INTERPOLATION IN A CLASSICAL HILBERT SPACE OF ENTIRE FUNCTIONS(1)

BY

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ABSTRACT. Let $H$ denote the Paley-Wiener space of entire functions of exponential type $\pi$ which belong to $L^2(-\infty, \infty)$ on the real axis. A sequence $(\lambda_n)$ of distinct complex numbers will be called an interpolating sequence for $H$ if $TH \supset I^2$, where $T$ is the mapping defined by $Tf = \{f(\lambda_n)\}$. If in addition $(\lambda_n)$ is a set of uniqueness for $H$, then $(\lambda_n)$ is called a complete interpolating sequence. The following results are established. If $\text{Re} (\lambda_{n+1}) - \text{Re} (\lambda_n) \geq \gamma > 1$ and if the imaginary part of $\lambda_n$ is sufficiently small, then $(\lambda_n)$ is an interpolating sequence. If $|\text{Re} (\lambda_n) - n| \leq L \leq (\log 2)/\pi (-\infty < n < \infty)$ and if the imaginary part of $\lambda_n$ is uniformly bounded, then $(\lambda_n)$ is a complete interpolating sequence and $(e^{i\lambda_n})$ is a basis for $L^2(-\pi, \pi)$. These results are used to investigate interpolating sequences in several related spaces of entire functions of exponential type.

Introduction. Let $H$ denote the Paley-Wiener space of entire functions $f$ of exponential type $\pi$ for which

$$
\|f\| = \left[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx \right]^{1/2} < \infty.
$$

A sequence $(\lambda_n)$ of distinct complex numbers will be called an interpolating sequence for $H$ if corresponding to each sequence $(c_n)$ in $l^2$ there is at least one function $f$ in $H$ satisfying $f(\lambda_n) = c_n (-\infty < n < \infty)$. (Unless otherwise stated, the term sequence in this paper will always mean a two-sided sequence.)

In §§2 and 3 of this paper we study both necessary and sufficient conditions for a sequence $(\lambda_n)$ to be interpolating. For the most part we require that the $\lambda_n$ lie in a strip parallel to the real axis. Under this condition we show that $l^2$ is the natural sequence space to interpolate with functions in $H$ and a necessary condition for interpolation is that the $\lambda_n$ be separated, that is, $|\lambda_n - \lambda_k| \geq \gamma$ for some constant $\gamma > 0$ and all $n \neq k$.

A classical theorem of Paley and Wiener [8, p. 13] shows that the Fourier transform of every function in $H$ vanishes almost everywhere outside $(-\pi, \pi)$. Thus $f$ belongs to $H$ if and only if it is of the form

$$
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{int} \, dt
$$

(1)

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97
for some $g$ in $L^2(-\pi, \pi)$. It follows that $\{\lambda_n\}$ is an interpolating sequence if and only if the trigonometric moment problem

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} \, dt \quad (-\infty < n < \infty)$$

has a solution $g$ in $L^2(-\pi, \pi)$ whenever $\sum |c_n|^2 < \infty$. The following result was first established by Boas [3] and later reproved in a more abstract setting by N. Bari [1]. (See also [10].)

**Lemma 1.** A necessary and sufficient condition that the trigonometric moment problem (2) have a solution in $L^2(-\pi, \pi)$ for every square summable sequence $\{c_n\}$ is that the inequality

$$A \sum |a_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum_n a_n e^{i\lambda_n t}|^2 \, dt$$

holds for some constant $A > 0$ and all finite sequences $\{a_n\}$.

Several years earlier, Ingham [7] had established the validity of (3) in the special case in which the $\lambda_n$ are real and satisfy the separation condition $\lambda_{n+1} - \lambda_n \geq \gamma > 1 \quad (-\infty < n < \infty)$. He showed that his result is the best possible, in the sense that $\gamma$ cannot be taken equal to 1, by demonstrating that the sequence $\{\lambda_n\}$ given by

$$\lambda_n = n - \frac{1}{4}, \quad \lambda_{-n} = -\lambda_n \quad (n = 1, 2, \ldots)$$

does not satisfy (3) for any positive $A$. By modifying Ingham's proof, we are able to show that $\{\lambda_n\}$ is an interpolating sequence whenever

$$\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1 \quad (-\infty < n < \infty)$$

and the imaginary part of $\lambda_n$ is sufficiently small. As a corollary we show that if $\{\lambda_n\}$ is a sequence of points lying in a strip parallel to the real axis and if $\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \to \infty$ as $n \to \pm \infty$, then for each $\{c_n\}$ in $l^2$ and each $\tau > 0$ there exists an entire function $f$ of exponential type $\tau$, square integrable on the real axis, for which $f(\lambda_n) = c_n \quad (-\infty < n < \infty)$.

In their classic treatise, Paley and Wiener showed [8, p.115] that every sequence of real numbers $\{\lambda_n\}$ which are close to the integers in the sense that $|\lambda_n - n| \leq L < 1/\pi^2 \quad (-\infty < n < \infty)$ is an interpolating sequence. Moreover, they showed that every function $g$ in $L^2(-\pi, \pi)$ has a *nonharmonic* Fourier series expansion

$$g(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{i\lambda_n t}$$

with $\sum |c_n|^2 < \infty$. These results were improved by Duffin and Eachus [5] who showed that $\lambda_n$ can be complex and that the constant $1/\pi^2$ can be replaced by $(\log 2)/\pi$. In the present paper we show that $\{\lambda_n\}$ is an interpolating sequence.
whenever \( |\text{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \) \((-\infty < n < \infty)\) and the imaginary part of \( \lambda_n \) is uniformly bounded, and that under these conditions every function in \( L^2(-\pi, \pi) \) has the representation (4) with some sequence \( \{c_n\} \) in \( l^2 \).

The main results of §§2 and 3 are used in §4 to investigate interpolating sequences in several related spaces of entire functions of exponential type. Specifically, we consider the spaces \( E^p_\tau \) of entire functions of exponential type \( \tau \) which belong to \( L^p(-\infty, \infty) \) on the real axis. We restrict attention mainly to the special cases \( p = 1 \) and \( p = \infty \).

1. Background material. Under the inner product

\[
(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx,
\]

\( H \) is a functional Hilbert space with the reproducing kernel

\[
K(\xi, z) = \sin \pi(\xi - z)/\pi(\xi - z).
\]

The functions \( K(\xi, n), -\infty < n < \infty, \) form a complete orthonormal system, so that for each \( f \) in \( H \)

\[
f(z) = \sum_{-\infty}^{\infty} f(n)\frac{\sin \pi(z - n)}{\pi(z - n)}.
\]

By Parseval's formula

\[
\|f\|^2 = \sum_{-\infty}^{\infty} |f(n)|^2 \quad (f \in H).
\]

If \( f \) is given by (1), then \( g \) is the Fourier transform of \( f \) and Plancherel's theorem gives

\[
\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)|^2\,dx.
\]

It follows easily from (1) and (6) that

\[
|f(x + iy)| \leq e^{\pi|x|} \|f\| \quad (f \in H).
\]

for every \( f \) belonging to \( H \). Similar estimates show that \( H \) is closed under differentiation and that

\[
\|f'\| \leq \pi\|f\| \quad (f \in H).
\]

2. Interpolation in \( H \). Let \( \{\lambda_n\} \) be a sequence of distinct complex numbers. For each \( f \) in \( H \) we define \( Tf \) to be the sequence \( Tf = \{f(\lambda_n)\} \) \((-\infty < n < \infty)\).

