{\alpha} - \alpha_{(q+1)}\) for some \(i > 0\) if \(n \in T\}\). Clearly \(\mathcal{A}\) is a collection of disjoint open sets covering \(X\); thus \(X\) is paracompact (normal) if and only if all members of \(\mathcal{A}\) are paracompact (normal). But \(\prod_{\alpha \in \omega_0} A_{\alpha} \in \mathcal{A}\) implies \(A_{\alpha}\) is either isomorphic to a compact ordinal or is a non-\(\sigma\)-compact ordinal. Hence we may assume \(T = \emptyset\).
D.

**Theorem 0.** Suppose \( \gamma \) and \( \beta \) are ordinals; \( \gamma \) is not \( \sigma \)-compact, and \( \beta > \text{cf}(\gamma) \). Then \( \gamma \times \beta \) is not normal.

**Proof.** Let \( \delta = \text{cf}(\gamma) \). Since \( \delta \times (\delta + 1) = D \) is homeomorphic to a closed subset of \( \gamma \times \beta \), it suffices to show that \( D \) is not normal. For \( \alpha < \delta \), let \( h_\alpha = (\alpha, \delta) \) and \( k_\alpha = (\alpha, \alpha) \). Then \( H = \{ h_\alpha \mid \alpha < \delta \} \) and \( K = \{ k_\alpha \mid \alpha < \delta \} \) are disjoint closed subsets of \( D \). Suppose \( U \) and \( V \) are disjoint sets open in \( D \) and \( U \supset H \) and \( V \supset K \). Let \( \Lambda = \{ \lambda < \delta \mid \lambda \text{ is a limit ordinal} \} \). Since \( k_\lambda \in V \) for each \( \lambda \in \Lambda \), there is a \( \beta_\lambda < \lambda \) with \( \{(\eta, \lambda) \mid \beta_\lambda \leq \eta \leq \lambda \} \subset V \). Since \( \text{cf}(\delta) = \delta \), there is a \( \beta < \delta \) such that \( \rho < \delta \) implies \( \{ \lambda \in \Lambda \mid \beta_\lambda = \beta \text{ and } \lambda > \rho \} \neq \emptyset \). Since \( h_\beta \in U \), there is a \( \rho < \delta \) such that \( \{(\beta, \lambda) \mid \rho \leq \lambda \leq \delta \} \subset U \). Choose \( \lambda \in \Lambda \) such that \( \beta_\lambda = \beta \) and \( \lambda \geq \rho \). Then \( (\beta, \lambda) \in U \cap V \). Thus \( X \) is not normal.

**Conjecture based on the proof of Theorem 0.** Let \( \alpha_0 = \omega_1 \) and \( \alpha_n = \omega_0 + 1 \) for \( n > 0 \). For \( \alpha < \omega_1 \) define \( h_\alpha \in X \) by \( h_\alpha(0) = \alpha \) and \( h_\alpha(n) = \omega_0 \) for \( n > 0 \). For \( \alpha < \omega_1 \) choose \( k_\alpha \in X \) so \( k_\alpha(0) = \alpha \) and, for \( n > 0 \), choose \( k_\alpha(n) < \omega_0 \) in such a way that \( \beta < \alpha \) implies there is an \( m \in \omega_0 \) so \( n > m \) gives \( k_\alpha(n) > k_\beta(n) \). Then \( H = \{ h_\alpha \mid \alpha < \omega_1 \} \) and \( K = \{ k_\alpha \mid \alpha < \omega_1 \} \) are again closed and disjoint subsets of \( X \). I conjecture that in some model of set theory including \([CH]\), \( K \) can be chosen in such a way that \( H \) and \( K \) cannot be separated. However, in a model with no scale of cardinality \( \aleph_1 \) (that is, for any \( K \) there is an \( x \in \omega_0 \times \omega_0 \times \cdots \) such that for all \( \alpha \) there is an \( m \) for which \( n > m \) implies \( x(n) > k_\alpha(n) \) I feel \( X \) must be normal. \([CH]\) implies there is a scale of cardinality \( \aleph_1 \).

E. The theorems are proved in this section and we need more notation. Assume \( \alpha_n \) is compact for all \( n \).

Let \( \mathcal{E} = \{ \prod_{n \in \omega_0} J_n \mid J_n \text{ is a closed subinterval of } \alpha_n \} \); we allow \( J_n = \emptyset \). See that \( X \in \mathcal{E} \) and the intersection of the members of any subset of \( \mathcal{E} \) is a member of \( \mathcal{E} \). If \( L \in \mathcal{E} \), define \( \prod_{n \in \omega_0} \sup\{ p(n) \mid p \in L \} \) to be the top of \( L \); observe that \( \emptyset \neq L \in \mathcal{E} \) implies the top of \( L \) belongs to \( L \).

