INTERPOLATION IN A CLASSICAL HILBERT SPACE OF ENTIRE FUNCTIONS

BY

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ABSTRACT. Let $H$ denote the Paley-Wiener space of entire functions of exponential type $\pi$ which belong to $L^2(-\infty, \infty)$ on the real axis. A sequence $\{\lambda_n\}$ of distinct complex numbers will be called an interpolating sequence for $H$ if $TH \subseteq l^2$, where $T$ is the mapping defined by $Tf = \{f(\lambda_n)\}$. If in addition $\{\lambda_n\}$ is a set of uniqueness for $H$, then $\{\lambda_n\}$ is called a complete interpolating sequence. The following results are established. If $\operatorname{Re}(\lambda_{n+1}) - \operatorname{Re}(\lambda_n) \geq \gamma > 1$ and if the imaginary part of $\lambda_n$ is sufficiently small, then $\{\lambda_n\}$ is an interpolating sequence. If $|\operatorname{Re}(\lambda_n) - n| \leq L \leq (\log 2)/\pi (-\infty < n < \infty)$ and if the imaginary part of $\lambda_n$ is uniformly bounded, then $\{\lambda_n\}$ is a complete interpolating sequence and $\{e^{\lambda_i x}\}$ is a basis for $l^2(\pi, \pi)$. These results are used to investigate interpolating sequences in several related spaces of entire functions of exponential type.

Introduction. Let $H$ denote the Paley-Wiener space of entire functions $f$ of exponential type $\pi$ for which

$$
\|f\| = \left[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx \right]^{1/2} < \infty.
$$

A sequence $\{\lambda_n\}$ of distinct complex numbers will be called an interpolating sequence for $H$ if for each sequence $\{c_n\}$ in $l^2$ there is at least one function $f$ in $H$ satisfying $f(\lambda_n) = c_n (-\infty < n < \infty)$. (Unless otherwise stated, the term sequence in this paper will always mean a two-sided sequence.)

In §§2 and 3 of this paper we study both necessary and sufficient conditions for a sequence $\{\lambda_n\}$ to be interpolating. For the most part we require that the $\lambda_n$ lie in a strip parallel to the real axis. Under this condition we show that $l^2$ is the natural sequence space to interpolate with functions in $H$ and a necessary condition for interpolation is that the $\lambda_n$ be separated, that is, $|\lambda_n - \lambda_k| \geq \gamma$ for some constant $\gamma > 0$ and all $n \neq k$.

A classical theorem of Paley and Wiener [8, p. 13] shows that the Fourier transform of every function in $H$ vanishes almost everywhere outside $(-\pi, \pi)$. Thus $f$ belongs to $H$ if and only if it is of the form

$$
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{izt} \, dt
$$

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for some \( g \) in \( L^2(-\pi, \pi) \). It follows that \( \{\lambda_n\} \) is an interpolating sequence if and only if the trigonometric moment problem

\[
(2) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{in \lambda} \, dt \quad (-\infty < n < \infty)
\]

has a solution \( g \) in \( L^2(-\pi, \pi) \) whenever \( \sum |c_n|^2 < \infty \). The following result was first established by Boas [3] and later reproved in a more abstract setting by N. Bari [1]. (See also [10].)

**Lemma 1.** A necessary and sufficient condition that the trigonometric moment problem (2) have a solution in \( L^2(-\pi, \pi) \) for every square summable sequence \( \{c_n\} \) is that the inequality

\[
(3) \quad A \sum |a_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum a_n e^{in \lambda}|^2 \, dt
\]

holds for some constant \( A > 0 \) and all finite sequences \( \{a_n\} \).

Several years earlier, Ingham [7] had established the validity of (3) in the special case in which the \( \lambda_n \) are real and satisfy the separation condition \( \lambda_{n+1} - \lambda_n \geq \gamma > 1 \) \((-\infty < n < \infty)\). He showed that his result is the best possible, in the sense that \( \gamma \) cannot be taken equal to 1, by demonstrating that the sequence \( \{\lambda_n\} \) given by

\[
\lambda_n = n - \frac{1}{4}, \quad \lambda_{-n} = -\lambda_n \quad (n = 1, 2, \ldots)
\]

does not satisfy (3) for any positive \( A \). By modifying Ingham’s proof, we are able to show that \( \{\lambda_n\} \) is an interpolating sequence whenever

\[
\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1 \quad (-\infty < n < \infty)
\]

and the imaginary part of \( \lambda_n \) is sufficiently small. As a corollary we show that if \( \{\lambda_n\} \) is a sequence of points lying in a strip parallel to the real axis and if \( \text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \to \infty \) as \( n \to \pm \infty \), then for each \( \{c_n\} \) in \( l^2 \) and each \( \tau > 0 \) there exists an entire function \( f \) of exponential type \( \tau \), square integrable on the real axis, for which \( f(\lambda_n) = c_n \) \((-\infty < n < \infty)\).

In their classic treatise, Paley and Wiener showed [8, p.115] that every sequence of real numbers \( \{\lambda_n\} \) which are close to the integers in the sense that \( |\lambda_n - n| < L < 1/\pi^2 \) \((-\infty < n < \infty)\) is an interpolating sequence. Moreover, they showed that every function \( g \) in \( L^2(-\pi, \pi) \) has a *nonharmonic* Fourier series expansion

\[
(4) \quad g(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{i\lambda_n t}
\]

with \( \sum |c_n|^2 < \infty \). These results were improved by Duffin and Eachus [5] who showed that \( \lambda_n \) can be complex and that the constant \( 1/\pi^2 \) can be replaced by \( (\log 2)/\pi \). In the present paper we show that \( \{\lambda_n\} \) is an interpolating sequence
whenever $|\text{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi$ ($-\infty < n < \infty$) and the imaginary part of $\lambda_n$ is uniformly bounded, and that under these conditions every function in $L^2(-\pi, \pi)$ has the representation (4) with some sequence $\{c_n\}$ in $l^2$.

The main results of §§2 and 3 are used in §4 to investigate interpolating sequences in several related spaces of entire functions of exponential type. Specifically, we consider the spaces $E_p^\tau$ of entire functions of exponential type $\tau$ which belong to $L^p(-\infty, \infty)$ on the real axis. We restrict attention mainly to the special cases $p = 1$ and $p = \infty$.

