A GENERALIZATION OF THE \( \cos \pi \rho \) THEOREM

BY

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ABSTRACT. Let \( f \) be an entire function, and let \( \beta \) and \( \lambda \) be positive numbers with \( \beta < \pi \) and \( \beta \lambda < \pi \). Let \( E(r) = \{ \theta : \log |f(re^{i\theta})| > \cos \beta \lambda \log M(r) \} \). It is proved that either there exist arbitrarily large values of \( r \) for which \( E(r) \) contains an interval of length at least \( 2\beta \), or else \( \lim_{r \to \infty} r^{-\lambda} \log M(r, f) \) exists and is positive or infinite. For \( \beta = \pi \) this is Kjellberg's refinement of the \( \cos \pi \rho \) theorem.

1. Introduction. Let \( f \) be an entire function. The classical \( \cos \pi \rho \) theorem (see [4, Chapter 3] for its history) asserts that if \( f \) has order \( \rho \), with \( 0 < \rho < 1 \), then

\[
\limsup_{r \to \infty} \frac{\log m(r)}{\log M(r)} \geq \cos \pi \rho,
\]

where \( M(r) \) and \( m(r) \) denote \( \sup |f(z)| \) and \( \inf |f(z)| \) on \( |z| = r \), respectively.

Kjellberg [11] proved a striking improvement of this theorem. He showed that, for any number \( \lambda \in (0, 1) \), either \( \log m(r) > \cos \pi \lambda \log M(r) \) holds for certain arbitrarily large values of \( r \) or else \( \lim_{r \to \infty} r^{-\lambda} \log M(r) \) exists and is positive or infinite. (The case \( \lambda = \frac{1}{2} \) had been proved earlier by Heins [7].) A consequence of Kjellberg's theorem is that if \( f \) has lower order \( \rho^* \in (0, 1) \) then the lim sup in (1) is \( \geq \cos \pi \rho^* \). We remark that in this theorem it is not necessary to make any assumption about the order of \( f \).

In this note I shall prove the following result:

Theorem 1. Let \( f \) be a nonconstant entire function. Let \( \beta \) and \( \lambda \) be numbers with \( 0 < \lambda < \infty \), \( 0 < \beta \leq \pi \), \( \beta \lambda < \pi \). Then either

(a) there exist arbitrarily large values of \( r \) for which the set of \( \theta \) such that \( \log |f(re^{i\theta})| > \cos \beta \lambda \log M(r) \) contains an interval of length at least \( 2\beta \), or else

(b) \( \lim_{r \to \infty} r^{-\lambda} \log M(r) \) exists, and is positive or infinite.

For \( \beta = \pi \) this is Kjellberg's theorem. For \( \beta = \pi/2\lambda \) the theorem provides a sharpening of results of Arima [1] and Heins [8, p. 121].

The possibility that there might be a result like the one in Theorem 1 was suggested to me by A. Weitsman. I would also like to acknowledge some very
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Drasin and Shea [5], [6] have characterized functions extremal for the \( \cos \pi p \) theorem, that is, those entire functions \( f \) of order \( p \in (0, 1) \) for which equality holds in (1). It would be interesting to determine what sort of functions are extremal, in a similar sense, for Theorem 1.

Theorem 1 suggests an analogous problem for meromorphic functions. For a given number \( \alpha \), what can we say about the size of the set

\[
E_\alpha(r) = \{ \theta : \log |f(re^{i\theta})| > \alpha T(r, f) \},
\]

where \( T(r, f) \) denotes the Nevanlinna characteristic of \( f \)? For \( \alpha = 0 \) the author has proved [2], [3] the "spread relation":

\[
\limsup_{r \to \infty} \text{meas } E_0(r) \geq \min \{ 2\pi, 4\rho^{-1} \sin^{-1}(\delta/2)^{1/2} \},
\]

where \( \rho \) is the lower order of \( f \) and \( \delta = \delta(\infty, f) \) is the Nevanlinna deficiency of \( f \) at \( \infty \). It is also known ([12], [13], [14], [15]) that certain hypotheses on \( \alpha \), \( \delta \), and \( \rho \) insure that \( \limsup_{r \to \infty} E_\alpha(r) = 2\pi \).

The proof of Theorem 1 depends on two key inequalities involving an auxiliary function \( \mu(r) \). In \S 2 we state the inequalities and then show how the conclusion of the theorem follows from them. In \S\S 3, 4 we obtain some results about harmonic functions which are needed to prove the inequalities, and in \S\S 5, 6 we prove the inequalities themselves.