Let \( \mathfrak{B} = \{ U \in \mathcal{E} \mid U(n) \text{ is both open and closed for all } n \} \); \( \mathfrak{B} \) is a basis for the topology of \( X \).

For \( p \in X \) and \( n \in \omega_0 \) let \( E^n(p) = \{ x \in X \mid x(m) = p(m) \text{ for all } m \geq n \} \). Let \( E(p) = \bigcup_{n \in \omega_0} E^n(p) \) and \( \mathcal{D} = \{ \bigcup_{p \in U} E(p) \mid U \in \mathfrak{B} \} \). For \( V \in \mathcal{D} \) let \( V^* \) denote a particular \( U \in \mathfrak{B} \) with \( V = \bigcup_{p \in U} E(p) \). Observe that the relation \( E \), where \( p E q \) means \( p(m) = q(m) \) for all but finitely many \( m \in \omega_0 \), partitions \( X \) into equivalence classes and that \( E(p) \) is the equivalence class to which \( p \) belongs. Also \( V \in \mathcal{D} \) implies

\[
V = \bigcup_{n \in \omega_0} \alpha_0 \times \alpha_1 \times \cdots \times \alpha_{n-1} \times V^*(n) \times V^*(n+1) \times \cdots .
\]

From this we see that \( V \in \mathcal{D} \) implies \( V \) is the union of disjoint members of \( \mathfrak{B} \).
because \( V = \bigcup_{n \in \omega_0} \{ \prod_{j \in \omega_0} J_j \mid J_j(i) = V^*(i) \text{ for } i \geq n \text{ and } J_j(i) \text{ is either } V^*(i) \text{ or a maximal subinterval of } \alpha_n - V^*(i) \} \).

Let \( \Omega \) be the set of all subsets \( \mathcal{A} \) of \( \mathcal{B} \) covering \( X \) such that \( V \in \mathcal{A} \) and \( U \in \mathcal{A} \) and \( V \supset U \) imply \( U \in \mathcal{A} \). For \( \mathcal{A} \in \Omega \) define \( \delta(\mathcal{A}) = \{ V \in \mathcal{A} \mid V \text{ is the union of a set of disjoint members of } \mathcal{A} \} \).

We now prove a sequence of lemmas.

**Lemma 1.** The intersection of the members of a countable subset of \( \mathcal{A} \) is the union of a set of disjoint members of \( \mathcal{A} \).

**Proof.** Assume \( \{ \mathcal{A}_n \}_{n \in \omega_0} \subset \mathcal{A} \).

For \( m \) and \( n \) in \( \omega_0 \) let \( \delta_{mn} = \{ I \mid I \text{ is either } V^*_m(n) \text{ or a maximal interval in } \alpha_n - V^*_m(n) \} \). Observe that \( \delta_{mn} \) partitions \( \alpha_n \) into three or fewer disjoint open and closed subintervals. Define \( \delta_n = \{ \bigcap_{m \leq n} I_m \mid I_m \in \delta_{mn} \} \). Let \( \xi = \{ \prod_{n \in \omega_0} I_n \mid I_n \in \delta_n \} \). Finally let \( \mathcal{K} = \{ \bigcup_{J \in E} E(p) \mid J \in \xi \} \) and \( \mathcal{K} = \{ V \in \mathcal{K} \mid V \subset \bigcap_{n \in \omega_0} V_n \} \). Since the terms of \( \delta_n \) are disjoint open and closed intervals of \( \alpha_n \) whose union is \( \alpha_n \), the terms of \( \xi \) are disjoint members of \( \mathcal{A} \) whose union is \( X \).

