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WEIGHTED NORM INEQUALITIES FOR FRACTIONAL INTEGRALS

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ABSTRACT. The principal problem considered is the determination of all nonnegative functions, $V(x)$, such that $\|T_\gamma f(x)V(x)\|_q \leq C\|f(x)V(x)\|_p$ where the functions are defined on R^n, $0 < \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, C is a constant independent of f and $T_\gamma f(x) = \int f(x-y)|y|^{\gamma-n}\,dy$. The main result is that $V(x)$ is such a function if and only if

$$\left(\frac{1}{|Q|}\int_Q [V(x)]^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q [V(x)]^{-p'}\,dx\right)^{1/p'} \leq K$$

where Q is any n dimensional cube, $|Q|$ denotes the measure of Q, $p' = p/(p-1)$ and K is a constant independent of Q. Substitute results for the cases $p = 1$ and $q = \infty$ and a weighted version of the Sobolev imbedding theorem are also proved.

1. **Introduction.** The first norm inequality for fractional integrals was the one proved by Hardy and Littlewood in [6] for the one dimensional case with $V(x) = 1$; they also proved a result for $V(x) = |x|^\alpha$. The result in n dimensions with $V(x) = 1$ was obtained by Sobolev in [8] and with $V(x) = |x|^\alpha$ by Stein and G. Weiss in [10]. T. Walsh in [12] obtained a result for other weight functions and with a more general operator but did not characterize all such V's.

A slightly stronger result is obtained here than stated in the abstract. It is shown that

$$\left(\frac{1}{|Q|}\int_Q [V(x)]^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q [V(x)]^{-p'}\,dx\right)^{1/p'} \leq K \tag{1.1}$$

implies the norm inequality for fractional integrals, but the necessity of (1.1) is shown with only the assumption of a weak type estimate on the fractional integrals.

The proof that (1.1) implies the norm inequality consists of two main parts. The first is the proof of a norm inequality between $T_\gamma f$ and a suitable maximal function, f^*_γ, defined in §2. This norm inequality is obtained in §2. We would like to acknowledge that the proof in §2 is an adaptation of certain proofs by Coifman and Fefferman in [1], and would like to thank them for showing us these proofs. In [1] Coifman and Fefferman greatly simplified the proofs in [5] and [7] to prove the principal result in [5].

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The rest of the proof that (1.1) implies the norm inequality for fractional integrals consists of proving a norm inequality for f_γ^*; this is done in §3. In §4 these results are combined to prove the asserted inequality and the necessity of (1.1) for the weak type inequality.

The cases $p = 1$ and $q = \infty$ are also considered here. In the case $p = 1$, (1.1) should be interpreted to mean

$$\left(\frac{1}{|Q|}\int_Q [V(x)]^q\,dx\right)^{1/q}\left(\operatorname*{ess\,sup}_{x\in Q}\frac{1}{V(x)}\right) \le K; \tag{1.2}$$

this is necessary and sufficient for a weak type inequality. This is also proved in §§2–4. For $q = \infty$ the proper interpretation of (1.1) is shown in §5 to imply that $T_\gamma f$ has a property resembling bounded mean oscillation; this generalizes the unweighted result contained in [11, Theorem 2(a), p. 341]. The necessity of (1.1) for this property is proved in §6. Finally, in §7 a weighted version of the Sobolev imbedding theorem is proved as an application of the main theorem.

Throughout this paper it is assumed that a fixed positive integer, n, has been taken as the dimension of the space and that functions are real-valued measurable functions on R^n. The letter C will denote a constant not necessarily the same at each occurrence, $0 \cdot \infty$ will be taken as 0, $|E|$ will denote the Lebesgue measure of E, $m_W(E) = \int_E W(x)\,dx$ and given a cube Q, mQ will denote the cube with the same center as Q and with sides parallel to those of Q and m times as long.

2. **Comparison of $T_\gamma f$ to f_γ^*.** In [1] norm inequalities for the conjugate function were obtained by comparing it to the Hardy-Littlewood maximal function. The same general procedure will be used here; the fractional integral of a function will be compared to a maximal function called f_γ^*.

Two definitions will be needed. First, given a real-valued function $f(x)$ on R^n and γ satisfying $0 < \gamma < n$, define

$$f_\gamma^*(x) = \sup_Q |Q|^{-1+\gamma/n}\int_Q |f(y)|\,dy$$

where the sup is taken over all cubes Q with center at x. Second, a nonnegative function $W(x)$ will be said to satisfy the condition A_∞ if given $\epsilon > 0$ there exists a $\delta > 0$ such that if Q is a cube, E is a subset of Q and $|E| \le \delta|Q|$, then $\int_E W(x)\,dx \le \epsilon \int_Q W(x)\,dx$. The main result of this section is the following.

Theorem 1. *If $W(x)$ satisfies the condition A_∞, $0 < q < \infty$ and $0 < \gamma < n$, then there is a C, independent of f, such that*

$$\int_{R^n} |T_\gamma f(x)|^q W(x)\,dx \le C\int_{R^n} [f_\gamma^*(x)]^q W(x)\,dx$$

and

$$\sup_{a>0} a^q \int_{|T_\gamma f|>a} W(x)\,dx \le C \sup_{a>0} a^q \int_{f_\gamma^*>a} W(x)\,dx.$$

Theorem 1 will be proved by first establishing the following lemma.