2. The auxiliary functions and key inequalities. Let \( f \) be entire and nonconstant. Consider the function \( u \) defined by

\[
u(r, \theta, \phi) = \int_{-\theta}^{\theta} \log |f(re^{i(\omega+\phi)}| \, d\omega
\]

where \( 0 \leq r < \infty \), \( 0 \leq \theta \leq \pi \), and \( \phi \) is any real number. This function was introduced by the author in [2], where it was shown (Statements (3.9) and (3.10)) that

\[
\text{for each fixed } \phi, \nu(r, \theta, \phi) \text{ is a subharmonic function of } re^{i\theta} \text{ in}
\]

\[
0 < \theta < \pi \text{ and, for each fixed } \theta \in [0, \pi], \nu(r, \theta, \phi) \text{ is a subharmonic function of } re^{i\phi} \text{ in the whole plane.}
\]

We remark that the statements (3.9) and (3.10) of [2] do not cover the cases when \( \theta \) is fixed and has the value zero or \( \pi \). However, for \( \theta = 0 \) we have \( u = 0 \) and for \( \theta = \pi \) we have

\[
u(r, \pi, \phi) = 2\pi \left[ N(r, 0, f) + k \log r + \log |c_k| \right]
\]

where \( N \) has its usual meaning and \( c_k \) is the first nonvanishing coefficient in the Maclaurin series of \( f \). The function in the brackets is a convex function of \( \log r \), hence is a subharmonic function of \( re^{i\phi} \). Thus (2) still holds when \( \theta = 0 \) and \( \theta = \pi \).
Now consider the function \( \nu(z) \) defined in the upper half plane by

\[
\nu(re^{i\theta}) = \sup_{\phi} u(r, \theta, \phi) \quad (0 \leq \theta \leq \pi).
\]

Alternatively,

\[
\nu(re^{i\theta}) = \sup_I \int I \log \left| f(re^{i\omega}) \right| \, d\omega
\]

where the sup is taken over all \( \omega \)-intervals \( I \) of length exactly \( 2\theta \). This \( \nu(z) \) is the same, except for a factor of \( 2\pi \), as the functions \( m_1(z) \) and \( T_1(z) \) considered by the author in [2].

\textbf{Proposition 1.} (a) For each fixed \( re^{i\theta} \) there exists an interval \( I \) of length \( 2\theta \) for which the sup in (3) is attained.

(b) \( \nu(z) \) is subharmonic in \( \text{Im } z > 0 \) and continuous on \( \text{Im } z \geq 0 \), except perhaps at \( z = 0 \).

(c) For each fixed \( \beta \in (0, \pi] \), \( \nu(re^{i\beta}) \) is a nondecreasing convex function of \( \log r \), \( 0 < r < \infty \).

(d) Define

\[
\nu_\beta(r) = \lim_{\theta \to 0^+} \frac{1}{\theta} \left[ \nu(re^{i\theta}) - \nu(r) \right] = \lim_{\theta \to 0^+} \frac{1}{\theta} \nu(re^{i\theta}).
\]

Then \( \nu_\beta(r) = 2 \log M(r, \beta) \) \( (0 < r < \infty) \).

\textbf{Proof.} (a) For \( re^{i\theta} \) fixed, \( u(r, \theta, \phi) \) is a continuous periodic function of \( \phi \). Take a \( \phi \) for which \( u(r, \theta, \phi) \) is maximal, and let \( I \) be the interval of length \( 2\theta \) centered at \( \phi \).

(b) The continuity statement follows from a routine argument. The definition (3), together with (2), shows that \( \nu(re^{i\theta}) \) is the supremum of a family of subharmonic functions of \( re^{i\theta} \). Such a function is always subharmonic, provided it is upper semicontinuous, and this is certainly the case here.

(c) The definition (3), together with (2), allows us to interpret \( \nu(re^{i\beta}) \) as the maximum modulus of a function of \( re^{i\phi} \) which is subharmonic in the whole plane. This implies the conclusion (c).

(d) For any interval \( I \) of length \( 2\theta \) we have \( \int_I \log \left| f(re^{i\omega}) \right| \, d\omega \leq 2\theta \log M(r) \). This implies

\[
\limsup_{\theta \to 0^+} \theta^{-1} \nu(re^{i\theta}) \leq 2 \log M(r).
\]

On the other hand, let \( re^{i\phi_0} \) be a point such that \( \log \left| f(re^{i\phi_0}) \right| = \log M(r) \). Then \( \nu(re^{i\theta}) \geq \int_{-\theta}^{\theta} \log \left| (r \exp \{ i(\phi_0 + \omega) \}) \right| \, d\omega \). Dividing by \( \theta \) and letting \( \theta \to 0 \) we obtain

\[
\liminf_{\theta \to 0^+} \theta^{-1} \nu(re^{i\theta}) \geq 2 \log M(r),
\]

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which with (5), proves (d). This completes the proof of Proposition 1.