1. **Background material.** Under the inner product

$$ (f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx, $$

$H$ is a functional Hilbert space with the reproducing kernel

$$ K(\xi, z) = \sin \pi(\xi - z)/\pi(\xi - z). $$

The functions $K(\xi, n), -\infty < n < \infty$, form a complete orthonormal system, so that for each $f$ in $H$

$$ f(z) = \sum_{-\infty}^{\infty} f(n) \frac{\sin \pi(z - n)}{\pi(z - n)}. $$

By Parseval's formula

$$ \|f\|^2 = \sum_{-\infty}^{\infty} |f(n)|^2 \quad (f \in H). $$

If $f$ is given by (1), then $g$ is the Fourier transform of $f$ and Plancherel's theorem gives

$$ \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)|^2 \, dx. $$

It follows easily from (1) and (6) that

$$ |f(x + iy)| \leq e^{\pi|y|} \|f\| $$

for every $f$ belonging to $H$. Similar estimates show that $H$ is closed under differentiation and that

$$ \|f'\| \leq \pi \|f\| \quad (f \in H). $$

2. **Interpolation in $H$.** Let $\{\lambda_n\}$ be a sequence of distinct complex numbers. For each $f$ in $H$ we define $Tf$ to be the sequence $Tf = \{f(\lambda_n)\}$ ($-\infty < n < \infty$).

**Definition.** The sequence $\{\lambda_n\}$ is said to be an interpolating sequence for $H$ if $TH \supset l^2$. It is not difficult to show that if $TH \supset l^2$, then the unit ball in $l^2$ can be interpolated in a uniformly bounded way. In fact, we have the following stronger result. For a proof see [9, p. 19].
Lemma 2. Let $X$ be a Banach space, $\{\mu_n\}$ a sequence in the dual space $X'$ and $T$ the mapping on $X$ defined by $Tx = (\mu_n(x))$. If $TX \supset l^p$ for some $p$, $1 \leq p \leq \infty$, then $TM$ covers the unit ball of $l^p$ for some bounded subset $M$ of $X$.

Definition. The sequence $\{\lambda_n\}$ is said to be separated if there is a constant $\gamma$ such that $|\lambda_n - \lambda_k| \geq \gamma > 0$ $(n \neq k)$.

It is well known [4, p. 101] that if the $\lambda_n$ are real and separated and if $f$ is an entire function of exponential type $\tau$ belonging to $L^p(-\infty, \infty)$ on the real axis $(0 < p < \infty)$, then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where $B$ depends only on $p$, $\tau$ and the separation constant $\gamma$. It is no more difficult to establish the following result, which we state without proof.

Lemma 3. Let $\{\lambda_n\}$ be a complex sequence satisfying

$$|\Im(\lambda_n)| \leq \alpha, \quad |\lambda_n - \lambda_k| \geq \gamma \quad (n \neq k)$$

for some positive constants $\alpha$ and $\gamma$. If $f$ is an entire function of exponential type $\tau$, belonging to $L^p(-\infty, \infty)$ on the real axis, then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where $B = B(p, \tau, \gamma, \alpha)$.

Proposition 1. Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TH \supset l^1$, then $\{\lambda_n\}$ is separated.

Proof. Since point-evaluations are continuous linear functionals on $H$, Lemma 2 shows that there are functions $f_n$ in $H$ and a constant $M > 0$ such that $f_n(\lambda_k) = \delta_{nk}$, $\|f_n\| \leq M$ (for all $n,k$). For $n \neq k$, we have

$$1 = f_n(\lambda_n) - f_n(\lambda_k) = \int_{\lambda_k}^{\lambda_n} f_n'(z) \, dz$$

so that

$$1 \leq \int_{\lambda_k}^{\lambda_n} |f_n'(z)| \, |dz| \leq \sup |f_n'(z)| \cdot |\lambda_n - \lambda_k|,$$

where the supremum is taken over the straight line segment from $\lambda_k$ to $\lambda_n$. If $|\Im(\lambda_n)| \leq \alpha$, then (7), (8) show that

$$1 \leq e^{\alpha n} \|f_n\| \cdot |\lambda_n - \lambda_k|$$

$$\leq \pi e^{\alpha n} \|f_n\| \cdot |\lambda_n - \lambda_k|$$

$$\leq M \pi e^{\alpha n} |\lambda_n - \lambda_k|,$$

and the result follows with $\gamma = e^{-\alpha n}/M\pi$. 

Corollary 1. Let \( \{ \lambda_n \} \) be a sequence of points lying in a strip parallel to the real axis. If \( TH \subseteq l^1 \), then \( TH \subseteq l^2 \).

Since separation is a necessary condition for a sequence of real numbers to be interpolating, the problem arises to determine if there is a constant \( \gamma \) with the property that \( \{ \lambda_n \} \) is an interpolating sequence whenever \( \lambda_{n+1} - \lambda_n \geq \gamma \). The following result will show that \( \{ \lambda_n \} \) is interpolating whenever \( \gamma > 1 \). The proof is a simple extension of an argument given by Ingham [7].

**Lemma 4.** Let \( f(t) = \sum_{n=-N}^{N} a_n e^{i\lambda_n t} \) where the \( \lambda_n \) satisfy

\[
\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1, \quad |\text{Im}(\lambda_n)| \leq \alpha.
\]

Then

\[
A \sum |a_n|^2 \leq \int_{-\pi}^{\pi} |f(t)|^2 dt,
\]

where \( A = 4\left[1/(1 + 16\alpha^2) - e^{2\alpha^2}\gamma^2\right] \).

**Proof.** If \( k \) belongs to \( L^1(-\infty, \infty) \) and \( K(z) = \int_{-\infty}^{\infty} k(t)e^{zt} dt \), then

\[
\int_{-\infty}^{\infty} k(t)|f(t)|^2 dt = \sum_{m,n} a_m \bar{a}_n K(\lambda_m - \lambda_n).
\]

Letting

\[
k(t) = \cos \frac{t}{2}, \quad |t| \leq \pi,
\]

\[
= 0, \quad |t| > \pi,
\]

we get \( K(z) = 4(\cos \pi z)/(1 - 4z^2) \). Since \( |k(t)| \leq 1 \), it follows from (10) that

\[
\int_{-\pi}^{\pi} |f(t)|^2 dt \geq \left| \sum a_n \bar{a}_n K(\lambda_m - \lambda_n) \right|.
\]

where the prime in the summation denotes omission of the term \( m = n \). The remainder of the proof is devoted to obtaining suitable estimates for the two sums in absolute value above. Since, by (9), \( |\lambda_m - \lambda_n| \geq |m - n|\gamma \geq 1 \) \( (m \neq n) \) and since \( |\cos(x + iy)| \leq e^{x^2} \), we have

\[
\sum_{m,n} |K(\lambda_m - \lambda_n)| \leq 4e^{2\alpha^2} \sum_{m,n} \frac{1}{4(m - n)^2 \gamma^2 - 1} \leq \frac{8e^{2\alpha^2}}{\gamma^2} \sum_{r=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{4e^{2\alpha^2}}{\gamma^2}.
\]

Since \( 2|a_m a_n| \leq |a_m|^2 + |a_n|^2 \), we get

\[
\sum_{m,n} a_m \bar{a}_n K(\lambda_m - \lambda_n) = \theta \sum_{m,n} \frac{|a_m|^2 + |a_n|^2}{2} |K(\lambda_m - \lambda_n)|,
\]

where \( |\theta| \leq 1 \), and since \( |K(z)| = |K(-z)| \).
\[ \sum_{m,n} a_m a_n K(\lambda_m - \lambda_n) = \theta \sum_n |a_n|^2 \left\{ \sum_m |K(\lambda_m - \lambda_n)| \right\} \]

\[ < \frac{4e^{2\pi}}{\gamma^2} \sum_n |a_n|^2. \]

If we set \( \beta_n = \text{Im}(\lambda_n) \), then

\[ K(2i\beta_n) = \frac{4 \cosh 2\pi \beta_n}{1 + 16\beta_n^2} \geq \frac{4}{1 + 16\alpha^2}, \]

and hence

\[ \sum_n |a_n|^2 K(\lambda_n - \lambda_n) \geq \frac{4}{1 + 16\alpha^2} \sum_n |a_n|^2, \]

and the proof is complete.