**Definition.** The sequence \( \{\lambda_n\} \) is said to be an interpolating sequence for \( H \) if \( TH \supset I_2 \). It is not difficult to show that if \( TH \supset I_2 \), then the unit ball in \( I_2 \) can be interpolated in a uniformly bounded way. In fact, we have the following stronger result. For a proof see [9, p. 19].

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Lemma 2. Let $X$ be a Banach space, $\{\mu_n\}$ a sequence in the dual space $X'$ and $T$ the mapping on $X$ defined by $Tx = (\mu_n(x))$. If $TX \supset I^p$ for some $p$, $1 \leq p \leq \infty$, then $TM$ covers the unit ball of $I^p$ for some bounded subset $M$ of $X$.

Definition. The sequence $\{\lambda_n\}$ is said to be separated if there is a constant $\gamma$ such that $|\lambda_n - \lambda_k| \geq \gamma > 0$ ($n \neq k$).

It is well known [4, p. 101] that if the $\lambda_n$ are real and separated and if $f$ is an entire function of exponential type $\tau$ belonging to $L^p(-\infty, \infty)$ on the real axis ($0 < p < \infty$), then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where $B$ depends only on $p$, $\tau$ and the separation constant $\gamma$. It is no more difficult to establish the following result, which we state without proof.

Lemma 3. Let $\{\lambda_n\}$ be a complex sequence satisfying

$$|\text{Im}(\lambda_n)| \leq \alpha, \quad |\lambda_n - \lambda_k| \geq \gamma \quad (n \neq k)$$

for some positive constants $\alpha$ and $\gamma$. If $f$ is an entire function of exponential type $\tau$, belonging to $L^p(-\infty, \infty)$ on the real axis, then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where $B = B(p, \tau, \gamma, \alpha)$.

Proposition 1. Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TH \supset I^1$, then $\{\lambda_n\}$ is separated.

Proof. Since point-evaluations are continuous linear functionals on $H$, Lemma 2 shows that there are functions $f_n$ in $H$ and a constant $M > 0$ such that $f_n(\lambda_k) = \delta_{nk}$, $\|f_n\| \leq M$ (for all $n, k$). For $n \neq k$, we have

$$1 = f_n(\lambda_n) - f_n(\lambda_k) = \int_{\lambda_k}^{\lambda_n} f'_n(z) \, dz$$

so that

$$1 \leq \int_{\lambda_k}^{\lambda_n} |f'_n(z)| \, |dz| \leq \sup |f'_n(z)| \cdot |\lambda_n - \lambda_k|,$$

where the supremum is taken over the straight line segment from $\lambda_k$ to $\lambda_n$. If $|\text{Im}(\lambda_n)| \leq \alpha$, then (7), (8) show that

$$1 \leq e^{\alpha} \|f'_n\| \cdot |\lambda_n - \lambda_k|$$

$$\leq \pi e^{\alpha} \|f_n\| \cdot |\lambda_n - \lambda_k|$$

$$\leq M \pi e^{\alpha} |\lambda_n - \lambda_k|,$$

and the result follows with $\gamma = e^{-\alpha}/M\pi$. 

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Corollary 1. Let \( \{\lambda_n\} \) be a sequence of points lying in a strip parallel to the real axis. If \( TH \supset I^1 \), then \( TH \subset I^2 \).

Since separation is a necessary condition for a sequence of real numbers to be interpolating, the problem arises to determine if there is a constant \( \gamma \) with the property that \( \{\lambda_n\} \) is an interpolating sequence whenever \( \lambda_{n+1} - \lambda_n \geq \gamma \). The following result will show that \( \{\lambda_n\} \) is interpolating whenever \( \gamma > 1 \). The proof is a simple extension of an argument given by Ingham [7].

Lemma 4. Let \( f(i) = \sum_{n=1}^{N} a_n e^{i\lambda_n} \) where the \( \lambda_n \) satisfy

\[
\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1, \quad |\text{Im}(\lambda_n)| \leq \alpha.
\]

Then

\[
A \sum |a_n|^2 \leq \int_{-\gamma}^{\gamma} |f(i)|^2 \, dt,
\]

where \( A = 4[1/(1 + 16\alpha^2)] - e^{2\pi \gamma^2} \).

Proof. If \( k \) belongs to \( L^1(-\infty, \infty) \) and \( K(z) = \int_{-\infty}^{\infty} k(t)e^{zt} \, dt \), then

\[
\int_{-\infty}^{\infty} k(t)|f(t)|^2 \, dt = \sum_{m,n} a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n).
\]

Letting

\[
k(t) = \cos \frac{1}{2}t, \quad |t| \leq \pi,
\]

\[
= 0, \quad |t| > \pi,
\]

we get \( K(z) = 4(\cos\pi z)/(1 - 4z^2) \). Since \( |k(t)| \leq 1 \), it follows from (10) that

\[
\int_{-\gamma}^{\gamma} |f(t)|^2 \, dt \geq \left| \sum_{m} |a_m|^2 K(\lambda_m - \bar{\lambda}_n) \right| - \left| \sum_{m} a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n) \right|,
\]

where the prime in the summation denotes omission of the term \( m = n \). The remainder of the proof is devoted to obtaining suitable estimates for the two sums in absolute value above. Since, by (9), \( |\lambda_m - \bar{\lambda}_n| \geq |m - n| \gamma > 1 \) \( (m \neq n) \) and since \( |\cos(x + iy)| \leq e^{2\pi y} \), we have

\[
\sum_{m} |K(\lambda_m - \bar{\lambda}_n)| \leq 4 e^{2\pi y} \sum_{m} \frac{1}{4(m - n)^2 \gamma^2 - 1} < \frac{8 e^{2\pi y}}{\gamma^2} \sum_{r=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{4 e^{2\pi y}}{\gamma^2}.
\]

Since \( 2|a_m a_n| \leq |a_m|^2 + |a_n|^2 \), we get

\[
\sum_{m,n} a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n) = \theta \sum_{m,n} |a_m|^2 + |a_n|^2 |K(\lambda_m - \bar{\lambda}_n)|,
\]

where \( |\theta| \leq 1 \), and since \( |K(z)| = |K(-\bar{z})| \),
\[
\sum_{m,n} a_m a_n K(\lambda_m - \lambda_n) = \theta \sum_n |a_n|^2 \left\{ \sum_m |K(\lambda_m - \lambda_n)| \right\} < \frac{4e^{2\mu}}{\gamma^2} \sum_n |a_n|^2.
\]

If we set \( \beta_n = \text{Im}(\lambda_n) \), then

\[
K(2i\beta_n) = \frac{4 \cosh 2\pi \beta_n}{1 + 16\beta_n^2} \geq \frac{4}{1 + 16\alpha^2},
\]

and hence

\[
\sum_n |a_n|^2 K(\lambda_n - \lambda_n) \geq \frac{4}{1 + 16\alpha^2} \sum_n |a_n|^2,
\]

and the proof is complete.