Fix $\beta \in (0, \pi]$. Let $I(\tau)$ be an interval of length $2\beta$ such that $v(re^{i\beta}) = \int_{I(\tau)} \log |(re^{i\omega})| \, d\omega$. Define

$$
\mu(\tau) = \inf \{ \log |(re^{i\omega})| : \omega \in I(\tau) \}.
$$

Then conclusion (a) of Theorem 1 will hold if

$$
(6) \quad \mu(\tau) > (\cos \beta) \log M(\tau)
$$

for arbitrarily large values of $\tau$.

In the inequalities below it is assumed that $\beta$ and $\gamma$ satisfy the hypotheses of Theorem 1.

**Key inequality I.** There exist positive constants $C_1, C_2$, depending only on $\beta$ and $\lambda$, such that whenever $f(0) = 1$, we have

$$
(7) \quad \int_0^s \frac{\mu(t) - (\cos \beta) \log M(t)}{r^{1+\lambda}} \, dt > C_1 \frac{\log M(r)}{r^\lambda} - C_2 \frac{\log M(2s)}{s^\lambda} \quad (0 < r < s < \infty).
$$

**Key inequality II.** Let

$$
Q(r, t) = 2\pi^{-2}(r^2 - t^2)^{-1} \log rt^{-1}, \quad \gamma = \beta/\pi.
$$

Then, if $\limsup_{r \to \infty} r^{-\lambda} \log M(r) < \infty$, we have

$$
(8) \quad \log M(r^\gamma) \leq \int_0^\infty \left[ \mu(t^\gamma) + \log M(t^\gamma) \right] Q(r, t) \, dt \quad (0 < r < \infty).
$$

Once we have these inequalities the proof of Theorem 1 is completed by exactly the same reasoning as that used by Kjellberg in [10] and [11]. Let

$$
A = \liminf_{r \to \infty} r^{-\lambda} \log M(r), \quad B = \limsup_{r \to \infty} r^{-\lambda} \log M(r).
$$

If $A = B = \infty$ then conclusion (b) of Theorem 1 holds. If $B = \infty$ and $A < \infty$ we can find arbitrarily large values of $r$ and $s$, with $r < s$, such that the right-hand side of (7) is positive. So, if $f(0) = 1$, then (6) holds for some $t > r$ and we are done. If $B = 0$ and $f(0) = 1$ then $r^{-\lambda} \log M(r) > 0$ for $r > 0$. For each fixed $r$ the right-hand side of (7) is positive for all sufficiently large $s$, and again we are done.

The restriction $f(0) = 1$ can be removed in the usual way. Let $g$ be the entire function with $g(0) = 1$ and $f(z) = cz^k g(z)$ $(c \neq 0)$. Then

$$
(9) \quad \log M(r, g) = \log M(r, f) - \log |c| - k \log r,
$$

and $\mu(r, g)$ can be chosen so that (9) holds with $\mu$ in place of $\log M$. Using (7) with $g$ in place of $f$ we easily deduce
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\[
\int r \frac{\mu(t, f) - \cos \beta \lambda \log \mathcal{M}(t, f)}{t^{1+\lambda}} \, dt > c_1 r^{-\lambda} (\log \mathcal{M}(r, f) - \log |c| - k \log r)
\]

\[
- c_2 s^{-\lambda} (\log \mathcal{M}(2s, f) - \log |c| - k \log 2s)
\]

\[
- (\lambda^2)^{-1} (1 - \cos \beta \lambda) \log |c|^{-1} \quad (1 \leq r < s < \infty).
\]

Arguing as above, with obvious modifications, we find that if $\beta = \infty$ and $A < \infty$, or if $B = 0$ and $f$ is not a polynomial, then (6) holds for arbitrarily large values of $t$. (For $f$ a polynomial (6) holds for all sufficiently large values of $t$.)

Now consider the case $0 < B < \infty$, so that (8) holds. If (6) is false for all sufficiently large $t$ then

\[
\mu(t) \leq (\cos \beta \lambda) \log \mathcal{M}(t) \quad (t \geq t_0).
\]

Dividing $f$ by a large positive constant, if necessary, we can assume that (10) holds for all $t > 0$. (See the argument on p. 6 of [11].) Putting (10) in (8) we obtain

\[
\log \mathcal{M}(r^\gamma) \leq \int_0^\infty (1 + \cos \beta \lambda) \log \mathcal{M}(t^\gamma) \varphi(t, t) \, dt.
\]

Proceeding as in §4 of [11], with $\gamma \lambda$ in place of $\lambda$, we arrive at

\[
\lim_{r \to \infty} \frac{\log \mathcal{M}(r^\gamma)}{r^{\gamma \lambda}} = B > 0
\]

so that (b) of Theorem 1 holds.

3. A class of harmonic functions. In this section $B(t)$ will always stand for a nondecreasing convex function of $\log t$ on $(0, \infty)$ satisfying

\[
B(0) = B(0^+) = 0, \quad B(t) = O(t^\rho) \quad (t \to \infty)
\]

for some $\rho \in (0, 1)$.