**Theorem 1.** Let \( \{\lambda_n\} \) be a complex sequence satisfying

\[ \text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1, \]

\[ |\text{Im}(\lambda_n)| < \alpha \quad (-\infty < n < \infty). \]

Then \( \{\lambda_n\} \) is an interpolating sequence for all sufficiently small values of \( \alpha \).

**Proof.** The result follows immediately from Lemmas 1 and 4 since the value of \( A \) given in the statement of Lemma 4 approaches \( 4(1 - 1/\gamma^2) > 0 \) as \( \alpha \to 0 \).

**Remark.** Shapiro and Shields have shown [10, p. 532] that if \( \{\lambda_n\} \) is a separated sequence of real numbers and if \( \sum k^2 |a_n|^2 < \infty \), then for each \( \alpha > 0 \) there corresponds a function \( f \) analytic in the strip \( D: |y| < \alpha \), with finite area norm

\[ \int_D |f(z)|^2 dx \, dy < \alpha, \]

such that \( f(\lambda_n) = c_n \) \( (-\infty < n < \infty) \). Since (11) is satisfied for each function in \( H \), an obvious extension of Theorem 1 shows that \( \tau \) may in fact be chosen to be entire of exponential type.

**Theorem 2.** Let \( \{\lambda_n\} \) be a sequence of distinct points lying in a strip parallel to the real axis and suppose that \( \text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \to \infty, n \to \pm \infty \). For each \( \tau > 0 \) and each sequence \( \{c_n\} \) in \( l^2 \) there exists an entire function \( f \) of exponential type \( \tau \), belonging to \( L^2(-\infty, \infty) \) on the real axis, for which \( f(\lambda_n) = c_n \) \( (-\infty < n < \infty) \).

**Proof.** Fix \( \tau > 0 \) and let \( H_\tau \) denote the space of entire functions of exponential type \( \tau \) which belong to \( L^2(-\infty, \infty) \) on the real axis. It follows from Theorem 1 that there exists a smallest integer \( N \geq 0 \) with the property that for each sequence \( \{c_n\} \) in \( l^2 \) there corresponds at least one function \( f \) in \( H_\tau \) such that \( f(\lambda_n) = c_n \) \( |n| \geq N \). If \( N = 0 \) there is nothing to show, so suppose that \( N \geq 1 \). Choose \( g \) in \( H_\tau \) such that
We begin by constructing a function \( h \) in \( H \) such that
\[
(12) \quad h(\lambda_{N-1}) = 1, \quad h(\lambda_n) = 0, \quad |n| \geq N.
\]
If \( g(\lambda_{N-1}) = 0 \), then for suitable constants \( c \) and \( m \geq 1 \) the function
\[
h(z) = \frac{g(z)}{c(z - \lambda_{N-1})^m} \left( z - \frac{\lambda_{N-1}}{N - \lambda_{N-1}} \right)
\]
satisfies (12). If \( g(\lambda_{N-1}) \neq 0 \), then \( h \) must be obtained by a different method. By a theorem of Titchmarsh [11] the zeros of \( g \) have a positive density, where it is understood that multiple zeros are counted as many times as their multiplicity warrants. It is easily shown that \( \{\lambda_n\} \) has density zero, so that some zero \( \lambda \) of \( g \) is either not in the sequence or else is in the sequence and is a multiple zero of \( g \). In either case, the function
\[
h(z) = \frac{\lambda_{N-1} - \lambda}{\lambda_{N-1} - \lambda_n} \left[ g(z) \frac{z - \lambda_{N-1}}{g(\lambda_{N-1})} \right]
\]
satisfies (12).

Now fix \( \{c\} \) in \( I^2 \) and choose \( f \) in \( H \) such that \( f(\lambda_n) = c_n \ (|n| \geq N) \). Let \( g \) and \( h \) be chosen as above and define
\[
f_1(z) = f(z) + [c_{N-1} - f(\lambda_{N-1})]h(z).
\]
Then \( f_1 \) belongs to \( H \) and \( f_1(\lambda_n) = c_n \ (|n| \geq N \text{ and } n = N - 1) \). The same construction gives \( f_2 \in H \) such that \( f_2(\lambda_n) = c_n \ (|n| \geq N - 1) \). But this contradicts the choice of \( N \); hence \( N \) must be equal to zero, and the proof is complete.

**Theorem 3.** Let \( \{\lambda_n\}, n = 1, 2, \ldots, \) be an interpolating sequence. There exist positive numbers \( \delta_n \) such that \( \{\mu_n\} \) is an interpolating sequence whenever \( |\lambda_n - \mu_n| < \delta_n \).

**Proof.** It follows from Lemma 2 that there is a constant \( M > 0 \) depending only on \( \{\lambda_n\} \) with the property that for each sequence \( \{c\} \) in the unit ball of \( I^2 \) there corresponds at least one function \( f \) in \( H \) for which \( f(\lambda_n) = c_n \ (n = 1, 2, \ldots) \) and \( \|f\| \leq M \).

Let \( \mathcal{F}_1 \) denote the family of all functions \( f \) in \( H \) for which \( \|f\| \leq M \). Then (7) shows that the functions in \( \mathcal{F}_1 \) are uniformly bounded on compacta. It follows that \( \mathcal{F}_1 \) is equicontinuous on compacta and hence in particular on the disk \( |z - \lambda_1| \leq 1 \). Thus, if \( 0 < \varepsilon_1 < 1 \), there is a corresponding \( \delta_1 = \delta_1(\varepsilon_1) \) such that \( |f(z) - f(\lambda_1)| < \varepsilon \ (|z - \lambda_1| < \delta_1) \), uniformly for all \( f \) in \( \mathcal{F}_1 \). In addition, \( \delta_1 \) may be chosen small enough so that the disk \( |z - \lambda_1| < \delta_1 \) intersects \( \{\lambda_n\} \) only at \( \lambda_1 \).
We are going to show that if $|\lambda_1 - \mu_1| < \delta_1$, then $\{\mu_1, \lambda_2, \lambda_3, \ldots\}$ is an interpolating sequence. Clearly, it will be enough to show that the unit ball of $l^2$ can be interpolated. More precisely, we will show that for each $\{c_n\}$ with $\sum |c_n|^2 \leq 1$ there corresponds a function $F_1$ in $H$ for which