Lemma 1. *If $0 < \gamma < n$, there exist constants B and K, depending only on γ and n such that if $a > 0$, $d > 0$, $b \geq B$, $f(x)$ is nonnegative, Q is a cube in R^n such that $T_\gamma f(x) \leq a$ at some point of Q and E is the subset of Q where both $T_\gamma f(x) > ab$ and $f^*_\gamma(x) \leq ad$, then $|E| \leq K|Q|[d/b]^{n/(n-\gamma)}$.*

To prove Lemma 1 let $g(x) = f(x)$ on $2Q$ and 0 elsewhere; let $h(x) = f(x) - g(x)$. Assume that there is a t in Q such that $f^*_\gamma(t) \leq ad$; otherwise the conclusion is trivial. By [9, Theorem 1, p. 119], there is a constant C, depending only on n and γ, such that for any positive a and b

$$|\{T_\gamma g(x) > ab/2\}| \leq C\left[\frac{1}{ab}\int_{R^n} g(x)\,dx\right]^{n/(n-\gamma)}. \tag{2.1}$$

Let P be the cube with center t and sides parallel to and three times as long as those of Q. Then since $2Q \subset P$,

$$\int_{R^n} g(x)\,dx \leq \int_P f(x)\,dx \leq f^*_\gamma(t)|P|^{(n-\gamma)/n} \leq ad|3Q|^{(n-\gamma)/n}.$$

Using this in (2.1) shows that

$$|\{T_\gamma g(x) > ab/2\}| \leq C3^n|Q|[d/b]^{n/(n-\gamma)}. \tag{2.2}$$

Now let s be a point of Q such that $T_\gamma f(s) \leq a$. There is a constant L, depending only on n and greater than 1, such that if x is in Q and y is not in $2Q$, then $|s - y| \leq L|x - y|$. Therefore, for x in Q,

$$\begin{aligned} T_\gamma h(x) &= \int_{R^n} \frac{h(y)dy}{|x-y|^{n-\gamma}} \leq L^{n-\gamma}\int_{R^n} \frac{h(y)dy}{|s-y|^{n-\gamma}} \\ &\leq L^n T_\gamma f(s) \leq L^n a. \end{aligned}$$

Let $B = 2L^n$. Then if $b \geq B$, $T_\gamma h(x) \leq ab/2$ for all x in Q and E is a subset of the set where $T_\gamma g(x) > ab/2$. The conclusion of Lemma 1 then follows from (2.2).

In the proof of Theorem 1 it may be assumed that $f(x)$ is nonnegative since replacing $f(x)$ by $|f(x)|$ only increases the left sides of the conclusions and does not affect the right sides. It can also be assumed that $f(x)$ is locally integrable since if it is not the conclusions are trivial. Local integrability of $W(x)$ can also be assumed; otherwise the right sides of the conclusions are infinite unless $f(x)$ is 0 almost everywhere.

Now assume that $f(x)$ has compact support. Given $a > 0$, decompose the set where $T_\gamma f(x) > a$ into cubes $\{Q_j\}$ with disjoint interiors such that, for every j, $T_\gamma f(x) \leq a$ at some point of $4Q_j$; this is possible by [9, Theorem 1, p. 167]. Let B and K be as in Lemma 1 and let $b = \max(1, B)$. Let δ correspond to $\epsilon = \frac{1}{2}b^{-q}$ in the definition of A_∞ for $W(x)$. Choose D so that $\delta = K4^n[D/b]^{n/(n-\gamma)}$. Let d

satisfy $0 < d \leq D$ and let E_j be the subset of Q_j where $T_\gamma f(x) > ab$ and $f_\gamma^*(x) \leq ad$.

By Lemma 1, $|E_j| \leq K|4Q_j|[d/b]^{n/(n-\gamma)} < \delta|Q_j|$ so by the definition of δ, $m_W(E_j) \leq \frac{1}{2}b^{-q}m_W(Q_j)$. Summing on j shows that

$$m_W(\{T_\gamma f > ab \text{ and } f_\gamma^* \leq ad\}) \leq \tfrac{1}{2}b^{-q}m_W(\{T_\gamma f > a\}).$$

This implies that

$$m_W(\{T_\gamma f > ab\}) \leq m_W(\{f_\gamma^* > ad\}) + \tfrac{1}{2}b^{-q}m_W(\{T_\gamma f > a\}) \tag{2.3}$$

for any d satisfying $0 < d \leq D$.

Now let Q be a cube such that $f(x) = 0$ for x outside Q. Given x outside $3Q$ let u be the point in Q closest to x and let P be the smallest cube with center at x and sides parallel to Q that contains Q. Then there is a constant, L, depending only on n and greater than 1 such that $|P| \leq L|x-u|^n$. Furthermore,

$$T_\gamma f(x) \leq |x-u|^{\gamma-n}\int_Q f(y)\,dy \leq \frac{|P|^{(n-\gamma)/n}}{|x-u|^{n-\gamma}}f_\gamma^*(x) \leq Lf_\gamma^*(x).$$