The function $B(t)$ is absolutely continuous. Let $B_1(t)$ denote its logarithmic derivative, $B_1(t) = t B'(t)$. Then $B_1$ exists a.e., and is a nonnegative nondecreasing function of $t$.

Since $B(2t) \geq \int_t^{2t} B_1(s) s^{-1} ds \geq B_1(t) \log 2$, it follows that

\[
B_1(t) = O(t^\rho) \quad (t \to \infty).
\]

Similarly, $B(t^{1/2}) - B(t) \geq B_1(t) (\log t^{1/2} - \log t)$, so

\[
\lim_{t \to 0} \left( \log \frac{1}{t} \right) B_1(t) = 0.
\]

The Poisson integral
\[ k(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty B(t) \frac{r \sin \theta}{t^2 + r^2 + 2tr \cos \theta} \, dt \]

is harmonic in the slit plane \(|\arg z| < \pi\), is zero on the positive axis and tends to \(B(r)\) as \(\theta \to \pi -\), the convergence being uniform on bounded subsets of \((0, \infty)\).

The purpose of this section is to obtain some results about \(b_\theta = \partial b / \partial \theta\). These results generalize known properties of entire functions. Let

\[ F(z) = \prod_{n=1}^\infty \left(1 + \frac{\pi}{a_n} \right) \]

where \(0 < a_n < a_{n+1}\) and \(n^{1/p} = O(a_n)\). Then \(B(t) = N(t, 0, f) = \sum \log^+(t/a_n)\) satisfies our hypotheses, and \(B_1(t) = n(t, 0, f)\). In this case the Poisson integral \(b\) has a representation \(b(re^{i\theta}) = \pi^{-1} \int_0^\infty \log |F(re^{i\phi})| \, d\phi\), since the right-hand side is a function harmonic in the upper half plane which has the same boundary values as \(b\). Thus \(b_\theta(re^{i\theta}) = \pi^{-1} \log F(re^{i\theta})\). In particular,

\[ b_\theta(r) = \pi^{-1} \log M(r, F), \quad b_\theta(re^{im}) = \pi^{-1} \log m(r, F). \]

The reader might find it helpful to keep this special case in mind in what follows.

**Proposition 2.**

\[ b_\theta(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty \log \left| 1 + \frac{r}{t} e^{i\theta} \right| \, dB_1(t) \quad (|\theta| < \pi). \]

This generalizes the well-known formula \(\log |F(re^{i\theta})| = \int_0^\infty \log |1 + n^{-1} e^{i\theta}| \, dn(t)\).

**Proof.** We differentiate the Poisson integral with respect to \(\theta\); use

\[ \frac{\partial}{\partial \theta} \left( \frac{r \sin \theta}{t^2 + r^2 + 2tr \cos \theta} \right) = -\frac{\partial}{\partial t} \Re \left( \frac{re^{i\theta}}{t + re^{i\theta}} \right) \]

and integrate by parts. The result is

\[ b_\theta(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty B_1(t) \Re \left( \frac{re^{i\theta}}{t + re^{i\theta}} \right) \, dt. \]

Using

\[ \Re \left( \frac{re^{i\theta}}{t + re^{i\theta}} \right) = -\frac{\partial}{\partial t} \log \left| 1 + \frac{re^{i\theta}}{t} \right|. \]

doing another integration by parts, and observing (11), (12), we obtain (13).

**Proposition 3.**

\[ \lim_{\theta \to \pi^-} \frac{B(t) - k(re^{i\theta})}{\pi - \theta} = \lim_{\theta \to \pi^-} b_\theta(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty \log \left| 1 - \frac{r}{t} \right| \, dB_1(t). \]
The above integral is always well defined, but it may be \(-\infty\) for some values of \(r\).

**Proof.** Fix \(r \in (0, \infty)\). Since \(\log |1 + re^{i\theta}/t|\) is a decreasing function of \(\theta\) on \((0, \pi)\), and since \(\log |1 + re^{i\theta}/t| \leq \log (1 + r/t)\) \((0 < \theta \leq \pi)\) with

\[
\int_0^\infty \log (1 + r/t) \, dB_1(t) = b_\theta(r) < \infty,
\]

the monotone convergence theorem shows that

\[
\lim_{\theta \to \pi^-} \frac{1}{\pi} \int_0^\infty \log \left| 1 + \frac{re^{i\theta}}{t} \right| \, dB_1(t) = \frac{1}{\pi} \int_0^\infty \log \left| 1 - \frac{r}{t} \right| \, dB_1(t).
\]

Because of (13), this proves the second equality in (14).