**Theorem 1.** Let \( \{\lambda_n\} \) be a complex sequence satisfying

\[
\text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \geq \gamma > 1,
\]

\[
|\text{Im}(\lambda_n)| < \alpha \quad (-\infty < n < \infty).
\]

Then \( \{\lambda_n\} \) is an interpolating sequence for all sufficiently small values of \( \alpha \).

**Proof.** The result follows immediately from Lemmas 1 and 4 since the value of \( A \) given in the statement of Lemma 4 approaches \( 4(1 - 1/\gamma^2) > 0 \) as \( \alpha \to 0 \).

**Remark.** Shapiro and Shields have shown [10, p. 532] that if \( \{\lambda_n\} \) is a separated sequence of real numbers and if \( \sum \beta_k^2 < \infty \), then for each \( \alpha > 0 \) there corresponds a function \( f \) analytic in the strip \( D: |y| < \alpha \), with finite area norm

\[
\iint_D |f(z)|^2 \, dx \, dy < \alpha,
\]

such that \( f(\lambda_n) = c_n \quad (-\infty < n < \infty) \). Since (11) is satisfied for each function in \( H \), an obvious extension of Theorem 1 shows that \( f \) may in fact be chosen to be entire of exponential type.

**Theorem 2.** Let \( \{\lambda_n\} \) be a sequence of distinct points lying in a strip parallel to the real axis and suppose that \( \text{Re}(\lambda_{n+1}) - \text{Re}(\lambda_n) \to \infty, n \to \pm \infty \). For each \( \tau > 0 \) and each sequence \( \{c_n\} \) in \( l^2 \) there exists an entire function \( f \) of exponential type \( \tau \), belonging to \( L^2(-\infty, \infty) \) on the real axis, for which \( f(\lambda_n) = c_n \quad (-\infty < n < \infty) \).

**Proof.** Fix \( \tau > 0 \) and let \( H_\tau \) denote the space of entire functions of exponential type \( \tau \) which belong to \( L^2(-\infty, \infty) \) on the real axis. It follows from Theorem 1 that there exists a smallest integer \( N \geq 0 \) with the property that for each sequence \( \{c_n\} \) in \( l^2 \) there corresponds at least one function \( f \) in \( H_\tau \) such that \( f(\lambda_n) = c_n \quad |n| \geq N \). If \( N = 0 \) there is nothing to show, so suppose that \( N \geq 1 \). Choose \( g \) in \( H_\tau \) such that
We begin by constructing a function $h$ in $H_r$ such that
\begin{equation}
(12) \quad h(X_{N-1}) = 1, \quad h(X_n) = 0, \quad |n| \geq N.
\end{equation}

If $g(X_{N-1}) = 0$, then for suitable constants $c$ and $m \geq 1$ the function
\[
h(z) = \frac{g(z)}{c(z - \lambda_{N-1})^m} \left[ \frac{z - \lambda_N}{\lambda_{N-1} - \lambda_N} \right]
\]
satisfies (12). If $g(X_{N-1}) \neq 0$, then $h$ must be obtained by a different method. By a theorem of Titchmarsh [11] the zeros of $g$ have a positive density, where it is understood that multiple zeros are counted as many times as their multiplicity warrants. It is easily shown that $\{X_n\}$ has density zero, so that some zero $\lambda$ of $g$ is either not in the sequence or else is in the sequence and is a multiple zero of $g$. In either case, the function
\[
h(z) = \frac{\lambda_{N-1} - \lambda}{\lambda_{N-1} - \lambda_N} \left[ \frac{g(z)}{g(X_{N-1})} \cdot \frac{z - \lambda_N}{z - \lambda} \right]
\]
satisfies (12).

Now fix $\{c_n\}$ in $l^2$ and choose $f$ in $H_r$ such that $f(X_n) = c_n$ ($|n| \geq N$). Let $g$ and $h$ be chosen as above and define
\[
f_1(z) = f(z) + [c_{N-1} - f(X_{N-1})]h(z).
\]
Then $f_1$ belongs to $H_r$ and $f_1(X_n) = c_n$ ($|n| \geq N$ and $n = N - 1$). The same construction gives $f_2 \in H_r$ such that $f_2(X_n) = c_n$ ($|n| \geq N - 1$). But this contradicts the choice of $N$; hence $N$ must be equal to zero, and the proof is complete.

**Theorem 3.** Let $\{\lambda_n\}$, $n = 1, 2, \ldots$, be an interpolating sequence. There exist positive numbers $\delta_n$ such that $\{\mu_n\}$ is an interpolating sequence whenever $|\lambda_n - \mu_n| < \delta_n$.

**Proof.** It follows from Lemma 2 that there is a constant $M > 0$ depending only on $\{\lambda_n\}$ with the property that for each sequence $\{c_n\}$ in the unit ball of $l^2$ there corresponds at least one function $f$ in $H$ for which $f(X_n) = c_n$ ($n = 1, 2, \ldots$) and $\|f\| \leq M$.

Let $F_l$ denote the family of all functions $f$ in $H$ for which $\|f\| \leq M$. Then (7) shows that the functions in $F_l$ are uniformly bounded on compacta. It follows that $F_l$ is equicontinuous on compacta and hence in particular on the disk $|z - \lambda_1| \leq 1$. Thus, if $0 < \epsilon_1 < 1$, there is a corresponding $\delta_1 = \delta_1(\epsilon_1)$ such that $|f(z) - f(\lambda_1)| < \epsilon$ ($|z - \lambda_1| < \delta_1$), uniformly for all $f$ in $F_l$. In addition, $\delta_1$ may be chosen small enough so that the disk $|z - \lambda_1| < \delta_1$ intersects $\{\lambda_n\}$ only at $\lambda_1$. 

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We are going to show that if \(|\lambda_1 - \mu_1| < \delta_1\), then \(\{\mu_1, \lambda_2, \lambda_3, \ldots\}\) is an interpolating sequence. Clearly, it will be enough to show that the unit ball of \(l^2\) can be interpolated. More precisely, we will show that for each \(\{c_n\}\) with \(\sum |c_n|^2 \leq 1\) there corresponds a function \(F_1\) in \(H\) for which

\[
F_1(\mu_1) = c_1,
\]

\[
F_1(\lambda_n) = c_n, \quad n = 2, 3, \ldots.
\]

\(\|F_1\| \leq M/(1 - \varepsilon_1)\).