$$F_1(\mu_1) = c_1,$$

$$F_1(\lambda_n) = c_n, \quad n = 2, 3, \ldots,$$

$$\|F_1\| \leq M/(1 - \epsilon_1),$$

Fix $\mu_i$ with $|\lambda_1 - \mu_i| < \delta_1$ and let $\sum |c_n|^2 \leq 1$. There exists a function $g$ in $H$ such that

$$g(\lambda_n) = c_n, \quad \|g\| \leq M, \quad n = 1, 2, \ldots.$$  

Since $g$ is in $\mathcal{G}_1$, $|g(\mu_i) - g(\lambda_i)| < \epsilon_1$. Also, there exists a function $f$ in $H$ such that

$$f(\lambda_i) = 1,$$

$$f(\lambda_n) = 0, \quad n = 2, 3, \ldots,$$

$$\|f\| \leq M,$$

Then $f$ is also in $\mathcal{G}_1$ so that $|f(\mu_i) - f(\lambda_i)| < \epsilon_1$, and hence $|f(\mu_i)| > 1 - \epsilon_1 > 0$. Now, set

$$F_1(z) = g(z) + \frac{c_1 - g(\mu_1)}{f(\mu_1)} f(z).$$

Clearly, $F_1$ is in $H$, $F_1(\mu_i) = c_1$, $F_1(\lambda_n) = c_n$ $(n > 1)$, and

$$\|F_1\| \leq \|g\| + \frac{|c_1 - g(\mu_1)| \|f\|}{\|f(\mu_1)\|}$$

$$\leq \|g\| + \frac{|g(\lambda_1) - g(\mu_1)| \|f\|}{\|f(\mu_1)\|}$$

$$\leq M/(1 - \epsilon_1/(1 - \epsilon_1)) = M/(1 - \epsilon_1).$$

The above argument can be repeated with $\{\lambda_n\}$ replaced by $\{\mu_i, \lambda_2, \lambda_3, \ldots\}$. Thus, we let $\mathcal{G}_2$ denote the family of all functions $f$ in $H$ for which $\|f\| \leq M/(1 - \epsilon_1)$. Then $\mathcal{G}_2$ is equicontinuous on each compact set and for $0 < \epsilon_2 < 1$ we find $\delta_2 = \delta_2(\epsilon_1, \epsilon_2)$ such that

$$|f(z) - f(\lambda_2)| < \epsilon_2 \quad (|z - \lambda_2| < \delta_2),$$

uniformly for all $f$ in $\mathcal{G}_2$. We note that $\delta_2$ is independent of $\mu_i$ and may be chosen so that the disks $|z - \lambda_1| < \delta_1$ and $|z - \lambda_2| < \delta_2$ are disjoint and intersect $\{\lambda_n\}$ only at $\lambda_1$ and $\lambda_2$, respectively. Just as before, we show that whenever $|\mu_2 - \lambda_2| < \delta_2$, the sequence $\{\mu_1, \mu_2, \lambda_3, \lambda_4, \ldots\}$ is interpolating, and that for each sequence $\{c_n\}$ in the unit ball of $l^2$ there corresponds a function $F_2$ in $H$ for which
The above process may be iterated. Thus, given a sequence \( \{\varepsilon_n\} \), \( 0 < \varepsilon_n < 1 \), we obtain a corresponding sequence \( \{\delta_n\} \), \( \delta_n = \varepsilon_n \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) \( > 0 \), with the property that for each positive integer \( N \) the sequence \( \{\mu_n, \mu_2, \ldots, \mu_N, \lambda_{N+1}, \lambda_{N+2}, \ldots\} \) is interpolating whenever \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots, N) \), and such that for every sequence \( \{c_n\} \) with \( \sum |c_n|^2 \leq 1 \) there exists a function \( F_N \) in \( H \) for which

\[
F_N(\mu_n) = c_n, \quad n = 1, 2, \ldots, N,
\]

\[
F_N(\lambda_n) = c_n, \quad n > N,
\]

\[
\|F_N\| \leq M \frac{1}{1 - \varepsilon_1} \frac{1}{1 - \varepsilon_2} \cdots \frac{1}{1 - \varepsilon_N}.
\]

Now, choose \( \{\varepsilon_n\} \), \( 0 < \varepsilon_n < 1 \), so that

\[
\varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{1 - \varepsilon_n} < \infty,
\]

and let the corresponding sequence \( \{\delta_n\} \) be determined as above. Fix \( \{\mu_n\} \) with \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots) \) and let \( \{c_n\} \) belong to the unit ball of \( L^2 \). For each positive integer \( N \) there exists a function \( F_N \) in \( H \) such that

\[
F_N(\mu_n) = c_n, \quad n = 1, 2, \ldots, N,
\]

\[
\|F_N\| \leq M \prod_{n=1}^{N} \left( 1 + \frac{\varepsilon_n}{1 - \varepsilon_n} \right)
\]

\[
\leq M \exp \left[ \sum_{n=1}^{\infty} \frac{\varepsilon_n}{1 - \varepsilon_n} \right] = Me^\varepsilon.
\]

Since \( \|F_N\| \) is uniformly bounded, a subsequence of \( \{F_N\} \) will converge weakly to a function \( F \) in \( H \) for which \( F(\mu_n) = c_n \) \( (n = 1, 2, \ldots) \). Thus, the unit ball of \( L^2 \) can be interpolated and the proof is complete.

3. Uniqueness: complete interpolating sequences.

**Proposition 2.** Let \( \{\lambda_n\} \), \( n = 1, 2, \ldots \), be an interpolating sequence. Each of the following statements implies the others.

(i) The set of relations \( f \in H \) and \( f(\lambda_n) = 0 \) \( (n = 1, 2, \ldots) \) imply that \( f = 0 \), that is, \( \{\lambda_n\} \) is a set of uniqueness for \( H \).

(ii) The exponentials \( \{e^{i\lambda_n}\} \) are complete in \( L^2(-\pi, \pi) \).

(iii) The sequence \( \{\lambda_n\} \) is contained in no larger interpolating sequence.
Proof. It follows immediately from the Paley-Wiener representation (1) that (i) and (ii) are equivalent. Since (ii) clearly implies (iii) it remains only to show that (iii) implies (ii).