The proof of the other equality seems to require a slightly more elaborate argument. Take \(\theta \in (0, \pi)\). From Proposition 2 we deduce

\[
b(\theta) = \int_0^\theta b_\theta(re^{i\phi}) \, d\phi = \frac{1}{\pi} \int_0^\infty dB_1(t) \int_0^\pi \log \left| 1 + \frac{re^{i\phi}}{t} \right| \, d\phi.
\]

Now

\[
B(\theta) = \int_0^\infty \log \left| 1 + \frac{re^{i\phi}}{t} \right| \, dB_1(t) = \frac{1}{\pi} \int_0^\infty dB_1(t) \int_0^\pi \log \left| 1 + \frac{re^{i\phi}}{t} \right| \, d\phi.
\]

So, setting \(A(r, \theta) = \pi^{-1} \int_0^\pi \log |1 + re^{i\phi}| \, d\phi\) we see that

\[
(15) \quad B(r) - b(re^{i\theta}) = \int_0^\infty A(r^{-1}, \theta) \, dB_1(t).
\]

A calculation shows

\[
\pi(\pi - \theta) \frac{\partial}{\partial \theta} \frac{A(r, \theta)}{\pi - \theta} = -\log |1 + re^{i\phi}| + \frac{1}{\pi - \theta} \int_\theta^{\pi} \log |1 + re^{i\phi}| \, d\phi.
\]

The monotonicity of \(\log |1 + re^{i\phi}|\) thus implies

\[
\frac{\partial}{\partial \theta} A(r, \theta) < 0 \quad (0 < \theta < \pi).
\]

Hence, for \(r\) and \(t\) fixed, \((\pi - \theta)^{-1} A(r/t, \theta) \searrow \) as \(\theta \nearrow \pi\). In particular,

\[
\frac{1}{\pi - \theta} A\left(\frac{r}{t}, \theta\right) < \frac{1}{\pi} A\left(\frac{r}{t}, \pi\right) = \log \left| 1 + \frac{r}{t} \right|.
\]

Now \(\int_0^\infty \log \left| 1 + \frac{re^{i\phi}}{t} \right| \, dB_1(t) = B(r) < \infty\), so, when we divide (15) by \(\pi - \theta\) and let \(\theta \nearrow \pi\) the monotone convergence theorem is again applicable. Since

\[
\lim_{\theta \to \pi^-} (\pi - \theta)^{-1} A(r/t, \theta) = \pi^{-1} \log |1 - r/t|,
\]

the other equality in (14) is thus established.

From now on we will denote the quantity appearing in (14) by \(b_\theta(-r)\).

**Proposition 4.** Let \(\sigma \in (0, 1)\). There exist positive constants \(k_1, k_2\), depending only on \(\sigma\), such that

\[
(16) \quad \int_r^s \frac{b_\theta(-r) - (\cos \sigma \theta) b_\theta(-s)}{t^{1+\sigma}} \, dt > k_1 \frac{b_\theta(r)}{r^\sigma} - k_2 \frac{b(s)}{s^\sigma} \quad (0 < r < s < \infty).
\]

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This generalizes the inequality (23) in Kjellberg's paper [10].

Proof. Let \( J \) be the integral in (16). Using Propositions 3 and 2, with \( \theta = 0 \), we deduce

\[
\int_0^\infty dB_t(u) \int_r^s t^{-(1+\sigma)}[\log|1-t/u| - \cos n\sigma \log(1+t/u)] \, dt.
\]

Kjellberg has shown [9], [10, p. 192] that

\[
\int_r^s t^{-(1+\sigma)}[\log|1-t/u| - \cos n\sigma \log(1+t/u)] \, dt
\]

\[
> k_1 r^{-\sigma} \log(1+r/u) - k_2 s^{-\sigma} \log(1+s/u) \quad (0 < u < \infty),
\]

where \( k_1, k_2 \) are positive and depend only on \( \sigma \). Putting this in (17), and using Proposition 2, with \( \theta = 0 \), we obtain (16).

Proposition 5.

(18) \( h \sigma t = \int_0^\infty [h \sigma t + h \sigma (-t)]Q(r, t) \, dt. \)

Here \( Q \) is as in the statement of key inequality II. This generalizes the identity (15) in Kjellberg's paper [11].