Now define $d = \min(D, 1/L)$; it follows immediately that

$$\{T_\gamma f > a\} \cap (3Q)^c \subset \{f_\gamma^* > ad\} \tag{2.4}$$

where $(3Q)^c$ denotes the complement of $3Q$. From (2.3) and (2.4) it follows that

$$m_W(\{T_\gamma f > ab\}) \leq 2m_W(\{f_\gamma^* > ad\}) + \tfrac{1}{2}b^{-q}m_W(\{T_\gamma f > a\} \cap 3Q). \tag{2.5}$$

Next, multiply both sides of (2.5) by a^{q-1} and integrate a from 0 to some positive N. After a change of variables the left side becomes

$$b^{-q}\int_0^{Nb} a^{q-1}m_W(\{T_\gamma f > a\})\,da. \tag{2.6}$$

Similarly, with a change of variables for the first integral on the right, the right side becomes

$$2d^{-q}\int_0^{Nd} a^{q-1}m_W(\{f_\gamma^* > a\})\,da + \tfrac{1}{2}b^{-q}\int_0^N a^{q-1}m_W(\{T_\gamma f > a\} \cap 3Q)\,da. \tag{2.7}$$

Since W has been assumed to be locally integrable, the second term in (2.7) is finite; it is also bounded by half of (2.6) since $b \geq 1$. Therefore,

$$\frac{1}{2}b^{-q}\int_0^{Nb} a^{q-1}m_W(\{T_\gamma f > a\})\,da \leq 2d^{-q}\int_0^{Nd} a^{q-1}m_W(\{f_\gamma^*(x) > a\})\,da. \tag{2.8}$$

Now let N approach ∞; (2.8) then reduces to

$$\frac{b^{-q}}{2q}\int_{R^n} |T_\gamma f(x)|^q W(x)\,dx \leq \frac{2d^{-q}}{q}\int_{R^n} [f_\gamma^*(x)]^q W(x)\,dx. \tag{2.9}$$

To prove (2.9) for an $f(x)$ that does not have compact support, let $f_m(x)$ equal $f(x)$ for $|x| \leq m$ and equal 0 for $|x| > m$. Then (2.9) can be applied to f_m; taking the limit as $m \to \infty$ and using the monotone convergence theorem then gives (2.9) for general $f(x)$. This completes the proof of the first part of Theorem 1.

To prove the second part of Theorem 1, multiply both sides of (2.5) by a^q. Given $N > 0$, take the sup of both sides for $0 < a < N$ and use the fact that $\sup(u + v) \leq \sup u + \sup v$. This shows that

$$\sup_{0<a<N} a^q m_W(\{T_\gamma f > ab\}) \tag{2.10}$$

is bounded by

$$\sup_{0<a<N} 2a^q m_W(\{f_\gamma^* > ad\}) + \sup_{0<a<N} \tfrac{1}{2} a^q b^{-q} m_W(\{T_\gamma f > a\} \cap 3Q). \tag{2.11}$$

Now (2.10) and (2.11) are equal respectively to

$$\sup_{0<a<bN} b^{-q} a^q m_W(\{T_\gamma f > a\}) \tag{2.12}$$

and

$$\sup_{0<a<Nd} 2a^q d^{-q} m_W(\{f_\gamma^* > a\}) + \sup_{0<a<N} \tfrac{1}{2} a^q b^{-q} m_W(\{T_\gamma f > a\} \cap 3Q). \tag{2.13}$$

The second term in (2.13) is finite and bounded by half of (2.12); therefore,

$$\tfrac{1}{2} b^{-q} \sup_{0<a<bN} a^q m_W(\{T_\gamma f > a\}) \leqslant 2d^{-q} \sup_{0<a<Nd} a^q m_W(\{f_\gamma^* > a\}). \tag{2.14}$$

Letting N approach ∞ in (2.14) shows that

$$\sup_{a>0} a^q \int_{T_\gamma f > a} W(x)\,dx \leq 4b^q d^{-q} \sup_{a>0} a^q \int_{f_\gamma^* > a} W(x)\,dx \tag{2.15}$$

for an $f(x)$ with compact support. For general $f(x)$ let $f_m(x)$ equal $f(x)$ for $|x| \leq m$ and equal 0 for $|x| > m$. Then (2.15) can be applied to f_m; taking the limit as $m \to \infty$ gives (2.15) for general f. This completes the proof of Theorem 1.

3. **Norm inequalities for f_γ^*.** This section consists of the proofs for the following two theorems.

Theorem 2. *If $0 \leq \gamma < n$, $1 \leq p < n/\gamma$, $1/q = 1/p - \gamma/n$, $a > 0$, E_a is the set where $f_\gamma^*(x) > a$, and $V(x)$ is a nonnegative function on R^n such that, for every cube Q, (1.1) holds with K independent of Q, then there is a C, independent of f, such that*

$$\left(\int_{E_a} [V(x)]^q\,dx\right)^{1/q} \leq \frac{C}{a}\left(\int_{R^n} |f(x)V(x)|^p\,dx\right)^{1/p}. \tag{3.1}$$

Theorem 3. *If $0 < \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$ and $V(x)$ is a nonnegative function on R^n such that, for every cube Q, (1.1) holds with K*

independent of Q, then there is a C, independent of f, such that

$$\left(\int_{R^n} [f_\gamma^*(x)V(x)]^q\,dx\right)^{1/q} \le C\left(\int_{R^n} |f(x)V(x)|^p\,dx\right)^{1/p}. \tag{3.2}$$