Suppose \( x \in \bigcap_{n \in \omega_0} V_n \); choose \( k_0 < k_1 < \cdots \) in \( \omega_0 \) so \( x(n) \in V^*_m(n) \) for all \( n \geq k_m \). Define \( J_n \) by \( x(n) \in J_n \in \delta_n \). Since \( k_m \geq m \), clearly \( J_n \subset V^*_m(n) \) for \( n \geq k_m \). Let \( J = \prod_{n \in \omega_0} J_n \) and \( V = \bigcup_{J \in E} E(p) \). Then \( x \in \mathcal{K} \). Assume \( y \in V \) and \( m \in \omega_0 \). There is a \( k \in \omega_0 \) so \( y(n) \in J_n \) for all \( n > k \); so \( y(n) \in V^*_m(n) \) for all \( n > k_m + k \). Choose a point \( p \) of \( V^*_m \) such that \( p(n) = y(n) \) for \( n > k_m + k \); then \( y \in E(p) \) so \( y \in V_m \). Thus \( V \in \mathcal{K} \). Hence \( \mathcal{K} \) is a set of disjoint members of \( \mathcal{A} \) whose union is \( \bigcap_{n \in \omega_0} V_n \).

**Lemma 2.** If \( x \in X \) and \( \mathcal{A} \in \Omega \), then \( x \in \bigcup \delta(\mathcal{A}) \).

**Proof.** By induction we define for each \( n \in \omega_0 \) a finite subset \( \mathcal{A}_n \) of \( \mathcal{A} \) such that \( V \in \mathcal{A}_n \) implies \( V \cap E^*(x) \neq \emptyset \); \( \mathcal{A}_n \) is a cover of \( E^*(x) \) by disjoint sets. Also there is an open-closed interval \( I_n \) of \( \alpha_n \) to which \( x(n) \) belongs which is \( V(n) \) for all \( V \subset \mathcal{A}_n \).

Choose \( W \in \mathcal{A} \) such that \( x \in W \) and let \( \mathcal{A}_0 = \{ W \} \) and \( I_0 = W(0) \).

Assume \( \mathcal{A}_{n-1} \) has been-chosen. Since \( \mathcal{A} \) is a basis for the topology of \( X \) and \( \mathcal{A}_{n-1} \) is finite and its members are closed, for each \( p \in \mathcal{A}_{n-1} \) there is a \( \mathcal{V}_p \in \mathcal{A} \) such that \( p \in \mathcal{V}_p \) and \( \mathcal{V}_p \cap (\bigcup \mathcal{A}_{n-1}) = \emptyset \). Since \( E^*(x) = \alpha_0 \times \alpha_1 \times \cdots \times \alpha_{n-1} \times \{ x(n) \} \times \{ x(n + 1) \} \times \cdots \), and each \( \alpha_i \) is compact, \( E^*(x) \) is compact. Hence \( E^*(x) \) contains a finite subset \( p_0, p_1, \ldots, p_k \) such that \( \{ V \mid i \leq k \} \) covers \( E^*(x) \) - (\( \bigcup \mathcal{A}_{n-1} \)). For \( i \leq k \) and \( j < n \) let \( \delta_{ij} = \{ K \mid K \text{ is } V_{n}^*(j) \text{ or a maximal subinterval of } \alpha_j - V_{n}^*(j) \} \). If \( j < n \) define \( \delta_{ij} = \{ \bigcap_{l \leq k} K_l \mid K_l \in \delta_{ij} \} \). And if \( j \geq n \) define \( \delta_{ij} = \{ J \mid J \subset V_{n}^*(j) \} \). Then define \( \mathcal{A}_n = \{ U \mid J \subset V_{n}^*(j) \text{ for some } i \leq k \} \) and \( J = \prod_{j \in \omega_0} J_j \) where \( J_j \in \delta_{ij} \) for \( j < n \).

Let \( I_n = \bigcap_{V \in \mathcal{A}_{n-1} \cup \mathcal{A}_n} V(n) \). Since \( V \in \mathcal{A}_{n-1} \cup \mathcal{A}_n \) implies \( V \cap E^*(x) \neq \emptyset \), \( x(n) \in I_n \). For \( V \in \mathcal{A}_{n-1} \cup \mathcal{A}_n \), define \( V \in \mathcal{B} \) by \( V(i) = V(i) \) for \( i \neq n \) and \( V(n) = I_n \). Then define \( \mathcal{A}_n = \{ V \mid V \in \mathcal{A}_{n-1} \cup \mathcal{A}_n \} \); clearly \( \mathcal{A}_n \) has the desired properties.
For all $r$ and $n$ in $\omega_0$ and $V \in \mathcal{A}_n$, define $V^m = V(0) \times \cdots \times V(n-1) \times I_n \times I_{n+1} \times \cdots \times I_{n+r} \times V(n+r+1) \times \cdots$; and define $V^n = V(0) \times \cdots \times V(n-1) \times I_n \times I_{n+1} \times \cdots$. Let $\mathcal{C} = \{V^n | n \in \omega_0 \text{ and } V \in \mathcal{A}_n\}$. Clearly $V \supset V^{n_0} \supset V^{n_1} \supset \cdots \supset V^n$, so $V \in \mathcal{A}_n \subset \mathcal{A}$ implies $V^n \in \mathcal{A}$. By an easy induction on $r$, $V^m \in \mathcal{A}_{n+r}$, thus the fact that, for all $m \in \omega_0$, the members of each $\mathcal{A}_m$ are disjoint, yields that $\mathcal{C}$ is a collection of disjoint members of $\mathcal{A}$.