Proof. The function \( h \sigma t \) is harmonic in the half plane \( |\theta| < \pi/2 \) and is continuous on the closure (we define \( h \sigma t(0) = 0 \)). From (13) it follows easily that \( 0 \leq h \sigma r(e^{i\theta}) \leq h \sigma r(0) = O(r^\rho) \, (|\theta| < \pi/2, r \to \infty) \). Thus \( h \sigma t \) can be represented in the half plane by the Poisson integral of its boundary values. Since \( h \sigma iy = h \sigma (-iy) \) for real \( y \), we have

(19) \( h \sigma t = \frac{2r}{\pi} \int_0^\infty h \sigma iy \frac{dy}{r^2 + y^2} \quad (0 < r < \infty). \)

Now we show that \( h \sigma t \) is also the Poisson integral of its boundary values on the real axis. Take \( \delta \in (0, \pi) \) and consider \( h \sigma r(e^{i(\theta - \delta)}) \) as a function of \( re^{i\theta} \) in the upper half plane. It follows easily from (13) that \( \sup_{\delta \leq \pi} |h \sigma r(e^{i(\theta - \delta)})| = O(r^\rho) \, (r \to \infty) \) for each fixed \( \delta \). Since \( h \sigma r(e^{i(\theta - \delta)}) \) is continuous in the closed half plane, we have

\[
\int_0^\infty [h \sigma (te^{-i\delta}) + h \sigma (te^{i(\pi - \delta)})] \frac{dt}{t^2 + y^2} \quad (0 < y < \infty).
\]

Let \( \delta \downarrow 0 \). Then \( h \sigma (te^{-i\delta}) \uparrow h \sigma t \) and \( h \sigma (te^{i(\pi - \delta)}) \downarrow h \sigma (-t) \), with \( h \sigma (te^{i(\pi - \delta)}) \leq h \sigma t \). Since \( h \sigma t \) is integrable with respect to \( (t^2 + y^2)^{-1} \, dt \), we can apply the monotone convergence theorem and conclude

(20) \( h \sigma t = \frac{2r}{\pi} \int_0^\infty [h \sigma t + h \sigma (-t)] \frac{dt}{t^2 + y^2} \quad (0 < y < \infty). \)

Putting (20) in (19) and changing the order of integration, we get (18). (To see that Fubini's theorem is applicable here, consider, for fixed \( r \),
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\[
F(t, y) = \left[ b^\theta(t) + b^{\pi - \theta}(t) \right] \frac{y}{(r^2 + y^2)(t^2 + y^2)}
\]

\[
= 2b^\theta(t) \frac{y}{(r^2 + y^2)(t^2 + y^2)} - \left[ b^\theta(t) - b^{\pi - \theta}(t) \right] \frac{y}{(r^2 + y^2)(t^2 + y^2)}
\]

\[
= F_1(t, y) - F_2(t, y).
\]

Then \( F_1 \geq 0 \), \( F_2 \geq 0 \), and it is easy to verify that \( \int_0^\infty dt \int_0^\infty F_1(t, y) dy < \infty \).

4. More results on harmonic functions.

**Proposition 6.** Let \( b \) be harmonic and bounded in \( |z| < R \). Let \( \alpha \in (0, 1) \). Then

\[
|b^\theta(z)| \leq k(\alpha) \left( \frac{|z|}{R} \right) \sup_{\phi} |b(Re^{i\phi})| \quad (|z| \leq \alpha R),
\]

where \( k(\alpha) \) depends only on \( \alpha \).

**Proof.** We have

\[
b(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} b(Re^{i\phi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\theta - \phi)} d\phi.
\]

Differentiation with respect to \( \theta \) and simple estimates show

\[
|b^\theta(re^{i\theta})| \leq \sup_{\phi} |b(Re^{i\phi})| \cdot \frac{(R^2 - r^2)2rR}{(R - r)^4} \quad (0 < r < R).
\]

If \( r \leq \alpha R \) then \( (R - r)^3 \geq (1 - \alpha)^3 R^3 \), so

\[
\frac{(R^2 - r^2)2rR}{(R - r)^4} \leq \frac{4rR^2}{(1 - \alpha)^3 R^3} = k(\alpha) \frac{r}{R}
\]

and we are done.

The next result is a local version of Proposition 4 in which no assumption is made about the growth of the boundary function \( B(t) \).

**Proposition 7.** Let \( B(t) \) be a nondecreasing convex function of \( \log t \) on \( (0, \infty) \) with \( B(0) = B(0+) = 0 \). Let \( g(\phi) \) be bounded and measurable on \( (0, \pi) \). Let \( b \) be the function which is bounded and harmonic in the half disk \( \{ z : |z| < R, \text{Im} z > 0 \} \) and has the following boundary values:

\[
b(Re^{i\phi}) = g(\phi), \quad b(\pi) = 0, \quad b(-\pi) = B(\pi) \quad (0 < r < R).
\]