To prove Theorem 2 fix $M > 0$ and let $E_{a,M}$ be the intersection of E_a and the sphere about the origin of radius M. For each x in $E_{a,M}$ there is a cube Q centered at x such that

$$|Q|^{-1+\gamma/n}\int_Q |f(x)|\,dx > a. \tag{3.3}$$

Using [2, Corollary 1.7, p. 304] pick a sequence $\{Q_k\}$ of these cubes such that $E_{a,M} \subset \bigcup Q_k$ (note that the opposite conclusion stated in conclusion a of this corollary is a misprint) and no point of R^n is in more than C of these cubes where C depends only on n. Then

$$\left(\int_{E_{a,M}} [V(x)]^q\,dx\right)^{p/q} \le \left(\sum_k \int_{Q_k} [V(x)]^q\,dx\right)^{p/q}, \tag{3.4}$$

and since $p/q \le 1$, the right side of (3.4) is bounded by

$$\sum_k \left(\int_{Q_k} [V(x)]^q\,dx\right)^{p/q}. \tag{3.5}$$

Since the Q_k are cubes that satisfy (3.3), (3.5) is bounded by

$$\sum_k \left(\int_{Q_k} [V(x)]^q\,dx\right)^{p/q}\left(a^{-1}|Q_k|^{-1+\gamma/n}\int_{Q_k} |f(x)|\,dx\right)^p.$$

Using Hölder's inequality on the last integral shows that this is bounded by

$$\sum_k \left(\int_{Q_k} [V(x)]^q\,dx\right)^{p/q} a^{-p}|Q_k|^{1-p-p/q}\left(\int_{Q_k} |f(x)V(x)|^p\,dx\right)\left(\int_{Q_k} [V(x)]^{-p'}\,dx\right)^{p/p'}.$$

Using the fact that V satisfies (1.1) shows that this is bounded by

$$Ca^{-p}\sum_k \int_{Q_k} |f(x)V(x)|^p\,dx, \tag{3.6}$$

and since no point of R^n is contained in more than a fixed number of Q_k's, (3.6) is bounded by

$$Ca^{-p}\int_{R^n} |f(x)V(x)|^p\,dx. \tag{3.7}$$

Therefore, the left side of (3.4) is bounded by (3.7), and since the C in (3.7) does not depend on M, (3.1) follows from the monotone convergence theorem.

To prove Theorem 3, define $W(x) = [V(x)]^q$ and note that (1.1) is equivalent to

$$\left(\frac{1}{|Q|}\int_Q W(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q [W(x)]^{-1/(r-1)}\,dx\right)^{r-1} \le C \tag{3.8}$$

where $r = 1 + q/p'$. Therefore, by the definition [7,p. 214], W satisfies the condition A_r. By Lemma 5 of [7] (extended to n dimensions on pp. 222–223 of [7]), there is an s satisfying $1 < s < r$ such that W is in A_s. Then there are numbers p_1 and q_1 such that $1/q_1 = 1/p_1 - \gamma/n$, $1 < p_1 < p$ and $s = 1 + q_1/p_1'$. By Theorem 2 there is a C such that

$$\left(\int_{E_a} W(x)\,dx\right)^{p_1/q_1} \le Ca^{-p_1}\int_{R^n} |f(x)|^{p_1} W(x)^{p_1/q_1}\,dx. \tag{3.9}$$

Now define a sublinear operator T by $Tg(x) = (g(x)[W(x)]^{\gamma/n})^*_\gamma$. Then with $f(x) = g(x)[W(x)]^{\gamma/n}$, (3.9) can be written in the form

$$\int_{Tg>a} W(x)\,dx \le Ca^{-q_1}\left(\int_{R^n} |g(x)|^{p_1} W(x)\,dx\right)^{q_1/p_1}. \tag{3.10}$$

Similarly, there is a p_2 satisfying $p < p_2 < n/\gamma$. Then with q_2 defined by $1/q_2 = 1/p_2 - \gamma/n$ it is immediate from Hölder's inequality that W satisfies A_t with $t = 1 + q_2/p_2'$ since $t > r$. Then Theorem 2 shows that (3.9) is true with p_1 and q_1 replaced by p_2 and q_2. The procedure used to derive (3.10) then shows that

$$\int_{Tg>a} W(x)\,dx \le Ca^{-q_2}\left(\int_{R^n} |g(x)|^{p_2} W(x)\,dx\right)^{q_2/p_2} \tag{3.11}$$

where T is the same operator as in (3.10). The Marcinkiewicz interpolation theorem, [13, Vol. II, p. 112], shows that

$$\left(\int_{R^n} [Tg(x)]^q W(x)\,dx\right)^{1/q} \le C\left(\int_{R^n} |g(x)|^p W(x)\,dx\right)^{1/p}. \tag{3.12}$$

Then letting $g(x) = f(x)[W(x)]^{-\gamma/n}$ and $W(x) = [V(x)]^q$ transforms (3.12) into (3.2). This completes the proof of Theorem 3.

4. The principal theorems. This section consists of the proof of the main result stated in the abstract and the substitute for $p = 1$. The theorems to be proved are as follows.