Define $Z = \{p \in X | \text{for some } n \in \omega_0, p(m) \in I_m \text{ for all } m > n\}$. Clearly $x \in Z \in \mathcal{A}$. We prove $Z = \bigcup \mathcal{C}$ and this proves the lemma.

Clearly each term of $\mathcal{C}$ is contained in $Z$, so we only need prove $Z \subset \bigcup \mathcal{C}$. Suppose $p \in Z$. There is an $n \in \omega_0$ with $p(m) \in I_m$ for all $m \geq n$. Let $q$ be the point of $E^n(x)$ with $q(m) = p(m)$ for $m < n$ and $q(m) = x(m)$ for $m \geq n$. Then $q \in V \in \mathcal{A}_n$. But also $p \in V$ and $p \in V^n \in \mathcal{C}$. So $Z \subset \bigcup \mathcal{C}$.

**Lemma 3.** If $\mathcal{A} \in \Omega$, then [CH] there is a set of disjoint members of $\mathcal{B}(\mathcal{A})$ covering $X$.

**Proof.** Define a one-to-one function $f: \omega_1 \times \omega_1 \rightarrow \omega_1$ such that $f(\beta, \alpha) > \beta$ for all $\beta$ and $\alpha$; $f$ need not be onto.

For each countable ordinal $\beta$ we define sets $\mathcal{K}_\beta$ and $\mathcal{K}_\beta$ by transfinite induction. Our induction hypotheses are:

1. $\mathcal{K}_\beta \subset \mathcal{B}(\mathcal{A})$ and $\mathcal{K}_\beta \subset \mathcal{A}$.
2. $\mathcal{K}_\beta \cup \mathcal{K}_\beta$ is a disjoint cover of $X$ and no term of $\mathcal{K}_\beta$ intersects a term of $\mathcal{K}_\beta$.
3. $\beta < \alpha$ and $V \in \mathcal{K}_\beta$ implies $V \neq \mathcal{K}_\beta$.
4. $\beta < \alpha$ and $V \in \mathcal{K}_\beta$ implies there is a $U \in \mathcal{K}_\beta$ with $U \supset V$.

We use some functions in the definitions and we define these before beginning the induction. Suppose $\beta < \omega_1$ and suppose $\{\mathcal{K}_\rho | \rho \leq \beta\}$ and $\{\mathcal{K}_\rho | \rho \leq \beta\}$ have been defined satisfying the induction hypotheses. Then define a function $g_\beta: (\mathcal{K}_\beta - \{\emptyset\}) \times \omega_1 \rightarrow \mathcal{A}$ as follows. Suppose $\emptyset \neq W \in \mathcal{K}_\beta$. If $\rho \leq \beta$, by (4), there is a $W(\rho) \in \mathcal{K}_\beta$ such that $W(\rho) \supset W$. By (2), the terms of $\mathcal{K}_\rho$ are disjoint so $W(\rho)$ is uniquely determined. For $\rho \leq \beta$ and $n \in \omega_0$, define $\xi_{\rho m} = \{I \in I \text{ is either } W(\rho)^*(n) \text{ or a maximal interval of } \alpha_n - W(\rho)^*(n)\}$. Let $\xi_n = \{\bigcap_{\xi \leq \beta} I \in \xi_m\}$ and $\xi = \{\Pi_{\alpha \in \omega_0} I_n | I_n \in \xi_m\}$. Since $\rho \leq \beta$ is countable, the cardinality of $\xi$ is at most that of the continuum. Hence [CH] there is a function $g_{\omega_1}$ from $\omega_1$ onto $\xi$. For $\alpha \in \omega_1$ define $g_\beta(W, \alpha) = g_{\omega_1}(\alpha)$.

We are now ready to begin our induction. Define $\mathcal{K}_0 = \emptyset$ and $\mathcal{K}_0 = \{X\}$.