Let \( \sigma \in (0, 1), \alpha \in (0, 1) \). Suppose \( 0 < r < s = \alpha R \). Then

\[
\int_r^s b^\theta(-\pi) - (\cos m\pi)h^\theta(\pi) \frac{dt}{t^{1+\sigma}} \geq k_1 \left( \frac{b^\theta(r)}{\sigma} - k(\alpha, \sigma) \frac{B(\alpha^{-1} R) + M_1}{s^\sigma} \right)
\]

where \( k_1 \) is as in Proposition 4, \( k(\alpha, \sigma) \) is a positive constant depending only on \( \alpha \) and \( \sigma \), and \( M_1 = \sup_{0 < \phi < \pi} |g(\phi)| \).
Proof. Define $B^*(t)$ by

$$B^*(t) = B(t) \quad (0 < t \leq R)$$

$$= B_1(R) \log(t/R) + B(R) \quad (R < t < \infty)$$

where $B_1(t) = tB'(t)$. Then $B^*$ satisfies the hypotheses of the $B$ in §3. Define $b_1$ in the slit plane $|\arg z| < \pi$ by

$$b_1(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty B^*(t) \frac{r \sin \theta}{t^2 + r^2 + 2tr \cos \theta} dt.$$

Define $b_2$ in the half disk by $b_2 = b - b_1$. Then $b_2(x) = 0$ for $-R < x < R$, so $b_2$ has a harmonic extension to the full disk $|z| < R$.

Let $J$ be the integral in (21), and let $J_1$, $J_2$ be the corresponding integrals with $b$ replaced by $b_1$ and $b_2$. Proposition 4 can be used to estimate $J_1$, so we have

$$J = J_1 + J_2 > k_1 \frac{(b_1)_\sigma(r)}{r^\sigma} - k_2 \frac{(b_1)_\sigma(s)}{s^\sigma} + J_2$$

(22)

$$= k_1 \frac{b_\sigma(r)}{r^\sigma} - k_2 \frac{(b_2)_\sigma(r)}{r^\sigma} - k_2 \frac{(b_1)_\sigma(s)}{s^\sigma} + J_2'$$

Let $M_2 = \sup_{0 < \phi < \pi} |b_2(Re^{i\phi})|$. By Proposition 6 we have

$$r^{-\sigma} |(b_2)_\sigma(r)| \leq k(\alpha) R^{-1} r^{-\sigma} M_2 < k(\alpha) M_2 R^{-\sigma}. \quad (23)$$

Another application of Proposition 6 gives

$$|J_2| \leq \int_r^s \frac{|(b_2)_\sigma(-\delta)| + |(b_2)_\sigma(\delta)|}{t^{1+\sigma}} dt$$

(24)

$$\leq 2k(\alpha) M_2 R^{-1} \int_r^s t^{-\sigma} dt < 2k(\alpha)(1 - \sigma)^{-1} M_2 R^{-1} s^{1-\sigma}$$

$$< 2k(\alpha)(1 - \sigma)^{-1} M_2 R^{-\sigma}. \quad \text{Using Proposition 2 with } \theta = 0,$$

and integrating by parts twice, we find

$$(b_1)_\sigma(s) = \frac{1}{\pi} \int_0^\infty \frac{s}{(t + s)^2} B^*(t) dt$$

$$= \frac{1}{\pi} \int_0^R \frac{s}{(t + s)^2} B(t) dt + \frac{1}{\pi} \int_R^\infty \frac{[B(R) + B_1(R) \log(t/R)]}{(t + s)^2} ds dt.$$

Since $B(t) \leq B(R)$ for $0 < t < R$ and $s = \alpha R$, we have
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\[
(b_1)_\theta(s) \leq \frac{1}{\pi} B(R) \int_0^\infty \frac{s}{(t + s)^2} \, dt + \frac{1}{\pi} B_1(R) \int_0^\infty \frac{(\log(t/R))}{(t + s)^2} \, dt
\]

\[
= \frac{1}{\pi} B(R) + \frac{1}{\pi} B_1(R) \int_1^\infty \log t \frac{\alpha}{(t + \alpha)^2} \, dt.
\]

Now \( \log t \alpha/(t + \alpha)^2 < t^{\alpha^2}/(1 + t)^2 \) \( (1 < t < \infty, \ 0 < \alpha < 1) \), so the last integral is < 2. Hence

\[
(25) \quad (b_1)_\theta(s) < \pi^{-1} (B(R) + 2B_1(R)).
\]

Using (23), (24), (25) in (22) and remembering \( s = \alpha R \), we obtain

\[
(26) \quad M_2 \leq M_1 + \sup_{0 \leq \theta} |b_1(Re^{i\theta})|.
\]

Putting (27) in (26), and using

\[
B(R) \leq B(\alpha^{-1} R), \quad B_1(R) \leq \frac{1}{(\log \alpha^{-1})} \int_R^{\alpha^{-1} R} B_1(t) \frac{dt}{t} \leq \frac{B(\alpha^{-1} R)}{(\log \alpha^{-1})},
\]

we obtain (21).

5. Proof of key inequality I. We are assuming \( f(0) = 1 \). This implies that \( \nu(z) \), defined by (3), is continuous in the closed upper half plane, with \( \nu(0) = 0 \).