Theorem 4. *Assume that* $0 < \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, $p' = p/(p-1)$ *and* $V(x)$ *is a nonnegative function on* R^n *such that, for every cube* Q, (1.1) *holds with* K *independent of* Q. *Then there is a* C, *independent of* f, *such that*

$$\left(\int_{R^n} |T_\gamma f(x)V(x)|^q\,dx\right)^{1/q} \le C\left(\int_{R^n} |f(x)V(x)|^p\,dx\right)^{1/p}. \tag{4.1}$$

Theorem 5. *Assume that* $0 < \gamma < n$, $q = n/(n-\gamma)$, $a > 0$, E_a *is the set where* $|T_\gamma f(x)| > a$ *and* $V(x)$ *is a nonnegative function on* R^n *such that, for every cube* Q, (1.2)*holds with* K *independent of* Q. *Then there is a* C *independent of* f *and* a *such that*

$$\int_{E_a} [V(x)]^q\,dx \le Ca^{-q}\left[\int_{R^n} |f(x)|V(x)\,dx\right]^q. \tag{4.2}$$

Theorem 6. *Assume that* $0 < \gamma < n$, $1 \le p < n/\gamma$, $1/q = 1/p - \gamma/n$, $p' = p/(p-1)$ *and let* $V(x)$ *be a nonnegative function on* R^n *such that for* $a > 0$

$$\left(\int_{|T_\gamma f|>a} [V(x)]^q\,dx\right)^{1/q} \le \frac{C}{a}\left(\int_{R^n} |f(x)V(x)|^p\,dx\right)^{1/p} \tag{4.3}$$

where C *is independent of* a *and* f. *Then, for every cube* Q, (1.1) *holds with* K *independent of* Q.

Since (4.1) implies (4.3) by Tchebycheff's inequality, this is a stronger result for $1 < p < n/\gamma$ than just showing that (4.1) implies (1.1).

To prove Theorems 4 and 5 observe that if $V(x)$ satisfies (1.1), then, by the definition [7, p. 214], $[V(x)]^q$ satisfies the condition A_r with $r = 1 + q/p'$. If $V(x)$ satisfies (1.2), $[V(x)]^q$ satisfies A_1. In either case by (3.19) of [7] (extended to n dimensions [7, pp. 222–223]), there is an $s > 1$ and a C, both independent of Q, such that

$$\int_Q [V(x)]^{sq}\,dx \le C|Q|^{1-s}\left(\int_Q [V(x)]^q\,dx\right)^s \tag{4.4}$$

for every cube Q. If E is a subset of Q, then by (4.4) and Hölder's inequality

$$\int_E [V(x)]^q\,dx \le |E|^{1/s'}\left(\int_E [V(x)]^{sq}\,dx\right)^{1/s} = C(|E|/|Q|)^{1/s'}\int_Q [V(x)]^q\,dx.$$

Therefore, $[V(x)]^q$ satisfies A_∞ and Theorems 1, 2 and 3 can be applied to prove Theorems 4 and 5.

To prove Theorem 6 for $p > 1$ fix a cube, Q, in R^n and let $A = \int_Q [V(x)]^{-p'}\,dx$. If $A = 0$, there is nothing to prove because of the convention $0 \cdot \infty = 0$. If $A = \infty$, then $1/V$ is not in $L^{p'}$ on Q so there is a nonnegative function, g, in L^p on Q such that $\int_Q g(x)/V(x)\,dx = \infty$. Let $f(x) = g(x)/V(x)$ on Q and 0 elsewhere. Then $T_\gamma f \equiv \infty$ and since $f(x)V(x) \le g(x)$

$$\int_Q [f(x)V(x)]^p\,dx \le \int_Q [g(x)]^p\,dx.$$

By (4.3), $\int_Q [V(x)]^q\,dx \le Ca^{-q}$ for all $a > 0$. Therefore, $\int_Q [V(x)]^q\,dx = 0$; this completes the proof if $A = \infty$.

If $0 < A < \infty$, let $f(x) = [V(x)]^{-p'}$ on Q and 0 elsewhere. On Q, $T_\gamma f \geq A|Q|^{-1+\gamma/n}$ so using this as a in (4.3) shows that

$$\int_Q [V(x)]^q\,dx \leq C[A|Q|^{-1+\gamma/n}]^{-q}\left[\int_Q ([V(x)]^{-p'}V(x))^p\,dx\right]^{q/p}.$$

This reduces easily to (1.1).

If $p = 1$, fix a cube, Q, in R^n and let $A = \operatorname{ess\,inf}_{y\in Q} V(y)$. If $A = \infty$, (1.1) is true. Otherwise, given $\epsilon > 0$ there is a subset, E, of Q with positive measure such that $V(x) < A + \epsilon$ for all x in E. Define $f(x) = 1$ on E and 0 elsewhere. Then for x in Q, $T_\gamma f(x) \geq |E|\,|Q|^{-1+\gamma/n}$ and (4.3) with $a = |E|\,|Q|^{-1+\gamma/n}$ shows that

$$\left(\int_Q [V(x)]^q\,dx\right)^{1/q} \leq C|E|^{-1}|Q|^{1-\gamma/n}\int_E V(x)\,dx. \tag{4.5}$$

Using the fact that $\int_E V(x)\,dx \leq |E|(A + \epsilon)$ in (4.5) shows that

$$\left(\int_Q [V(x)]^q\,dx\right)^{1/q} \leq C|Q|^{1-\gamma/n}(A + \epsilon).$$

Since ϵ is arbitrary, this completes the proof of (1.1).