Fix \(\mu_i\) with \(|\lambda_1 - \mu_i| < \delta_1\) and let \(\sum |c_n|^2 \leq 1\). There exists a function \(g\) in \(H\) such that

\[
g(\lambda_n) = c_n, \quad \|g\| \leq M, \quad n = 1, 2, \ldots.
\]

Since \(g\) is in \(K_1\), \(|g(\mu_i) - g(\lambda_n)| < \varepsilon_1\). Also, there exists a function \(f\) in \(H\) such that

\[
f(\lambda_n) = 0, \quad n = 2, 3, \ldots.
\]

\(\|f\| \leq M\).

Then \(f\) is also in \(K_1\) so that \(|f(\mu_i) - f(\lambda_n)| < \varepsilon_1\), and hence \(|f(\mu_i)| > 1 - \varepsilon_1 > 0\).

Now, set

\[
F_1(z) = g(z) + \frac{c_1 - g(\mu_i)}{|f(\mu_i)|} f(z) / f(\mu_i).
\]

Clearly, \(F_1\) is in \(H\), \(F_1(\mu_i) = c_1, F_1(\lambda_n) = c_n\) \((n > 1)\), and

\[
\|F_1\| \leq \|g\| + \frac{|c_1 - g(\mu_i)|}{|f(\mu_i)|} \|f\|
\]

\[
\leq \|g\| + \frac{|g(\lambda_n) - g(\mu_i)|}{|f(\mu_i)|} \|f\|
\]

\[
\leq M(1 + \varepsilon_1/(1 - \varepsilon_1)) = M/(1 - \varepsilon_1).
\]

The above argument can be repeated with \(\{\lambda_n\}\) replaced by \(\{\mu_1, \lambda_2, \lambda_3, \ldots\}\). Thus, we let \(K_2\) denote the family of all functions \(f\) in \(H\) for which \(\|f\| \leq M/(1 - \varepsilon_1)\). Then \(K_2\) is equicontinuous on each compact set and for \(0 < \varepsilon_2 < 1\) we find \(\delta_2 = \delta_2(\varepsilon_1, \varepsilon_2)\) such that

\[
|f(z) - f(\lambda_2)| < \varepsilon_2 \quad (|z - \lambda_2| < \delta_2),
\]

uniformly for all \(f\) in \(K_2\). We note that \(\delta_2\) is independent of \(\mu_i\) and may be chosen so that the disks \(|z - \lambda_1| < \delta_1\) and \(|z - \lambda_2| < \delta_2\) are disjoint and intersect \(\{\lambda_n\}\) only at \(\lambda_1\) and \(\lambda_2\), respectively. Just as before, we show that whenever \(|\mu_2 - \lambda_2| < \delta_2\), the sequence \(\{\mu_1, \mu_2, \lambda_3, \lambda_4, \ldots\}\) is interpolating, and that for each sequence \(\{c_n\}\) in the unit ball of \(l^2\) there corresponds a function \(F_2\) in \(H\) for which
\[ F_2(\mu_n) = c_n, \quad n = 1, 2, \]
\[ F_2(\lambda_n) = c_n, \quad n > 2, \]
\[ \|F_2\| \leq M/(1 - \epsilon_1)(1 - \epsilon_2). \]

The above process may be iterated. Thus, given a sequence \( \{\epsilon_n\}, 0 < \epsilon_n < 1, \)
we obtain a corresponding sequence \( \{\delta_n\}, \delta_n = \delta_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) > 0, \)
with the property that for each positive integer \( N \) the sequence \( \{\mu_n, \mu_2, \ldots, \mu_N, \lambda_{N+1}, \lambda_{N+2}, \ldots\} \)
is interpolating whenever \( |\lambda_n - \mu_n| < \delta_n \quad (n = 1, 2, \ldots, N), \) and such
that for every sequence \( \{c_n\} \) with \( \sum |c_n|^2 \leq 1 \) there exists a function \( F_N \) in \( H \) for
which
\[ F_N(\mu_n) = c_n, \quad n = 1, 2, \ldots, N, \]
\[ F_N(\lambda_n) = c_n, \quad n > N, \]
\[ \|F_N\| \leq M \frac{1}{1 - \epsilon_1} \frac{1}{1 - \epsilon_2} \cdots \frac{1}{1 - \epsilon_N}. \]

Now, choose \( \{\epsilon_n\}, 0 < \epsilon_n < 1, \) so that
\[ \epsilon = \sum_{n=1}^{\infty} \frac{\epsilon_n}{1 - \epsilon_n} < \infty, \]
and let the corresponding sequence \( \{\delta_n\} \) be determined as above. Fix \( \{\mu_n\} \) with
\( |\lambda_n - \mu_n| < \delta_n \quad (n = 1, 2, \ldots) \) and let \( \{c_n\} \) belong to the unit ball of \( l^2. \) For each
positive integer \( N \) there exists a function \( F_N \) in \( H \) such that
\[ F_N(\mu_n) = c_n, \quad n = 1, 2, \ldots, N, \]
\[ \|F_N\| \leq M \prod_{n=1}^{N} \left( 1 + \frac{\epsilon_n}{1 - \epsilon_n} \right) \]
\[ \leq M \exp \left[ \sum_{n=1}^{\infty} \frac{\epsilon_n}{1 - \epsilon_n} \right] = Me^\epsilon. \]

Since \( \|F_N\| \) is uniformly bounded, a subsequence of \( \{F_N\} \) will converge weakly to
a function \( F \) in \( H \) for which \( F(\mu_n) = c_n \quad (n = 1, 2, \ldots). \) Thus, the unit ball of \( l^2 \)
can be interpolated and the proof is complete.

3. Uniqueness: complete interpolating sequences.

Proposition 2. Let \( \{\lambda_n\}, n = 1, 2, \ldots, \) be an interpolating sequence. Each of the
following statements implies the others.

(i) The set of relations \( f \in H \) and \( f(\lambda_n) = 0 \quad (n = 1, 2, \ldots) \) imply that \( f = 0, \)
that is, \( \{\lambda_n\} \) is a set of uniqueness for \( H. \)

(ii) The exponentials \( \{e^{i\lambda_n}\} \) are complete in \( L^2(-\pi, \pi). \)

(iii) The sequence \( \{\lambda_n\} \) is contained in no larger interpolating sequence.
Proof. It follows immediately from the Paley-Wiener representation (1) that (i) and (ii) are equivalent. Since (ii) clearly implies (iii) it remains only to show that (iii) implies (ii).