Assume $\mathcal{K}_\beta$ and $\mathcal{K}_\beta$ satisfying the induction hypotheses have been defined for all $\beta < \gamma$ where $0 < \gamma < \omega_1$.

We first define $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma$ in the case $\gamma = \delta + 1$ for some $\delta < \omega_1$. Observe that part of our assumption in this case is that $g_\beta$ has been defined for all $\beta \leq \delta$. Let $Q_\delta = \{U \in \mathcal{K}_\delta | \text{there are } \alpha \in \omega_1, \beta < \delta, \text{ and } W \in \mathcal{K}_\beta \text{ such that } f(\beta, \alpha) = \delta, \text{ } U \subset W, \text{ and the top of } g_\beta(W, \alpha) \text{ belongs to } U\}$.

Fix $U \in Q_\delta$. Since $f$ is one-to-one, $\alpha$ and $\beta$ are uniquely determined. By (2) there can be at most one $W \in \mathcal{K}_\beta$ such that $U \subset W$. So $g_\beta(W, \alpha)$ and the top $t_U$ of $g_\beta(W, \alpha)$ are uniquely determined by $U$. 

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Choose \( Z \in \mathcal{E}(\mathcal{G}) \) with \( t_U \subseteq Z \) as guaranteed by Lemma 2; keep in mind that \( Z \) is a function of \( U \). Define \( H_U = H \cap Z \). For \( n \in \omega_0 \) define \( \mathcal{F}_n = \{ I \cap J \mid I \) is either \( U^*(n) \) or a maximal subinterval of \( \alpha_n - U^*(n) \) and \( J \) is either \( Z^*(n) \) or a maximal subinterval of \( \alpha_n - Z^*(n) \). Then define \( \mathcal{G} = \{ \prod_{n \in \omega_0} F_n \mid F_n \in \mathcal{F}_n \) and, for infinitely many \( n, F_n \subseteq Z^*(n) \}. Let \( \mathcal{K}_U = \{ \cup_{p \in F} E(p) \mid F \in \mathcal{G} \) and \( F \subseteq U \}. \) The terms \( \mathcal{K}_U \) are disjoint and their union is \( U - Z \).

Now define \( \mathcal{K}_y = \mathcal{K}_y \cup \{ H_U \mid U \in \mathcal{Q}_y \) and \( \mathcal{K}_y = (\mathcal{K}_y - \mathcal{Q}_y) \cup \cup \{ \mathcal{K}_U \mid U \in \mathcal{Q}_y \}. Using only the preceding paragraph and the facts that \( \delta + 1 = \gamma \), the induction hypotheses are satisfied for \( \beta < \gamma \), \( \mathcal{Q}_y \subseteq \mathcal{K}_y \) and that for each \( U \in \mathcal{Q}_y \), a unique term \( t_U \) of \( U \) has been chosen, it is easy to check that the induction hypotheses hold for \( \gamma \). The messy definitions of \( g_\beta \) and \( \mathcal{Q}_y \) are only used later. But we need \( \mathcal{K}_\gamma \) and \( \mathcal{K}_y \) chosen in this complicated way in order to prove the lemma.

If \( \gamma \) is a limit ordinal, define \( \mathcal{K}_y = \mathcal{K}_y \cup \bigcup \{ H_U \mid U \in \mathcal{Q}_y \) and \( \mathcal{E}_y = (\mathcal{K}_y - \mathcal{Q}_y) \cup \cup \{ \mathcal{K}_U \mid U \in \mathcal{Q}_y \}. By Lemma 1, \( \mathcal{E} \neq V \in \mathcal{E}_y \) implies \( V \) is the union of a set \( \mathcal{K}_\gamma \) of disjoint members of \( \mathcal{G} \). Define \( \mathcal{K}_\gamma = \bigcup \{ \mathcal{K}_U \mid V \in \mathcal{E}_y \}. The induction hypotheses clearly thus hold for \( \gamma \).

Define \( \mathcal{K} = \bigcup_{\beta < \omega} \mathcal{K}_\beta \). The members of \( \mathcal{K} \) are certainly disjoint terms of \( \mathcal{E}(\mathcal{G}) \) so the lemma is proved if \( \mathcal{K} \) covers \( X \).