Suppose then that \{e^{i\lambda_n}\} is not complete in \(L^2(-\pi, \pi)\) and let \(\lambda_0\) be distinct from the \(\lambda_n\). We show that \(\{\lambda_n\}, n = 0, 1, 2, \ldots\), is an interpolating sequence. Since \(\{e^{i\lambda_n}\}\) is incomplete there is a function \(g\) in \(L^2(-\pi, \pi)\), \(g \neq 0\), such that

\[
\int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt = 0, \quad n = 1, 2, \ldots.
\]

Setting \(h(z) = \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt\), we have \(h \in H, h \neq 0\), and \(h(\lambda_n) = 0 (n = 1, 2, \ldots)\). If \(h(\lambda_0) \neq 0\), we set \(F(z) = h(z)/h(\lambda_0)\), while if \(h(\lambda_0) = 0\), we take \(F(z) = h(z)/A(z - \lambda_0)^m\), where \(A\) and \(m\) are chosen so that \(F(\lambda_0) = 1\). In either case, \(F(\lambda_0) = 1\) and \(F(\lambda_n) = 0 (n = 1, 2, \ldots)\).

Fix \(\{c_n\}, n = 0, 1, 2, \ldots\), in \(L^2\). There is a function \(G\) in \(H\) with \(G(\lambda_n) = c_n (n = 1, 2, \ldots)\). Let

\[
f(z) = G(z) + [c_0 - G(\lambda_0)] F(z).
\]

Then \(f\) is in \(H\) and \(f(\lambda_n) = c_n (n = 0, 1, 2, \ldots)\).

Definition. An interpolating sequence satisfying any one of the conditions listed in Proposition 2 will be called a complete interpolating sequence.

Theorem 4. Let \(\{\lambda_n\}\) be a sequence of distinct points lying in a strip parallel to the real axis. If \(\{\Re(\lambda_n)\}\) is a complete interpolating sequence, then \(\{\lambda_n\}\) is a complete interpolating sequence.

The proof of Theorem 4 requires the following lemma.

Lemma 5. Let \(\lambda_n = \alpha_n + i\beta_n\), where \(\alpha_n\) and \(\beta_n\) are real and satisfy

\[
\alpha_{n+1} - \alpha_n \geq \gamma > 0, \quad |\beta_n| \leq \beta, \quad -\infty < n < \infty.
\]

If \(\{e^{i\lambda_n}\}\) is complete in \(L^2(-\pi, \pi)\), then \(\{e^{i\alpha_n}\}\) is also complete in \(L^2(-\pi, \pi)\).

Proof of Lemma 5. An equivalent problem is to show that the completeness of \(\{e^{i(\alpha_n + i)\lambda}\}\) implies that of \(\{e^{i(\alpha_n + i)\lambda}\}\). For this it is enough to show that the only function in \(H\) which vanishes at every point \(\alpha_n + i\) is identically zero. Arguing by contradiction, we suppose that for some \(f\) in \(H\), \(f \neq 0\), \(f(\alpha_n + i) = 0 (-\infty < n < \infty)\). Without any loss of generality we may suppose that no \(\alpha_n\) is an integer and that \(f(0) = 1\). We are going to exhibit a function \(g\) in \(H\) with \(g(\alpha_n + i) = 0 (-\infty < n < \infty)\) and \(g(0) = 1\), thereby contradicting the completeness of \(\{e^{i(\alpha_n + i)\lambda}\}\). Set

\[
f_N(z) = f(z) \prod_{n=-N}^{N} \frac{1 - z/(\alpha_n + i)}{1 - z/(\alpha_n + i)}, \quad N = 1, 2, \ldots.
\]

For each \(N\) we have \(f_N \in H, f_N(\alpha_n + i) = 0 (|n| \leq N)\), and \(f_N(0) = 1\). By (5),
We show that the products
\[
\sum_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 = \sum_{n=-N}^{N} \left| \frac{\alpha_n + i}{\lambda_n + i} \right|^2 \left| \frac{\lambda_n + i - k}{\alpha_n + i - k} \right|^2
\]
are uniformly bounded in \(N\) and \(k\). Simple calculations show that
\[
\left| \frac{\alpha_n + i}{\lambda_n + i} \right|^2 = \frac{\alpha_n^2 + 1}{\alpha_n^2 + (\beta_n + 1)^2} = 1 - \frac{\beta_n^2 + 2\beta_n}{\alpha_n^2 + (\beta_n + 1)^2} \leq 1 + \frac{\beta_n^2 + 2|\beta_n|}{\alpha_n^2 + (\beta_n + 1)^2} \leq 1 + \frac{\beta^2 + 2\beta}{\alpha_n^2} = 1 + \frac{A}{\alpha_n^2},
\]
and
\[
\left| \frac{\lambda_n + i - k}{\alpha_n + i - k} \right|^2 = \frac{(\alpha_n - k)^2 + (\beta_n + 1)^2}{(\alpha_n - k)^2 + 1} \leq \frac{(\alpha_n - k)^2 + (\beta + 1)^2}{(\alpha_n - k)^2 + 1} \leq 1 + \frac{\beta^2 + 2\beta}{(\alpha_n - k)^2 + 1} = 1 + \frac{B}{(\alpha_n - k)^2 + 1}.
\]
Therefore
\[
\sum_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 \leq \left[ 1 + \frac{A}{\alpha_n^2} \right] \left[ 1 + \frac{B}{(\alpha_n - k)^2 + 1} \right] \leq \exp \left\{ \sum_{n=-\infty}^{\infty} \left[ \frac{A}{\alpha_n^2} + \frac{B}{(\alpha_n - k)^2 + 1} \right] \right\}.
\]
Since \(\{\alpha_n\}\) is separated and no \(\alpha_n\) vanishes, the series \(\sum \alpha_n^{-2}\) converges and
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(\alpha_n - k)^2 + 1} < 2 + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} \quad (-\infty < k < \infty).
\]
It follows that \(\|f_N\| < \infty\), so that a subsequence of \(\{f_n\}\) converges weakly to a function \(g\) in \(H\) for which \(g(\lambda_n + i) = 0\) \((-\infty < n < \infty)\) and \(g(0) = 1\).

**Proof of Theorem 4.** Let \(\lambda_n = \alpha_n + i\beta_n\). Since \(\{\alpha_n\}\) is a complete interpolating sequence, it follows from Lemma 3 and Proposition 1 that the mapping \(T: H \to l^2\) given by \(Tf = \{f(\alpha_n)\}\) is continuous, one-to-one, and onto. By the open mapping theorem, \(T\) has a continuous inverse. Thus, there exists a positive constant \(A\) such that
\[
A\|f\|^2 \leq \sum |f(\alpha_n)|^2 \quad (f \in H).
\]
By a theorem of Duffin and Schaeffer [6, p. 355] we have
\[
B\|f\|^2 \leq \sum |f(\lambda_n)|^2
\]
for some constant $B > 0$ and all $f$ in $H$. Now there exist functions $g_n$ in $H$ such that $g_n(\alpha_k) = \delta_{nk}$. It follows from Lemma 5 that for each $k$ the sequence of functions \{\epsilon^{i\alpha_k}\} is incomplete in $L^2(-\pi, \pi)$. Therefore, we can find functions $f_n$ in $H$ such that $f_n(\lambda_k) = \delta_{nk}$. Fix \{\epsilon_n\} in $l^2$ and set $F_n(z) = \sum_{n=-N}^{N} c_n f_n(z)$ ($N = 1, 2, \ldots$). Since $F_n(\lambda_k)$ is equal to $c_k$ when $|k| \leq N$ and has the value 0 for $|k| > N$, (13) gives
\[
\|F_n\|^2 \leq \frac{1}{B} \sum_{k=-\infty}^{\infty} |F_n(\lambda_k)|^2 \leq \frac{1}{B} \sum_{k=-\infty}^{\infty} |c_k|^2
\]
so that a subsequence of $\{F_n\}$ converges weakly to a function $F \in H$ for which $F(\lambda_k) = c_k (-\infty < k < \infty)$.