Suppose then that \{e^{i\lambda_n t}\} is not complete in \(L^2(-\pi, \pi)\) and let \(\lambda_0\) be distinct from the \(\lambda_n\). We show that \((\lambda_n), n = 0, 1, 2, \ldots,\) is an interpolating sequence. Since \{e^{i\lambda_n t}\} is incomplete there is a function \(g\) in \(L^2(-\pi, \pi), g \neq 0,\) such that

\[
\int_{-\pi}^{\pi} g(t)e^{i\lambda_n t} dt = 0, \quad n = 1, 2, \ldots.
\]

Setting \(h(z) = \int_{-\pi}^{\pi} g(t)e^{it} dt,\) we have \(h \in H, h \neq 0,\) and \(h(\lambda_n) = 0\) \((n = 1, 2, \ldots).\) If \(h(\lambda_0) \neq 0,\) we set \(F(z) = h(z)/h(\lambda_0),\) while if \(h(\lambda_0) = 0,\) we take \(F(z) = h(z)/A(z - \lambda_0)^m,\) where \(A\) and \(m\) are chosen so that \(F(\lambda_0) = 1.\) In either case, \(F(\lambda_0) = 1\) and \(F(\lambda_n) = 0\) \((n = 1, 2, \ldots).\)

Fix \{c_n\}, \(n = 0, 1, 2, \ldots,\) in \(L^2.\) There is a function \(G\) in \(H\) with \(G(\lambda_n) = c_n\) \((n = 1, 2, \ldots).\) Let

\[
f(z) = G(z) + [c_0 - G(\lambda_0)]F(z).
\]

Then \(f\) is in \(H\) and \(f(\lambda_n) = c_n\) \((n = 0, 1, 2, \ldots).\)

Definition. An interpolating sequence satisfying any one of the conditions listed in Proposition 2 will be called a complete interpolating sequence.

Theorem 4. Let \(\{\lambda_n\}\) be a sequence of distinct points lying in a strip parallel to the real axis. If \(\{\Re(\lambda_n)\}\) is a complete interpolating sequence, then \(\{\lambda_n\}\) is a complete interpolating sequence.

The proof of Theorem 4 requires the following lemma.

Lemma 5. Let \(\lambda_n = \alpha_n + i\beta_n,\) where \(\alpha_n\) and \(\beta_n\) are real and satisfy

\[
\alpha_{n+1} - \alpha_n \geq \gamma > 0, \quad |\beta_n| \leq \beta, \quad -\infty < n < \infty.
\]

If \(\{e^{i\lambda_n t}\}\) is complete in \(L^2(-\pi, \pi),\) then \(\{e^{i\alpha_n t}\}\) is also complete in \(L^2(-\pi, \pi).\)

Proof of Lemma 5. An equivalent problem is to show that the completeness of \(\{e^{i(\alpha_n+\gamma) t}\}\) implies that of \(\{e^{i(\alpha_n+i) t}\}.\) For this it is enough to show that the only function in \(H\) which vanishes at every point \(\alpha_n + i\) is identically zero. Arguing by contradiction, we suppose that for some \(f\) in \(H, f \neq 0, f(\alpha_n + i) = 0\) \((-\infty < n < \infty).\) Without any loss of generality we may suppose that no \(\alpha_n\) is an integer and that \(f(0) = 1.\) We are going to exhibit a function \(g\) in \(H\) with \(g(\lambda_n + i) = 0\) \((-\infty < n < \infty)\) and \(g(0) = 1,\) thereby contradicting the completeness of \(\{e^{i(\alpha_n+i) t}\}.\) Set

\[
f_N(z) = f(z) \prod_{n=-N}^{N} \frac{1 - z/(\lambda_n + i)}{1 - z/(\alpha_n + i)}, \quad N = 1, 2, \ldots.
\]

For each \(N\) we have \(f_N \in H, f_N(\lambda_n + i) = 0 \((|n| \leq N),\) and \(f_N(0) = 1.\) By (5),
We show that the products
\[
\prod_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 = \prod_{n=-N}^{N} \left| \frac{\alpha_n + i - k}{\lambda_n + i} \right|^2 \left| \frac{\lambda_n + i - k}{\alpha_n + i - k} \right|^2
\]
are uniformly bounded in \( N \) and \( k \). Simple calculations show that
\[
\frac{1}{\lambda_n + i} \left( \frac{\alpha_n + i}{\lambda_n + i} \right)^2 = 1 - \frac{\beta_n^2 + 2\beta_n}{\alpha_n^2 + (\beta_n + 1)^2} \leq 1 + \frac{\beta_n^2 + 2|\beta_n|}{\alpha_n^2 + (\beta_n + 1)^2}
\]
and
\[
\frac{1}{\alpha_n + i - k} \left( \frac{\alpha_n + i - k}{\alpha_n + i} \right)^2 = 1 + \frac{\beta_n^2 + 2\beta_n}{\alpha_n^2 + (\beta_n + 1)^2} \leq 1 + \frac{\beta_n^2 + 2|\beta_n|}{\alpha_n^2 + (\beta_n + 1)^2}.
\]
Therefore
\[
\prod_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 \leq \prod_{n=-N}^{N} \left[ 1 + \frac{A}{\alpha_n^2} \right] \left[ 1 + \frac{B}{\alpha_n^2} \right]
\]
\[
\leq \exp\left\{ \sum_{n=-\infty}^{\infty} \left[ \frac{A}{\alpha_n^2} + \frac{B}{\alpha_n^2} \right] \right\}.
\]
Since \( \{\alpha_n\} \) is separated and no \( \alpha_n \) vanishes, the series \( \sum \alpha_n^{-2} \) converges and
\[
\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^2} < \infty \quad \text{for} \quad \alpha_n < 0 < \infty.
\]
It follows that \( \sup \|f_n\| < \infty \), so that a subsequence of \( \{f_n\} \) converges weakly to a function \( g \) in \( H \) for which \( g(\lambda_n + i) = 0 \) \((-\infty < n < \infty)\) and \( g(0) = 1 \).

**Proof of Theorem 4.** Let \( \lambda_n = \alpha_n + i\beta_n \). Since \( \{\alpha_n\} \) is a complete interpolating sequence, it follows from Lemma 3 and Proposition 1 that the mapping \( T: H \to \ell^2 \) given by \( Tf = \{ f(\alpha_n) \} \) is continuous, one-to-one, and onto. By the open mapping theorem, \( T \) has a continuous inverse. Thus, there exists a positive constant \( A \) such that
\[
A\|f\|^2 \leq \sum |f(\alpha_n)|^2 \quad (f \in H).
\]
By a theorem of Duffin and Schaeffer [6, p. 355] we have
\[
B\|f\|^2 \leq \sum |f(\lambda_n)|^2
\]
for some constant $B > 0$ and all $f$ in $H$. Now there exist functions $g_n$ in $H$ such that $g_n(\alpha_k) = \delta_{nk}$. It follows from Lemma 5 that for each $k$ the sequence of functions $\{e^{i\lambda_n \alpha_k}\}$ is incomplete in $L^2(-\pi, \pi)$. Therefore, we can find functions $f_n$ in $H$ such that $f_n(\lambda_k) = \delta_{nk}$. Fix $\{c_n\}$ in $l^2$ and set $F_n(z) = \sum_{k=1}^{N} c_n f_n(z)$ $(N = 1, 2, \ldots)$. Since $F_n(\lambda_k)$ is equal to $c_k$ when $|k| \leq N$ and has the value 0 for $|k| > N$, (13) gives

$$\|F_n\|^2 \leq \frac{1}{B} \sum_{k=-\infty}^{\infty} |F_n(\lambda_k)|^2 \leq \frac{B}{2} \sum_{k=-\infty}^{\infty} |c_k|^2$$

so that a subsequence of $\{F_n\}$ converges weakly to a function $F \in H$ for which $F(\lambda_k) = c_k$ ($-\infty < k < \infty$).