Assume \( p \in X - \bigcup \mathcal{K} \). Then for each \( \beta < \omega_1 \), there is a unique \( U_\beta \in \mathcal{K}_\beta \) with \( p \in U_\beta \). For \( n \in \omega_0 \) and \( \beta < \omega_1 \), let \( s_\beta(n) = \{ I \subseteq \alpha_n \mid I \) is either \( U_\beta^*(n) \) or a maximal subinterval of \( \alpha_n - U_\beta^*(n) \}. For \( n \in \omega_0 \) and \( \beta < \omega_1 \), let \( t_\beta(n) \) be the term of \( s_\beta(n) \) to which \( p(n) \) belongs, and for each \( \delta < \omega_1 \), let \( J_\beta = \bigcap_{\delta < \beta} J_\beta \). Define \( J_\beta = \prod_{n \in \omega_0} J_\beta(n) \). Clearly \( p \in J_\beta \subseteq U_\beta \) and \( \beta < \delta \) implies \( J_\beta \subseteq J_\beta \). Let \( t_\delta \) be the top of \( J_\delta \); that is, \( t_\delta = \prod_{n \in \omega_0} \sup \{ x(n) \mid x \in J_\delta \}. Clearly \( t_\delta \subseteq J_\delta \) and \( \beta < \delta \) implies \( t_\beta(n) \leq t_\delta(n) \) for all \( n \). Since \( \omega_1 \) and \( \alpha_n \) are well ordered, there is, for each \( n \in \omega_0 \), a smallest \( \beta_n < \omega_1 \) such that \( t_\beta(n) = \inf \{ t_\beta(n) \mid \beta < \omega_1 \}. Define \( \beta = \sup \{ \beta_n \mid n \in \omega_0 \}. Then t_\beta = t_\beta \) for all \( \beta > \beta \). Look again at the definition of \( g_\beta \). If \( U_\beta = W \), then \( \rho \leq \beta \) yields \( U_\rho = W(\rho) \) and \( s_\rho = s_\beta \). So \( J_\rho \subseteq J_\beta \) and \( J_\rho \subseteq \mathcal{J} \). Hence, \( J_\beta = g_\beta(U_\beta, \alpha) \) for some \( \alpha < \omega_1 \). Let \( f(\beta, \alpha) = \delta \); look at the definition of \( \mathcal{K}_\gamma \) and \( \mathcal{K}_y \) in the case \( \gamma = \delta + 1 \). Clearly \( U_\delta \subseteq \mathcal{Q}_y \) and \( i_\delta = i_\delta \). Thus \( i_\delta \) belongs to a term of \( \mathcal{K}_y \), but this contradicts \( t_\delta = t_\gamma \subseteq U_\gamma \subseteq \mathcal{K}_\gamma \). Hence \( \mathcal{K} \) covers \( X \).

**Theorem 1.** The continuum hypothesis implies the box product of countably many \( \sigma \)-compact ordinals is paracompact. In fact every open cover of such a product has a refinement consisting of disjoint open-closed sets.

**Proof.** By Lemma 0 we assume a space \( X = \prod_{\alpha_n \in \omega_0} \alpha_n \) where each \( \alpha_n \) is compact. Let \( \mathcal{G} \) be an open cover of \( X \). Define \( \mathcal{G} = \{ V \in \mathcal{G} \mid \) for some \( G \in \mathcal{G}, V \subseteq G \}. Obviously \( \mathcal{G} \in \Omega \). So, by Lemma 3, [CH] there is a set \( \mathcal{K} \) of disjoint members of \( \mathcal{E}(\mathcal{G}) \) covering \( X \). For \( H \in \mathcal{K} \), let \( \mathcal{E}_H \) denote a set of disjoint members of \( \mathcal{G} \) whose union is \( H \). Then \( \bigcup_{H \in \mathcal{K}} \mathcal{E}_H \) is a set of disjoint open sets refining \( \mathcal{G} \) and covering \( X \).
Theorem 2. Suppose that for each $n \in \omega_0$, $\gamma_n$ is an ordinal and $Y$ is the box product of $\{\gamma_n \mid n \in \omega_0\}$. Suppose $\delta$ is an uncountable ordinal and $S = \{n \in \omega_0 \mid \text{cf}(\gamma_n) = \delta\} \neq \emptyset$. Suppose also that $n \in \omega_0 - S$ implies $\gamma_n$ is $\sigma$-compact. Define $\kappa = \sup\{\text{cardinality of } \prod_{n \in \omega_0 - S} \beta_n \mid s \in S, \beta_n \leq \gamma_n, \text{ and } \beta_n \text{ is compact}\}$. Then [CH] and $\delta > \kappa$ imply $Y$ is normal.

An analogous proof shows $Y$ is collectionwise normal.