5. A sufficiency result for $q = \infty$.

Theorem 7. *Assume that $0 < \gamma < n$, $p = n/\gamma$, $p' = n/(n-\gamma)$ and $V(x)$ is a nonnegative function on R^n such that for every cube Q*

$$\left(\operatorname*{ess\,sup}_{x\in Q} V(x)\right)\left(\frac{1}{|Q|}\int_Q [V(x)]^{-p'}\,dx\right)^{1/p'} \leq K \tag{5.1}$$

where K is independent of Q. Then for every cube Q

$$\left(\operatorname*{ess\,sup}_{x\in Q} V(x)\right)\frac{1}{|Q|}\int_Q |T_\gamma f(x) - (T_\gamma f)_Q|\,dx \leq C\left(\int_{R^n} |f(x)V(x)|^p\,dx\right)^{1/p} \tag{5.2}$$

where C is independent of Q and $(T_\gamma f)_Q = (1/|Q|)\int_Q T_\gamma f(x)\,dx$.

This generalizes [11, Theorem 2(a), p. 341]; in [11], however, the conclusion contains the $L(p,\infty)$ norm of f instead of the L^p norm. Theorem 7 can be strengthened in this way; a sketch of how this could be done is given after the proof of Theorem 7.

To prove Theorem 7 fix a cube, Q, and let S be the cube with the same center and orientation as Q with sides twice as long. Let T be the complement of S. Then $T_\gamma f(x)$ is the sum of

$$\int_S f(y)|x-y|^{\gamma-n}\,dy \tag{5.3}$$

and

$$\int_T f(y)|x-y|^{\gamma-n}\,dy \tag{5.4}$$

and it is sufficient to prove (5.2) with $T_\gamma f$ replaced by (5.3) and (5.4).

Let $E = \operatorname{ess\,sup}_{x\in Q} V(x)$. Then the left side of (5.2) with $T_\gamma f$ replaced by (5.3) is bounded by

$$\frac{E}{|Q|}\int_Q\left[\int_S \frac{|f(y)|dy}{|x-y|^{n-\gamma}}\right]dx + \frac{E}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q\left[\int_S\frac{|f(y)|dy}{|t-y|^{n-\gamma}}\right]dt\right)dx.$$

This is bounded by

$$\frac{2E}{|Q|}\int_S |f(y)|\left[\int_Q \frac{dx}{|x-y|^{n-\gamma}}\right]dy; \tag{5.5}$$

performing the inner integration shows that (5.5) is bounded by

$$CE|Q|^{-1+\gamma/n}\int_S |f(y)|\,dy. \tag{5.6}$$

Hölder's inequality shows that (5.6) is bounded by

$$CE|Q|^{-1/p'}\left[\int_S |f(x)V(x)|^p\,dx\right]^{1/p}\left[\int_S [V(x)]^{-p'}\,dx\right]^{1/p'}. \tag{5.7}$$

Condition (5.1) then shows that (5.7) is bounded above by the right side of (5.2) as desired.

The left side of (5.2) with $T_\gamma f$ replaced by (5.4) can be written in the form

$$\frac{E}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q\left[\int_T f(y)(|x-y|^{\gamma-n}-|t-y|^{\gamma-n})\,dy\right]dt\right|dx. \tag{5.8}$$

Since x and t are in Q and y is in T,

$$\big||x-y|^{\gamma-n}-|t-y|^{\gamma-n}\big| \le C|Q|^{1/n}|x-y|^{\gamma-n-1}.$$

Using this fact in (5.8) shows that (5.8) is bounded by

$$\frac{CE}{|Q|}\int_Q\left[\int_T \frac{|f(y)|\,|Q|^{1/n}dy}{|x-y|^{n-\gamma+1}}\right]dx. \tag{5.9}$$

Interchanging the order of integration and performing the inner integration shows that (5.9) is bounded by

$$CE\int_T \frac{|Q|^{1/n}|f(y)|}{|q-y|^{n-\gamma+1}}\,dy \tag{5.10}$$

where q is the center of Q. A use of Hölder's inequality shows that (5.10) is bounded by the product of the right side of (5.2) and

$$E|Q|^{1/n}\left(\int_T |q-y|^{(n-\gamma+1)(-p')}[V(y)]^{-p'}\,dy\right)^{1/p'}. \tag{5.11}$$

The proof of Theorem 7 can be completed by showing that (5.11) has a bound independent of Q.

Let Q_k be the cube with the same orientation and center as Q and sides 2^k times as long. Then $T = \bigcup_{k=2}^{\infty}[Q_k - Q_{k-1}]$ and (5.11) is bounded by

$$CE|Q|^{1/n}\sum_{k=2}^{\infty}\left(\int_{Q_k-Q_{k-1}}[2^k|Q|^{1/n}]^{(n-\gamma+1)(-p')}[V(y)]^{-p'}\,dy\right)^{1/p'}. \tag{5.12}$$

The facts that $|Q_k| = 2^{kn}|Q|$ and $n-\gamma = n/p'$ show that (5.12) is bounded by

$$CE\sum_{k=2}^{\infty}2^{-k}\left(\frac{1}{|Q_k|}\int_{Q_k}[V(y)]^{-p'}\,dy\right)^{1/p'}. \tag{5.13}$$

The hypothesis (5.1) then shows that (5.13) is bounded.