**Corollary 2.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis, and suppose that

$$|\Re(\lambda_n) - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty).$$

Then $\{\lambda_n\}$ is a complete interpolating sequence.

**Proof.** It was shown by Duffin and Eachus [5] that the inequality

$$\|\sum c_n (e^{i\Re(\lambda_n)x} - e^{i\lambda_n})\|_{L^2(-\pi, \pi)}^2 \leq \theta^2 \sum |c_n|^2$$

holds for some constant $\theta$, $0 \leq \theta < 1$, and every sequence $\{c_n\}$ in $l^2$. By a theorem of Paley and Wiener [8, p. 100], $\{\Re(\lambda_n)\}$ is a complete interpolating sequence. The result then follows from Theorem 4.

**Theorem 5.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $\{\lambda_n\}$ is a complete interpolating sequence, then each function in $L^2(-\pi, \pi)$ has a unique expansion of the form

$$g(t) = \lim_{N \to \infty} \sum_{n=1}^{N} c_n e^{i\lambda_n t}.$$ 

Moreover, $A \sum |c_n|^2 \leq \|g\|^2_{L^2(-\pi, \pi)} \leq B \sum |c_n|^2$, where $A$ and $B$ are positive constants independent of $f$.

**Proof.** It follows from Lemma 1 that the inequality

$$A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|_{L^2(-\pi, \pi)}^2$$

holds for some $A > 0$ and all finite sequences $\{c_n\}$. Since $\{\lambda_n\}$ is separated (Proposition 1), Lemma 3 shows that

$$\sum |f(\lambda_n)|^2 \leq B \|f\|^2$$

for some $B > 0$ and every $f$ in $H$.

If $K_n$ denotes the reproducing function at $\lambda_n$,

$$K_n(z) = K(z, \lambda_n) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)},$$
then (15) may be rewritten as

\[(16) \quad \sum |(f, K_n)|^2 \leq B \|f\|^2.\]

Let us set \( f = \sum c_n K_n \) where \( \{c_n\} \) is a finite sequence. By (16) and the Cauchy-Schwarz inequality

\[\|f\|^2 = \langle f, \sum c_n K_n \rangle = \sum c_n \langle f, K_n \rangle \leq \left( \sum |c_n|^2 \right)^{1/2} \left( \sum |\langle f, K_n \rangle|^2 \right)^{1/2} \leq \left( \sum |c_n|^2 \right)^{1/2} B^{1/2} \|f\|,\]

so that \( \|f\|^2 \leq B \sum |c_n|^2 \). Taking the Fourier transform of \( f \) we get

\[(17) \quad A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|_{L^2(-\pi, \pi)}^2 \leq B \sum |c_n|^2\]

for every finite sequence \( \{c_n\} \) and hence for every sequence in \( l^2 \). It is a simple consequence of (17) that each function in \( L^2(-\pi, \pi) \) which lies in the closed linear span of \( \{e^{i\lambda_n t}\} \) has a unique expansion of the form \( \lim_{N \to \infty} \sum_{-N}^{N} c_n e^{i\lambda_n t} \) with \( \{c_n\} \) in \( l^2 \). Since the exponentials \( e^{i\lambda_n t} \) are complete in \( L^2(-\pi, \pi) \), the result follows.

**Theorem 6.** Let \( \{\lambda_n\} \) be a complete interpolating sequence. There exist positive numbers \( \delta_n \) such that \( \{\mu_n\} \) is a complete interpolating sequence whenever \( |\lambda_n - \mu_n| < \delta_n \).

**Proof.** Since \( \{\lambda_n\} \) is interpolating there is a constant \( A > 0 \) such that

\[(18) \quad A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|_{L^2(-\pi, \pi)}^2 \]

for every finite sequence \( \{c_n\} \). If \( \delta_n > 0 \) is chosen small enough so that

\[\sum_{n=1}^{\infty} \|e^{i\lambda_n t} - e^{i\mu_n t}\|_{L^2(-\pi, \pi)}^2 \leq \frac{A}{2}\]

whenever \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots) \), then for every finite sequence \( \{c_n\} \)

\[\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|_{L^2(-\pi, \pi)}^2 \leq \left( \sum |c_n|^2 \right) \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2 \leq \left( \sum |c_n|^2 \right) \left( \sum \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2 \right) \leq \frac{A}{2} \sum |c_n|^2\]

Combining (18) and (19) we get

\[\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|_{L^2(-\pi, \pi)}^2 \leq \frac{1}{2} \|\sum c_n e^{i\lambda_n t}\|_{L^2(-\pi, \pi)}^2\]

whenever \( |\lambda_n - \mu_n| < \delta_n \) \( (n = 1, 2, \ldots) \). Since \( \{e^{i\lambda_n t}\} \) is complete in \( L^2(-\pi, \pi) \), it follows from a theorem of Boas [2, p. 469] that \( \{e^{i\mu_n t}\} \) is also complete. In Theorem 3 it was shown that \( \{\mu_n\} \) is interpolating whenever the \( \lambda_n \) are sufficiently small, whence the result follows.
4. Interpolation in $E^p_t$. We use the standard notation $E^p_t$ to denote the space of entire functions of exponential type $\tau$ ($0 < \tau < \infty$) which belong to $L^p(-\infty, \infty)$ on the real axis. For the properties of the spaces $E^p_t$ see [4]. For $0 < p < \infty$, let

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

while for $p = \infty$, let $\|f\|_\infty = \sup |f(x)|$ (x real).

**Definition.** A sequence $\{\lambda_n\}$ of distinct complex numbers is called an interpolating sequence for $E^p_t$ if $TE^p_t \supset l^p$. Here we continue to denote by $T$ the mapping $f \rightarrow \{f(\lambda_n)\}$.

The following results are derived from Lemmas 2 and 3 in essentially the same way as Proposition 1 and Corollary 1. The proofs are therefore omitted.