Proof. Using Lemma 0, we assume $n \in \omega_0 - S$ implies $\gamma_n$ is compact. Suppose $A$ and $B$ are disjoint closed subsets of $Y$. Define $X$ to be the box product of $\{\alpha_n\}_{n \in \omega_0}$ where $\alpha_n = \gamma_n$ when $n \in \omega_0 - S$ and $\alpha_n = \gamma_n + 1$ when $n \in S$. Observe that $\alpha_n$ is compact for each $n$, and $Y$ is a subspace of $X$. We now use the notation set up at the beginning of §E for $X$; recall $\mathfrak{B}$ is a basis for $X$. Define $\mathfrak{D} = \{W \in \mathfrak{B} \mid \text{either } W \cap A = \emptyset \text{ or } W \cap B = \emptyset\}$. Lemma 4 below proves $\mathfrak{D} \in \mathfrak{B}$. Then Lemma 3 proves [CH] there is a set $\mathfrak{C}$ of disjoint members of $\mathfrak{D}(\mathfrak{D})$ covering $X$. For $H \in \mathfrak{X}$ let $\mathfrak{D}_H$ denote a set of disjoint members of $\mathfrak{D}$ whose union is $H$. Let $U = \bigcup\{W \in \mathfrak{D}_H \mid H \in \mathfrak{X} \text{ and } W \cap A \neq \emptyset\}$ and $V = \bigcup\{W \in \mathfrak{D}_H \mid H \in \mathfrak{X} \text{ and } W \cap B \neq \emptyset\}$. Then $U \supset A$, $V \supset B$, and $U \cap V = \emptyset$. Thus $Y$ is proved normal.

Lemma 4. Assume $y$, $S$, $\delta$, $\kappa$, $A$, $B$, $X$, and $\mathfrak{D}$ as above. Suppose $x \in X$ and $\kappa < \delta$. Then $x \in \bigcup \mathfrak{D}$.

Proof. Let $R = \{n \in S \mid x(n) = \gamma_n\}$, $Z = \prod_{n \in R} \gamma_n$, $W = \prod_{n \in \omega_0 - R} (x(n) + 1)$; then $(W \times Z) \subset Y$.

If $R = \emptyset$ then $x \in Y$ and, since $A$ and $B$ are closed and disjoint in $Y$, the lemma is true. Assume $R \neq \emptyset$ for the rest of the proof. Observe that $R \neq \emptyset$ and $\kappa < \delta$ imply the cardinality of $W$ is less than $\delta$. We have two similar major cases.

Case (1). $R$ has more than one member. In this case, by the definition of $\kappa$ and $\kappa < \delta$, $\gamma_n = \delta$ for all $n \in R$. For $\sigma < \delta$, define $\zeta_\sigma$ to be the point of $Z$ all of whose coordinates are $\sigma$ and define $Z_\sigma$ to be the set of all points of $Z$ all of whose coordinates are greater than $\sigma$. We use $\{\zeta_\sigma \mid \sigma < \delta\}$ and $\{Z_\sigma \mid \sigma < \delta\}$ to help us choose a special ordinal $\lambda < \delta$.

Case (1a). $R \neq \omega_0$. In this case we wish to choose $\lambda < \delta$ such that, for all $p \in W$, one of the following hold:

(i) $(p, Z_\lambda) \cap B = \emptyset$ and $(p, Z_\lambda) \in A$,
(ii) $(p, Z_\lambda) \cap A = \emptyset$ and $(p, Z_\lambda) \in B$, or
(iii) $(p, Z_\lambda) \cap (A \cup B) = \emptyset$.

Let $P = \{p \in W \mid \text{there is a } \sigma < \delta \text{ such that } q \in Z_\sigma \implies (p, q) \in A \cup B\}$. Since the cardinality of $W$ is less than $\delta$, there is a $\beta < \delta$ such that $p \in P$ and $q \in Z_\beta \implies (p, q) \in A \cup B$. Any $\lambda$ chosen with $\beta \leq \lambda < \delta$ yields (iii) for all $p \in P$. If $W \subset P$, define $\lambda = \beta$ and (iii) holds for $p \in W$. Otherwise we have (1a*) or (1a**).

Case (1a*). $W - P \neq \emptyset$ and $\delta \neq \omega_1$. 

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Suppose $p \in W - P$ and define $\Delta_p = \{ \sigma \in \delta \mid (p, z_\sigma) \in (A \cup B) \}$.