The strengthened version of Theorem 7 consists of showing that the left side of (5.2) is bounded by a constant times $\|f(x)V(x)\|_{p,\infty}$; for the definition of $\|\ \|_{p,q}$ see [4]. To do this the following lemma will be needed.

Lemma 2. *If $V(x)$ satisfies the hypothesis of Theorem 7, then for every cube Q*

$$\|\chi_Q(x)/V(x)\|_{p',1} \le C\left(\int_Q [V(x)]^{-p'}\,dx\right)^{1/p'} \tag{5.14}$$

where C is independent of Q and χ_Q denotes the characteristic function of Q.

To prove this let $g(x)$ be the nonincreasing rearrangement of $\chi_Q(x)/V(x)$. By [4, p. 258], the left side of (5.14) is bounded by a constant times

$$\int_0^{|Q|} g(x)x^{-1/p}\,dx. \tag{5.15}$$

By the definition (3.8), $[V(x)]^{-p'}$ satisfies condition A_1. By [7, Lemma 6, p. 214], there is an $s > p'$ such that $[V(x)]^{-s}$ also satisfies condition A_1. By Hölder's inequality (5.15) is bounded by

$$\left(\int_0^{|Q|}[g(x)]^s\,dx\right)^{1/s}\left(\int_0^{|Q|}x^{-s'/p}\,dx\right)^{1/s'}. \tag{5.16}$$

Using the fact that $s' < p$ and the definition of g shows that (5.16) is bounded by

$$C|Q|^{1/s}\left(\frac{1}{|Q|}\int_Q [V(x)]^{-s}\,dx\right)^{1/s}|Q|^{-1/p+1/s'}. \tag{5.17}$$

The fact that $[V(x)]^{-s}$ satisfies A_1 shows that (5.17) is bounded by

$$C|Q|^{1/p'}\left(\operatorname*{ess\,sup}_{x\in Q}[V(x)]^s\right)^{-1/s}$$

and this is clearly bounded by the right side of (5.14).

The proof of the strengthened version of Theorem 7 is the same as that of the original Theorem 7 except at two points. In passing from (5.6) to the appropriate version of (5.7) the $L(p,q)$ Hölder inequality [4, p. 262], is used to show that (5.6) is bounded by

$$CE|Q|^{-1/p'}\|f(x)V(x)\|_{p,\infty}\|\chi_S(x)/V(x)\|_{p',1};$$

the desired inequality for (5.3) then follows from Lemma 2.

Similarly the estimation of (5.4) is the same up through (5.10); (5.11) is replaced by

$$E|Q|^{1/n}\|\chi_T(y)/|q-y|^{n-\gamma+1}V(y)\|_{p',1}. \tag{5.18}$$

Since these norms satisfy the triangle inequality [4, p. 259], (5.18) is bounded by

$$CE|Q|^{1/n}\sum_{k=2}^{\infty}\left\|\frac{\chi_{Q_k-Q_{k-1}}(y)}{[2^k|Q|^{1/n}]^{(n-\gamma+1)}V(y)}\right\|_{p',1}$$

and this is bounded by

$$CE\sum_{k=2}^{\infty}2^{-k}|Q_k|^{-1/p'}\left\|\frac{\chi_{Q_k}(y)}{V(y)}\right\|_{p',1}.$$

The proof is then completed by using Lemma 2 and (5.1).

6. Necessity for $q = \infty$.

Theorem 8. *Assume that $0 < \gamma < n$, $p = n/\gamma$, $p' = n/(n-\gamma)$ and $V(x)$ is a nonnegative function on R^n such that* (5.2) *is true with C independent of Q and f. Then* (5.1) *is true with K independent of Q.*

To prove this it will first be shown that there is a $k > 1$ such that for every cube, Q, and every y in Q,

$$\int_{kQ}|x-y|^{\gamma-n}dx \le \frac{k^n}{2}\int_Q|x-y|^{\gamma-n}dx, \tag{6.1}$$

where kQ denotes the cube with the same orientation and center as Q and sides k times as long. To prove the existence of k let S be the sphere with center y and radius equal to the diameter of kQ. Then $kQ \subset S$ and the radius of S is $kn^{1/2}|Q|^{1/n}$. The ieft side of (6.1) is bounded by the integral over S of the same integrand and a simple computation with polar coordinates shows that the left side of (6.1) is bounded by $(A/\gamma)[kn^{1/2}|Q|^{1/n}]^\gamma$ where A is the "area" of the unit sphere in R^n. Similarly the integral on the right side of (6.1) is bounded below by 2^{-n} times the integral of $|x-y|^{\gamma-n}$ over the sphere with center y and radius $\frac{1}{2}|Q|^{1/n}$. This shows that the integral on the right side of (6.1) is bounded below by $(A/\gamma)2^{-n}[\frac{1}{2}|Q|^{1/n}]^\gamma$. The ratio of the integral on the left side of (6.1) to the one on the right is then bounded by $[2n^{1/2}k]^\gamma 2^n$ and for k sufficiently large this is bounded by $\frac{1}{2}k^n$.