**Proposition 3.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE^p_t \supset l^1$ ($1 < p < \infty$), then $\{\lambda_n\}$ is separated.

**Corollary 2.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE^p_t \supset l^1$ ($1 < p < \infty$), then $TE^p_t \subset l^p$.

The remainder of this section is devoted to interpolation in $E^p_t$ in the special cases $p = 1$ and $p = \infty$.

**Theorem 7.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that there exist functions $f_n$ in $E^\infty_p$ satisfying

$$f_n(\lambda_k) = \delta_{nk}, \quad \|f_n\|_\infty \leq M, \quad (\text{all } n, k).$$

Then $TE^\infty_t \supset l^\infty$ whenever $\tau > \mu$.

**Proof.** It is well known [4, p. 82] that for every $f$ in $E^\infty_t$

$$|f(x + iy)| \leq \|f\|_\infty e^{\mu|y|},$$

so that $TE^\infty_t \subset l^\infty$ for each $\tau > 0$.

Fix $\{c_n\}$ in $l^\infty$, $\tau > \mu$ and let $\varepsilon = (\tau - \mu)/2$. We show that the function

$$f(z) = \sum_{n=0}^{\infty} c_n f_n(z) \left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)} \right]^2$$

belongs to $E^\infty_t$. Clearly, $f(\lambda_n) = c_n$ ($-\infty < n < \infty$). Let $\lambda_n = \alpha_n + i\beta_n$ and suppose that $|\beta_n| \leq \alpha$ and $|c_n| \leq N$. For $m = 0, 1, 2, \ldots$, let $S_m$ be the set of integers $n$ for which $m - 1 \leq |\lambda_n| \leq m + 2$ and $T_m$ the set of $n$ for which $|\lambda_n| < m - 1$ or $|\lambda_n| > m + 2$. The method of proof of Proposition 1 shows that $\{\lambda_n\}$ is separated. Since $\{\beta_n\}$ is bounded there is a constant $K$, independent of $m$, such that the number of integers in $S_m$ is at most $K$. For $m \leq |z| \leq m + 1$, write

$$f(z) = \sum_{n \in S_m} c_n f_n(z) \left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)} \right]^2 + \sum_{n \in T_m} c_n f_n(z) \left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)} \right]^2.$$
Since \((\sin z)/z\) is entire of exponential type 1 and is bounded by 1 on the real axis, (20) shows that \(|(\sin z)/z| \leq e^{|\Im z|}\). Therefore, setting \(z = x + iy\), we have

\[
|f(x)| \leq NM \exp(\mu |y|) \sum_{n \in \mathbb{N}_m} \exp(2e|y - \beta_n|)
\]

\[
+ \frac{NM \exp(\mu |y|)}{e^2} \sum_{n \in \mathbb{Z}} \frac{\exp(2e|y - \beta_n|)}{|z - \lambda_n|^2}
\]

\[
\leq KMN \exp((\mu + 2e)|y| + 2e|y - \beta_n|)
\]

\[
+ e^{-2NM} \exp((\mu + 2e)|y| + 2e|y - \beta_n|) \sum_{n \in \mathbb{Z}} \frac{1}{|z - \lambda_n|^2}.
\]

We claim that the sums \(\sum_{n \in \mathbb{Z}} |z - \lambda_n|^{-2}\) have a uniform upper bound for all \(m \geq 0\) and \(m \leq |z| \leq m + 1\). Since \(\{\lambda_n\}\) is a separated sequence, our assertion is immediate when each \(\lambda_n\) is real, while in the general case the existence of an upper bound follows readily from the boundedness of \(\Im \lambda_n\). It follows from (22) that the series in (21) converges uniformly in each disk \(|z| \leq m\) \((m = 1, 2, \ldots)\) and that, for some constant \(A\), \(|f(x)| \leq A \exp[(\mu + 2e)|y|]\). Since \(\tau = \mu + 2e\), \(f\) belongs to \(E^\infty_r\) and the proof is complete.

**Theorem 8.** If \(\{\lambda_n\}\) is a real sequence with \(\lambda_{n+1} - \lambda_n \geq 1\) \((-\infty < n < \infty)\), then \(TE^\infty_r = l^\infty\) whenever \(\tau > \pi\).

**Proof.** That \(TE^\infty_r \subset l^\infty\) is clear. It follows readily from Theorem 1 that \(\{\lambda_n\}\) is an interpolating sequence for \(E^2_\mu\) whenever \(\mu > \pi\). Indeed, if we set \(\mu_n = (\mu/\pi)\lambda_n\) then \(\mu_{n+1} - \mu_n \geq \mu/\pi > 1\) and Theorem 1 shows that \(\{\mu_n\}\) is an interpolating sequence for \(E^2_\mu\). Therefore, given \(\{c_n\} \in l^2\) there exists a function \(g\) in \(E^2_\mu\) such that \(g(\mu_n) = c_n\) for all \(n\). Setting \(f(x) = g((\mu/\pi)x)\) we see that \(f\) belongs to \(E^2_\mu\) and \(f(\lambda_n) = c_n\) (all \(n\)), and this establishes our assertion. Let us now fix \(\mu\) with \(\pi < \mu < \tau\). Lemma 2 shows that there exist functions \(f_n\) in \(E^2_\mu\) for which

\[
f_n(\lambda_k) = \delta_{nk}, \quad \sup_n \|f_n\|_2 < \infty, \quad (all\ n, k).
\]

From (7) it follows that \(|f(x)|^2 \leq (\mu/\pi)\|f\|_2^2\) for all \(f\) in \(E^2_\mu\) and all real \(x\), so that \(\sup_n \|f_n\|_\infty < \infty\). The conclusion now follows from Theorem 7.

In the same way we get the following result.

**Theorem 9.** Let \(\{\lambda_n\}\) be a sequence of points lying in a strip parallel to the real axis and suppose that

\[
|\Re(\lambda_n) - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty).
\]

Then \(TE^\infty_r = l^\infty\) whenever \(\tau > \pi\).

**Theorem 10.** The integers are not an interpolating sequence for \(E^\infty_r\).

**Proof.** We show that the sequence \(\{c_n\}\) given by
cannot be interpolated. Suppose first that \( \{w_n\}\) is an arbitrary sequence in \( l^\infty \) and that \( w_0 = 0 \). If there is an \( f \) in \( E^\infty_{\omega} \) with \( f(n) = w_n \ ( -\infty < n < \infty ) \), then
\[
g(z) = \frac{f(z)}{z} \quad \text{is in } E^2_{\omega} \quad \text{and} \quad ng(n) = w_n \ ( -\infty < n < \infty ) .
\]
If \( h \) is any other function in \( E^2_{\omega} \) for which \( zh(z) \) belongs to \( E^\infty_{\omega} \), then \( nh(n) = w_n \). Thus
\[
g(z) = \frac{f(z)}{z} = \sum_{n=0}^{\infty} \frac{w_n}{n} \frac{\sin \pi(z - n)}{(z - n)} + \frac{\alpha \sin \pi \omega}{\pi \omega}
\]
is the most general function in \( E^\infty_{\omega} \) with \( f(n) = w_n \ ( n \neq 0 ) \) and \( f(0) = 0 \).