Suppose $\{z_\eta \mid \eta \in \Omega_1 \}$ is a monotone subset of $\delta$, and $q_\eta \in Z_{z_\eta} - Z_{z_{\eta + 1}}$, and $(p, q_\eta) \in A \cup B$ for all $\eta \in \Omega_1$. Let $\sigma = \sup\{\sigma_\eta \mid \eta \in \Omega_1 \}$. Since $R$ is countable, $(p, q_\eta)$ is a limit point of $\{(p, q_\eta) \mid \eta \in \Omega_1 \}$. So $(p, q_\eta) \in A \cup B$ and $\sigma \in \Delta_p$ and only countably many $(p, q_\eta)$ can belong to the one of $A$ and $B$ to which $(p, q_\eta)$ does not belong.

Thus by the preceding paragraph there is a $\sigma_\eta < \delta$ such that either $\{(p, q) \mid q \in Z_{z_\eta}\} \cap A = \emptyset$ or $\{(p, q) \mid q \in Z_{q_\eta}\} \cap B = \emptyset$. Let $\delta^* = \{ \sigma \in \delta \mid \c (\sigma) \geq \omega_1 \}$. By the preceding paragraph and the definition of $P$, since $p \in W - P$, $\Delta_p \cap \delta^*$ is a cardinality $\delta$ closed subset of $\delta^*$. Thus $\Delta_p = \{ \rho \in \Delta_p \cap \delta^* \mid \rho \geq \sigma_p \}$ and $\rho \geq \beta$ is a cardinality $\delta$ closed subset of $\delta^*$. Recall that $\delta$ is an uncountable ordinal and $\c (\delta) = \delta$. Thus it is standard set theory that the intersection of any family of cardinality less than $\delta$ of closed subsets of $\delta^*$, each of cardinality $\delta$, is nonempty. Therefore, since the cardinality of $W$ is less than $\delta$, there is a $\lambda \in \cap_{p \in W - P} \Delta_p$. Clearly $p \in W - P$ implies (i) or (ii) holds for $\lambda$; and, since $\lambda > \beta$, $p \in P$ implies (iii) holds.

**Case (1a**) $W - P \neq \emptyset$ and $\delta = \omega_1$. Observe that cardinality less than $\delta$ means countable here. Thus $R$ is finite. So we can prove the existence of a $\delta$ with the desired properties using exactly the proof given in Case (1a) if we replace $\omega_1$ in the proof by $\omega_0$ and countable by finite. In this case $\delta = \delta^*$.

**Case (1b).** $R = \emptyset$. In this case we wish to choose $\lambda < \delta$ such that, for all $p \in W$, either $Z_\lambda \cap A = \emptyset$ or $Z_\lambda \cap B = \emptyset$. Such a $\lambda$ can be shown to exist using a simplified version of the argument given in Case (1a) where all references to $P$, $\beta$, and $p$ are omitted.

Having chosen $\lambda$, we are now ready to prove the lemma in Case 1. Define $A' = \{ p \in W - P \mid (p, z_\lambda) \in A \}$ and $B' = \{ p \in W - P \mid (p, z_\lambda) \in B \}$. Clearly $A'$ and $B'$ are closed and disjoint in $W$. If $x'$ is the point of $W$ such that $x'(n) = x(n)$ for all $n \in \omega_0 - R$, then $x' \notin A' \cap B'$. Hence, for each $n \in \omega_0 - R$, there is an open and closed subinterval $I_n$ of $x(n) + 1$ containing $x'(n)$ such that either $A' \cap \cap_{n \in \omega_0 - R} I_n = \emptyset$ or $B' \cap \cap_{n \in \omega_0 - R} I_n = \emptyset$. For $n \in R$ define $I_n = Z_\lambda$. Then $x \in \cap_{n \in \omega_0} I_n \in A$.

**Case 2.** $R$ has only one member. Say $R = \{ r \}$.

Since $\c (\gamma_r) = \delta$, there is a closed subset $\{ \sigma \mid \sigma < \delta \}$ of $\gamma_r$, such that $\sigma < \eta < \delta$ implies $z_\sigma < z_\eta$ and $\beta < \gamma_r$ implies $\beta < z_\sigma$ for some $\sigma < \delta$. Since $R = \{ r \}$, $z_\sigma \in Z$. Define $Z_\sigma = \{ \beta \in \gamma_r \mid \beta > z_\sigma \}$. Then using precisely the same argument given in Case 1 after the definitions of $z_\sigma$ and $Z_\sigma$, one shows $x \in \cup A$.

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