**Corollary 2.** Let \{\lambda_n\} be a sequence of points lying in a strip parallel to the real axis, and suppose that
\[
|\text{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty).
\]
Then \{\lambda_n\} is a complete interpolating sequence.

**Proof.** It was shown by Duffin and Eachus [5] that the inequality
\[
\|\sum c_n (\epsilon^{i\text{Re}(\lambda_n)} - \epsilon^{i\alpha_n})\|^2_{L^2(-\pi, \pi)} \leq \theta^2 \sum |c_n|^2
\]
holds for some constant $\theta$, $0 \leq \theta < 1$, and every sequence \{\epsilon_n\} in $l^2$. By a theorem of Paley and Wiener [8, p. 100], \{\text{Re}(\lambda_n)\} is a complete interpolating sequence. The result then follows from Theorem 4.

**Theorem 5.** Let \{\lambda_n\} be a sequence of points lying in a strip parallel to the real axis. If \{\lambda_n\} is a complete interpolating sequence, then each function in $L^2(-\pi, \pi)$ has a unique expansion of the form
\[
g(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n \epsilon^{i\lambda_n}.
\]
Moreover, $A \sum |c_n|^2 \leq \|g\|^2_{L^2(-\pi, \pi)} \leq B \sum |c_n|^2$, where $A$ and $B$ are positive constants independent of $f$.

**Proof.** It follows from Lemma 1 that the inequality
\[
A \sum |c_n|^2 \leq \|\sum c_n \epsilon^{i\alpha_n}\|^2_{L^2(-\pi, \pi)}
\]
holds for some $A > 0$ and all finite sequences \{\epsilon_n\}. Since \{\lambda_n\} is separated (Proposition 1), Lemma 3 shows that
\[
\sum |f(\lambda_n)|^2 \leq B \|f\|^2
\]
for some $B > 0$ and every $f$ in $H$.

If $K_n$ denotes the reproducing function at $\lambda_n$,
\[
K_n(z) = K(z, \lambda_n) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)},
\]
then (15) may be rewritten as

\[(16) \quad \sum |(f, K_n)|^2 \leq B \|f\|^2.\]

Let us set \( f = \sum c_n K_n \) where \( \{c_n\} \) is a finite sequence. By (16) and the Cauchy-Schwarz inequality

\[
\|f\|^2 = (f, \sum c_n K_n) = \sum c_n (f, K_n)
\leq [\sum |c_n|^2]^{1/2} [\sum |(f, K_n)|^2]^{1/2}
\leq [\sum |c_n|^2]^{1/2} B^{1/2} \|f\|,
\]

so that \( \|f\|^2 \leq B \sum |c_n|^2 \). Taking the Fourier transform of \( f \) we get

\[(17) \quad A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi, \pi)} \leq B \sum |c_n|^2\]

for every finite sequence \( \{c_n\} \) and hence for every sequence in \( l^2 \). It is a simple consequence of (17) that each function in \( L^2(-\pi, \pi) \) which lies in the closed linear span of \( \{e^{i\lambda_n t}\} \) has a unique expansion of the form \( \lim_{N \to \infty} \sum_{n=1}^N c_n e^{i\lambda_n t} \) with \( \{c_n\} \) in \( l^2 \). Since the exponentials \( e^{i\lambda_n t} \) are complete in \( L^2(-\pi, \pi) \), the result follows.

**Theorem 6.** Let \( \{\lambda_n\} \) be a complete interpolating sequence. There exist positive numbers \( \delta_n \) such that \( \{\mu_n\} \) is a complete interpolating sequence whenever \( |\lambda_n - \mu_n| < \delta_n \).

**Proof.** Since \( \{\lambda_n\} \) is interpolating there is a constant \( A > 0 \) such that

\[(18) \quad A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi, \pi)}\]

for every finite sequence \( \{c_n\} \). If \( \delta_n > 0 \) is chosen small enough so that

\[
\sum_{n=1}^\infty \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2_{L^2(-\pi, \pi)} \leq \frac{A}{2}
\]

whenever \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots) \), then for every finite sequence \( \{c_n\} \)

\[
\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|^2_{L^2(-\pi, \pi)} \leq [\sum |c_n| \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2]
\leq [\sum |c_n|^2] [\sum \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2]
\leq \frac{A}{2} \sum |c_n|^2.
\]

Combining (18) and (19) we get

\[
\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|^2_{L^2(-\pi, \pi)} \leq \frac{1}{2} \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi, \pi)}
\]

whenever \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots) \). Since \( \{e^{i\lambda_n t}\} \) is complete in \( L^2(-\pi, \pi) \), it follows from a theorem of Boas [2, p. 469] that \( \{e^{i\mu_n t}\} \) is also complete. In Theorem 3 it was shown that \( \{\mu_n\} \) is interpolating whenever the \( \lambda_n \) are sufficiently small, whence the result follows.
4. Interpolation in $E^p_t$. We use the standard notation $E^p_t$ to denote the space of entire functions of exponential type $\tau$ ($0 < \tau < \infty$) which belong to $L^p(-\infty, \infty)$ on the real axis. For the properties of the spaces $E^p_t$ see [4]. For $0 < p < \infty$, let

$$\|f\|_p = \left[ \int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p},$$

while for $p = \infty$, let $\|f\|_\infty = \sup |f(x)|$ (x real).

**Definition.** A sequence $\{\lambda_n\}$ of distinct complex numbers is called an interpolating sequence for $E^p_t$ if $TE^p_t \supset I^p$. Here we continue to denote by $T$ the mapping $f \rightarrow \{f(\lambda_n)\}$.

The following results are derived from Lemmas 2 and 3 in essentially the same way as Proposition 1 and Corollary 1. The proofs are therefore omitted.

**Proposition 3.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE^p_t \supset I^1$ ($1 < p \leq \infty$), then $\{\lambda_n\}$ is separated.

**Corollary 2.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE^p_t \supset I^1$ ($1 < p \leq \infty$), then $TE^p_t \subset I^p$.

The remainder of this section is devoted to interpolation in $E^p_t$ in the special cases $p = 1$ and $p = \infty$.