A necessary condition that \( zg(z) \) belong to \( E^\infty_{\omega} \) is that its derivative \( zg'(z) \) + \( g(z) \) be bounded on the real axis \([4, \text{p. 206}]\) and hence, in particular, that
\[
ng'(n) + g(n) \quad \text{be bounded uniformly in } n .
\]
Since \( g \) belongs to \( E^2_{\omega} \), \( g(n) \to 0 \) as \( |n| \to \infty \), so that \( \{ng'(n)\} \) must be bounded.

Now, let \( \{c_n\} \) be given by (23) and let
\[
g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \sin \pi(z - n)}{n} \frac{\sin \pi \omega}{\pi \omega}
\]

We will show that \( zg(z) \) is not bounded on the real axis by showing that
\[
|ng'(n)| \to \infty \quad \text{as } n \to \infty .
\]
By the preceding remarks the integers cannot be interpolating for \( E^\infty_{\omega} \).

We have
\[
g'(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{\pi^2(z - n) \cos \pi z - \pi \sin \pi z}{\pi^2(z - n)^2}
\]
so that for \( k > 0 \),
\[
g'(k) = \sum_{n=1; n \neq k}^{\infty} \frac{1}{n} \frac{\cos \pi k}{k - n} = \sum_{n=1; n \neq k}^{\infty} \frac{1}{n} \frac{(-1)^k}{k - n} .
\]
Thus
\[
k g'(k) = (-1)^k \sum_{n=1; n \neq k}^{\infty} \left( \frac{1}{n} - \frac{1}{n - k} \right) .
\]
It is not difficult to show that
\[
\sum_{n=1; n \neq k}^{\infty} \left( \frac{1}{n} - \frac{1}{n - k} \right) = \left( 1 + \frac{1}{2} + \ldots + \frac{1}{k} \right) - \frac{2}{k}
\]
so that \( |kg'(k)| \to \infty \) as \( k \to \infty \).

**Theorem 11.** If \( \{\lambda_n\} \) is a real sequence with \( \lambda_{n+1} - \lambda_n \geq 1 \ ( -\infty < n < \infty ) \), then \( TE^1_{\tau} = l^1 \) whenever \( \tau > \pi \).
Proof. Lemma 3 shows that $TE_1^i \subseteq l^1$. It follows just as in the proof of Theorem 8 that for $\pi < \mu < \tau$ there exist functions $g_n$ in $E_1^\mu$ with $g_n(\lambda_k) = \delta_{nk}$ and $\sup_n \|g_n\|_2 < \infty$. If we set $\epsilon = \tau - \mu$ and let

$$f_n(z) = g_n(z)(\sin \epsilon(z - \lambda_n))/\epsilon(z - \lambda_n),$$

then $f_n \in E_1^\epsilon$ and $f_n(\lambda_k) = \delta_{nk}$. Hölder's inequality shows that

$$\|f_n\|_1 \leq \|g_n\|_2 \left|\frac{\sin \epsilon(z - \lambda_n)}{\epsilon(z - \lambda_n)}\right|_2$$

and it follows that $\sup_n \|f_n\| < \infty$.

Now, choose $\{c_n\}$ in $l^1$ and set

(24) $$f(z) = \sum_{n=0}^\infty c_n f_n(z).$$

Since $\sum \|c_n f_n\| < \infty$, $f$ belongs to $E_1^\epsilon$, and Lemma 3 implies that the convergence in (24) is uniform in each horizontal strip. Therefore, $f(\lambda_n) = c_n$ ($-\infty < n < \infty$) and the proof is complete.

It is easy to see that this result is best possible, in the sense that $\tau$ cannot always be taken equal to $\pi$. Indeed, the integers are not an interpolating sequence for $E_1^\mu$ for the trivial reason that the nonzero integers are a set of uniqueness. However, we have the following stronger result.

**Theorem 12.** The nonzero integers are not an interpolating sequence for $E_1^\mu$.

Proof. Lemma 3 shows that point evaluations are continuous linear functionals on $E_1^\mu$. By Lemma 2 it is enough to show that the unit ball of $l^1$ cannot be interpolated in a uniformly bounded way. Since

$$f_n(z) = n(\sin \pi(z - n))/\pi z(z - n) \quad (n \neq 0)$$

is the unique function in $E_1^\mu$ with the property that $f_n(k) = \delta_{nk}$, it is sufficient to show that $\|f_n\|_1 \to \infty$ as $n \to \infty$. For $n > 0$,

$$\|f_n\|_1 = \int_{-\infty}^\infty |f_n(x)| \, dx > \frac{n}{\pi} \int_1^\infty \left|\frac{\sin \pi x}{x(x + n)}\right| \, dx$$

$$> \frac{n}{\pi} \sum_{k=1}^{\infty} \int_{k+1/4}^{k+3/4} \frac{\sin \pi x}{x(x + n)} \, dx$$

$$> \frac{\sqrt{2}}{2\pi} \sum_{k=1}^{\infty} \int_{k+1/4}^{k+3/4} \left(\frac{1}{x} - \frac{1}{x + n}\right) \, dx$$

$$= \frac{\sqrt{2}}{2\pi} \log \prod_{k=1}^{\infty} \left[\frac{k + 3/4}{k + n + 3/4} \cdot \frac{k + n + 1/4}{k + 1/4}\right].$$

Using the relation $\Gamma(x + 1) = x\Gamma(x)$ it easily follows that the infinite product
above is equal to
\[
\lim_{N \to \infty} \frac{(1 + 3/4)(2 + 3/4) \cdots (N + 3/4)}{(1 + n + 3/4)(2 + n + 3/4) \cdots (N + n + 3/4)} \cdot \frac{(1 + n + 1/4) \cdots (N + n + 1/4)}{1 + 1/4 \cdots (N + 1/4)} = \frac{\Gamma(1 + 1/4) \cdots \Gamma(n + 1 + 3/4)}{\Gamma(1 + 3/4) \cdots \Gamma(n + 1 + 1/4)}.
\]

From the estimate $\Gamma(x + 1) \sim (2\pi)^{1/2} x^{x+1/2} e^{-x}$ (as $x \to \infty$) we conclude that $\Gamma(n + 1 + 3/4)/\Gamma(n + 1 + 1/4) \to \infty$ (as $n \to \infty$), and the proof is complete.

The proof of the next theorem is similar to that of Theorem 11 and is therefore omitted.

**Theorem 13.** Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that
\[
|\text{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \quad (-\infty < n < \infty).
\]
Then $TE_n^\tau = l^1$ whenever $\tau > \pi$.

**BIBLIOGRAPHY**


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