Now fix a cube, Q, let $f(x)$ be a nonnegative function integrable on Q and 0 off Q and let k be a number for which (6.1) is true. Then $(T_\gamma f)_Q - (T_\gamma f)_{kQ}$ equals

$$\frac{1}{|Q|}\int_Q\left[\int_Q|x-y|^{\gamma-n}f(y)\,dy\right]dx - \frac{k^{-n}}{|Q|}\int_{kQ}\left[\int_Q|x-y|^{\gamma-n}f(y)\,dy\right]dx.$$

Fubini's theorem shows that this equals

$$\frac{1}{|Q|}\int_Q f(y)\left[\int_Q |x-y|^{\gamma-n}\,dx - k^{-n}\int_{kQ}|x-y|^{\gamma-n}\,dx\right]dy.$$

Then (6.1) shows that this is bounded below by

$$\frac{1}{2|Q|}\int_Q f(y)\left[\int_Q |x-y|^{\gamma-n}\,dx\right]dy$$

and this is just $\frac{1}{2}(T_\gamma f)_Q$. Therefore,

$$(T_\gamma f)_Q \le 2[(T_\gamma f)_Q - (T_\gamma f)_{kQ}]. \tag{6.2}$$

By Minkowski's inequality the right side of (6.2) is bounded by

$$2\left(\frac{1}{|Q|}\int_Q |(T_\gamma f)_Q - T_\gamma f(x)|\,dx + \frac{1}{|Q|}\int_Q |T_\gamma f(x) - (T_\gamma f)_{kQ}|\,dx\right). \tag{6.3}$$

Changing the range of integration in the second integral to kQ and then using (5.2) on both integrals shows that (6.3) is bounded by

$$C\left[\operatorname*{ess\,sup}_{x\in Q} V(x)\right]^{-1}\left(\int_{R^n}[f(x)V(x)]^p\,dx\right)^{1/p}.$$

Since f is 0 off Q this finally shows that

$$(T_\gamma f)_Q \le C\left[\operatorname*{ess\,sup}_{x\in Q} V(x)\right]^{-1}\left(\int_Q [f(x)V(x)]^p\,dx\right)^{1/p}. \tag{6.4}$$

Now if $\int_Q [V(x)]^{-p'}\,dx = 0$, (5.1) is immediate because of the convention $0\cdot\infty = 0$. If $0 < \int_Q [V(x)]^{-p'}\,dx < \infty$, let $f(x) = [V(x)]^{-p'}$ on Q and 0 elsewhere. From the fact that

$$\frac{1}{|Q|}\int_Q\left[\int_Q \frac{[V(y)]^{-p'}}{|Q|^{(n-\gamma)/n}}dy\right]dx \le \frac{C}{|Q|}\int_Q\left[\int_Q \frac{[V(y)]^{-p'}}{|x-y|^{n-\gamma}}dy\right]dx$$

it follows that

$$|Q|^{-1+\gamma/n}\int_Q [V(y)]^{-p'}\,dy \le C(T_\gamma f)_Q. \tag{6.5}$$

Then combining (6.4) and (6.5) and replacing the f on the right side by its definition leads immediately to (5.1).

If $\int_Q [V(x)]^{-p'}\,dx = \infty$, then there is a nonnegative function, $g(x)$, such that $[g(x)]^p$ is integrable on Q but $g(x)/V(x)$ is not. Choose a positive integer, m, and let $f(x)$ be the smaller of m and $g(x)/V(x)$ on Q and 0 outside Q. Then the left side of (6.4) approaches ∞ as m approaches ∞. Since the integral on the right side of (6.4) is a bounded function of m, $\operatorname{ess\,sup}_{x\in Q} V(x) = 0$ and (5.1) is proved in this case also.

7. A weighted version of the Sobolev imbedding theorem. The usual version of Sobolev's theorem can be found, for example, on p. 124 of [9]. The theorem stated below is a weighted version of part of Sobolev's theorem; more general results could be proved as mentioned at the end of this section.

Theorem 9. *If $1 < p < n$, $1/q = 1/p - 1/n$ and $V(x)$ satisfies* (1.1), *then*

$$\|f(x)V(x)\|_q \le C\left(\|f(x)V(x)\|_p + \sum_{j=1}^n \left\|\frac{\partial f(x)}{\partial x_j}V(x)\right\|_p\right)$$

where C is independent of f and $\partial f(x)/\partial x_j$ is taken in the sense of distributions.

Suppose f is infinitely differentiable with compact support. As shown on p.125 of [9],

$$f(x) = \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \int_{R^n} \frac{\partial f(x-y)}{\partial x_j} \frac{y_j}{|y|^n} dy$$

where ω_{n-1} is the "area" of the unit sphere S^{n-1} in R^n. Therefore, for such f,

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \int_{R^n} \left| \frac{\partial f(x-y)}{\partial x_j} \right| \frac{dy}{|y|^{n-1}}.$$

Now multiplying both sides by $V(x)$ and taking L^q norms, it follows that

$$\|f(x)V(x)\|_q \leq \frac{1}{\omega_{n-1}} \sum_{j=1}^{n} \left\| T_1\left(\left| \frac{\partial f(x)}{\partial x_j} \right| \right) V(x) \right\|_q.$$

By Theorem 4

$$\|f(x)V(x)\|_q \leq C \sum_{j=1}^{n} \left\| \frac{\partial f(x)}{\partial x_j} V(x) \right\|_p.$$

A weighted version of Proposition 1, p. 122 of [9] then completes the proof.

An induction argument can be used to generalize Theorem 9 to functions f with derivatives up to order k in weighted L^p spaces, $k < n$.

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