**Theorem 7.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that there exist functions $f_n$ in $E^\infty$ satisfying

$$f_n(\lambda_k) = \delta_{nk}, \quad \|f_n\|_\infty \leq M, \quad (all \ n, k).$$

Then $TE^\infty_t = I^\infty$ whenever $\tau > \mu$.

**Proof.** It is well known [4, p. 82] that for every $f$ in $E^\infty_t$

$$(20) \quad |f(x + iy)| \leq \|f\|_\infty e^{\epsilon |y|},$$

so that $TE^\infty_t \subset I^\infty$ for each $\tau > 0$.

Fix $\{c_n\}$ in $I^\infty$, $\tau > \mu$ and let $\epsilon = (\tau - \mu)/2$. We show that the function

$$(21) \quad f(z) = \sum_{-\infty}^{\infty} c_n f_n(z) \left[ \frac{\sin \epsilon(z - \lambda_n)}{\epsilon(z - \lambda_n)} \right]^2$$

belongs to $E^\infty_t$. Clearly, $f(\lambda_n) = c_n$ ($-\infty < n < \infty$). Let $\lambda_n = \alpha_n + i\beta_n$ and suppose that $|\beta_n| \leq \alpha$ and $|c_n| \leq N$. For $m = 0, 1, 2, \ldots$, let $S_m$ be the set of integers $n$ for which $m - 1 \leq |\lambda_n| \leq m + 2$ and $T_m$ the set of $n$ for which $|\lambda_n| < m - 1$ or $|\lambda_n| > m + 2$. The method of proof of Proposition 1 shows that $\{\lambda_n\}$ is separated. Since $\{\beta_n\}$ is bounded there is a constant $K$, independent of $m$, such that the number of integers in $S_m$ is at most $K$. For $m \leq |z| \leq m + 1$, write

$$f(z) = \sum_{n \in S_m} c_n f_n(z) \left[ \frac{\sin \epsilon(z - \lambda_n)}{\epsilon(z - \lambda_n)} \right]^2 + \sum_{n \in T_m} c_n f_n(z) \left[ \frac{\sin \epsilon(z - \lambda_n)}{\epsilon(z - \lambda_n)} \right]^2.$$
Since \((\sin z)/z\) is entire of exponential type 1 and is bounded by 1 on the real axis, (20) shows that 
\[ |(\sin z)/z| < e^{|\text{Im} z|}. \]
Therefore, setting \(z = x + iy\), we have
\[
|/(z)| < \frac{1}{2\pi} \exp \left(\frac{2e|y - \lambda_n|}{|x - \lambda_n|^2}\right) 
+ \frac{NM \exp(\mu|y|)}{e^2} \sum_{n \in \mathbb{Z}_+} \frac{1}{2} \exp(2e|y - \beta_n|) 
\leq KMN \exp((\mu + 2e)|y| + 2\alpha) 
+ e^{-2NM} \exp((\mu + 2e)|y| + 2\alpha) \sum_{n \in \mathbb{Z}_+} \frac{1}{2} \exp(\frac{(\mu + 2e)|y| + 2\alpha}{|x - \lambda_n|^2}).
\]
We claim that the sums \(\sum_{n \in \mathbb{Z}_+} |z - \lambda_n|^2\) have a uniform upper bound for all \(m \geq 0\) and \(m \leq |z| \leq m + 1\). Since \(\{\lambda_n\}\) is a separated sequence, our assertion is immediate when each \(\lambda_n\) is real, while in the general case the existence of an upper bound follows readily from the boundedness of \(\text{Im} \lambda_n\). It follows from (22) that the series in (21) converges uniformly in each disk \(|z| < m\) \((m = 1, 2, \ldots)\) and that, for some constant \(A\), \(|f(z)| \leq A \exp((\mu + 2e)|y|)\). Since \(\tau = \mu + 2e, f\) belongs to \(E^\tau_\infty\) and the proof is complete.

**Theorem 8.** If \(\{\lambda_n\}\) is a real sequence with \(\lambda_{n+1} - \lambda_n \geq 1\) \((-\infty < n < \infty)\), then \(TE^\tau_\infty = l^\infty\) whenever \(\tau > \pi\).

**Proof.** That \(TE^\tau_\infty \subset l^\infty\) is clear. It follows readily from Theorem 1 that \(\{\lambda_n\}\) is an interpolating sequence for \(E^2_\mu\) whenever \(\mu > \pi\). Indeed, if we set \(\mu_n = (\mu/\pi)\lambda_n\) then \(\mu_{n+1} - \mu_n \geq \mu/\pi > 1\) and Theorem 1 shows that \(\{\mu_n\}\) is an interpolating sequence for \(E^2_\mu\). Therefore, given \(\{c_n\} \in l^2\) there exists a function \(g\) in \(E^2_\mu\) such that \(g(\mu_n) = c_n\) for all \(n\). Setting \(f(z) = g((\mu/\pi)z)\) we see that \(f\) belongs to \(E^2_\mu\) and \(f(\lambda_n) = c_n\) \((\text{all } n)\), and this establishes our assertion. Let us now fix \(\mu\) with \(\pi < \mu < \tau\). Lemma 2 shows that there exist functions \(f_n\) in \(E^2_\mu\) for which
\[
f_n(\lambda_k) = \delta_{mk}, \quad \sup_n \|f_n\|_2 < \infty, \quad (\text{all } n, k).
\]
From (7) it follows that \(|f(x)|^2 \leq (\mu/\pi)\|f\|_2^2\) for all \(f\) in \(E^2_\mu\) and all real \(x\), so that \(\sup_n \|f_n\|_\infty < \infty\). The conclusion now follows from Theorem 7.

In the same way we get the following result.

**Theorem 9.** Let \(\{\lambda_n\}\) be a sequence of points lying in a strip parallel to the real axis and suppose that
\[
|\text{Re} \lambda_n - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty).
\]
Then \(TE^\tau_\infty = l^\infty\) whenever \(\tau > \pi\).

**Theorem 10.** The integers are not an interpolating sequence for \(E^\tau_\infty\).

**Proof.** We show that the sequence \(\{c_n\}\) given by
cannot be interpolated. Suppose first that \( \{w_n\} \) is an arbitrary sequence in \( l^\infty \) and that \( w_0 = 0 \). If there is an \( f \) in \( E^\infty_\alpha \) with \( f(n) = w_n \) \((-\infty < n < \infty)\), then \( g(z) = f(z)/z \) is in \( E^2_\alpha \) and \( ng(n) = w_n \). If \( h \) is any other function in \( E^2_\alpha \) for which \( zh(z) \) belongs to \( E^\infty_\alpha \) and \( nh(n) = w_n \) \((-\infty < n < \infty)\), then \( h(z) = g(z) + \alpha(\sin \pi z)/\pi z \) for some complex number \( \alpha \) [4, p. 221]. Thus

\[
f(z) = z \left[ \sum_{n=0}^{\infty} \frac{w_n \sin \pi(z - n)}{(z - n)} + \frac{\alpha \sin \pi z}{\pi z} \right]
\]

is the most general function in \( E^\infty_\alpha \) with \( f(n) = w_n \) \((n \neq 0)\) and \( f(0) = 0 \).

A necessary condition that \( zg(z) \) belong to \( E^\infty_\alpha \) is that its derivative \( zg'(z) \) be bounded on the real axis [4, p. 206] and hence, in particular, that \( ng'(n) + g(n) \) be bounded uniformly in \( n \). Since \( g \) belongs to \( E^2_\alpha \), \( g(n) \to 0 \) as \( |n| \to \infty \), so that \( \{ng'(n)\} \) must be bounded.

Now, let \( \{c_n\} \) be given by (23) and let

\[
g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n \sin \pi(z - n)}{n(z - n)}.
\]

We will show that \( zg(z) \) is not bounded on the real axis by showing that \( |ng'(n)| \to \infty \) as \( n \to \infty \). By the preceding remarks the integers cannot be interpolating for \( E^\infty_\alpha \).

We have

\[
g'(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{\pi(z - n) \cos \pi z - \pi \sin \pi z}{\pi^2(z - n)^2}
\]

so that for \( k > 0 \),

\[
g'(k) = \sum_{n=1; n \neq k}^{\infty} \frac{1}{n} \frac{\cos \pi k}{k - n} = \sum_{n=1; n \neq k}^{\infty} \frac{1}{nk} \frac{(-1)^k}{n - k}.
\]

Thus

\[
k^g'(k) = (-1)^k \sum_{r=1; r \neq k}^{\infty} \left( \frac{1}{n} - \frac{1}{n - k} \right) = \left( 1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) - \frac{2}{k}.
\]

It is not difficult to show that

\[
\sum_{n=1; n \neq k}^{\infty} \left( \frac{1}{n} - \frac{1}{n - k} \right) = \left( 1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) - \frac{2}{k}
\]

so that \( |kg'(k)| \to \infty \) as \( k \to \infty \).

**Theorem 11.** If \( \{\lambda_n\} \) is a real sequence with \( \lambda_{n+1} - \lambda_n \geq 1 \) \((-\infty < n < \infty)\), then \( TE^1_\tau = l^1 \) whenever \( \tau > \pi \).
INTERPOLATION IN A CLASSICAL HILBERT SPACE

Proof. Lemma 3 shows that $TE^1_\ell \subset l^1$. It follows just as in the proof of Theorem 8 that for $\pi < \mu < \tau$ there exist functions $g_n$ in $E^2_\ell$ with $g_n(\lambda_k) = \delta_{nk}$ and $\sup_n ||g_n||_2 < \infty$. If we set $\varepsilon = \tau - \mu$ and let

$$f_\varepsilon(z) = g_n(z) (\sin \varepsilon(z - \lambda_n))/\varepsilon(z - \lambda_n),$$

then $f_\varepsilon \in E^1_\ell$ and $f_\varepsilon(\lambda_k) = \delta_{nk}$. Hölder's inequality shows that

$$||f_\varepsilon|| \leq ||g_n||_2 \left| \frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)} \right|_2$$

and it follows that $\sup_n ||f_\varepsilon|| < \infty$.

Now, choose $\{c_n\}$ in $l^1$ and set

$$(24) \quad f(z) = \sum_{-\infty}^{\infty} c_n f_n(z).$$

Since $\sum ||c_n f_n|| < \infty$, $f$ belongs to $E^1_\ell$, and Lemma 3 implies that the convergence in (24) is uniform in each horizontal strip. Therefore, $f(\lambda_n) = c_n (-\infty < n < \infty)$ and the proof is complete.

It is easy to see that this result is best possible, in the sense that $\tau$ cannot always be taken equal to $\pi$. Indeed, the integers are not an interpolating sequence for $E^1_\ell$ for the trivial reason that the nonzero integers are a set of uniqueness. However, we have the following stronger result.

Theorem 12. The nonzero integers are not an interpolating sequence for $E^1_\ell$.

Proof. Lemma 3 shows that point evaluations are continuous linear functionals on $E^1_\ell$. By Lemma 2 it is enough to show that the unit ball of $l^1$ cannot be interpolated in a uniformly bounded way. Since

$$f_n(z) = n(\sin \pi(z - n))/\pi z(z - n) \quad (n \neq 0)$$

is the unique function in $E^1_\ell$ with the property that $f_n(k) = \delta_{nk}$, it is sufficient to show that $||f_n|| \to \infty$ as $n \to \infty$. For $n > 0$,

$$||f_n|| = \int_{-\infty}^{\infty} |f_n(x)| dx \geq \frac{n}{\pi} \int_{1}^{\infty} \left| \frac{\sin \pi x}{x(x + n)} \right| dx$$

$$> \frac{n}{\pi} \sum_{k=1}^{\infty} \int_{k+3/4}^{k+1/4} \left| \frac{\sin \pi x}{x(x + n)} \right| dx$$

$$> \frac{\sqrt{2}}{2\pi} \sum_{k=1}^{\infty} \int_{k+1/4}^{k+3/4} \frac{1}{x - \frac{1}{x + n}} dx$$

$$= \frac{\sqrt{2}}{2\pi} \log \prod_{k=1}^{\infty} \left[ \frac{k + 3/4}{k + n + 3/4} \frac{k + n + 1/4}{k + 1/4} \right].$$

Using the relation $\Gamma(x + 1) = x\Gamma(x)$ it easily follows that the infinite product
above is equal to

\[
\lim_{N \to \infty} \left[ \frac{(1 + 3/4)(2 + 3/4) \cdots (N + 3/4)}{(1 + n + 3/4)(2 + n + 3/4) \cdots (N + n + 3/4)} \right] \cdot \left[ \frac{(1 + n + 1/4) \cdots (N + n + 1/4)}{(1 + 1/4) \cdots (N + 1/4)} \right] = \left[ \frac{1 + 3/4}{1 + 1/4} \frac{2 + 3/4}{2 + 1/4} \cdots \frac{n + 3/4}{n + 1/4} \right]
\]

From the estimate \( \Gamma(x + 1) \sim (2\pi)^{1/2} x^{x + 1/2} e^{-x} \) (as \( x \to \infty \)) we conclude that \( \Gamma(n + 1 + 3/4)/\Gamma(n + 1 + 1/4) \to \infty \) (as \( n \to \infty \)), and the proof is complete.

The proof of the next theorem is similar to that of Theorem 11 and is therefore omitted.

Theorem 13. Let \( \{\lambda_n\} \) be a sequence of points lying in a strip parallel to the real axis and suppose that

\[ |\text{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty). \]

Then \( T E_1^\tau = l^1 \) whenever \( \tau > \pi \).

BIBLIOGRAPHY


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