Fix \( R > 0 \). Define \( D \) by \( D = \{z: 0 < |z| < R, \ 0 < \arg z < \beta\} \). Let \( H \) be the bounded harmonic function in \( D \) which has the following boundary values:

\[
H(r) = 0, \quad H(re^{i\theta}) = \nu(re^{i\theta}) \quad (0 < r < R),
\]

\[
H(Re^{i\theta}) = \begin{cases} 
2\pi \log M(R, f) & (0 < \theta < \frac{1}{2} \beta), \\
4\pi \log M(R, f) & (\frac{1}{2} \beta < \theta < \beta).
\end{cases}
\]

Let \( \gamma = \beta/\pi \), and define \( b(z) \) in the upper half disk of radius \( R^{1/\gamma} \) by \( b(z) = H(z \gamma) \) \( (0 < |z| < R^{1/\gamma}, \ 0 < \arg z < \pi) \). Then \( b \) is the function considered in Proposition 7, with \( B(t) = \nu(\gamma e^{i\beta}) \), the \( R \) there replaced by \( R^{1/\gamma} \), and

\[
(27) \quad g(\phi) = 2\pi \log M(R, f) \quad (0 < \phi < \pi/2),
\]

\[
= 4\pi \log M(R, f) \quad (\pi/2 < \phi < \pi).
\]

The function \( B(t) \) satisfies the hypothesis of Proposition 7, by virtue of Proposition 1.

Let \( s = 2^{-\frac{1}{2}} R \) and let \( 0 < r < s \). Using (21), with \( \sigma = \gamma \lambda \ (\beta \lambda/\pi < 1) \) and \( \alpha = 2^{-\frac{1}{2}} \gamma \), we obtain
\[
\begin{align*}
\int_\tau^{1/\gamma} & b_\theta(-t) - \cos \pi \lambda b_\theta(t) \\
& \frac{dt}{t^{1+\sigma}} \\
& > k_1 \frac{b_\theta(r^{1/\gamma})}{r^\lambda} - k_2 \frac{B(2^{\frac{1}{2}} \gamma R^{1/\gamma}) + 4\pi \log M(R)}{s^\lambda},
\end{align*}
\]

where \( k_1, k_2 \) depend on \( \beta \) and \( \lambda \). Now \( b_\theta(t) = \gamma H_\theta(t) \), \( b_\theta(-t) = \gamma H_\theta(t) e^{i\beta} \).

Changing variables in (28), and using \( \beta(2M \gamma R 1/\gamma) = \gamma(2\log M(2\gamma s)) \), we obtain

\[
\int_\tau^{1/\gamma} H_\theta(te^{i\beta}) - \cos \pi \lambda H_\theta(t) \\
\frac{dt}{t^{1+\lambda}} > k_1 \frac{\gamma H_\theta(t)}{r^\lambda} - k_2 \frac{v(2se^{i\beta}) + 4\pi \log M(2\gamma s)}{s^\lambda}.
\]

Since \( v(2se^{i\beta}) \leq 2\pi \log M(2s) \), \( \log M(2\gamma s) \leq \log M(2s) \), we have

\[
\int_\tau^{1/\gamma} H_\theta(te^{i\beta}) - \cos \pi \lambda H_\theta(t) \\
\frac{dt}{t^{1+\lambda}} > k_1 \frac{\gamma H_\theta(t)}{r^\lambda} - k_2 \frac{\log M(2s)}{s^\lambda},
\]

where \( C_1, C_2 \) are positive constants depending on \( \beta \) and \( \lambda \).

By Proposition 1, \( v \) is subharmonic in \( D \). The harmonic function \( H \) majorizes \( v \) on the boundary of \( D \) (since \( v(r) = 0 \) for \( r \geq 0 \) and \( v(Re^{-i\beta}) \leq 2\pi \log M(R) \)). It follows that

\[
(30) \quad u(x) \leq H(x) \quad \text{for all } x \in D.
\]

Since \( v(r) = H(r) = 0 \) for \( r > 0 \), it follows from (30) and Proposition 1 that

\[
(31) \quad H_\theta(r) \geq v_\theta(r) = 2\log M(r) \quad (0 < r < R).
\]

Here, and in what follows, \( H_\theta(r) \) and \( H_\theta(te^{i\beta}) \) are understood to be one-sided derivatives computed from inside \( D \).

I claim that the following inequality also holds:

\[
(32) \quad H_\theta(te^{i\beta}) + H_\theta(t) \leq 2(\mu(t) + \log M(t)) \quad (0 < t < R).
\]

Let us assume (32). Using it together with (31), we find

\[
H_\theta(te^{i\beta}) - \cos \beta \lambda H_\theta(t) = [H_\theta(te^{i\beta}) + H_\theta(t)] - (1 + \cos \beta \lambda) H_\theta(t)
\]

\[
\leq 2(\mu(t) + \log M(t)) - 2(1 + \cos \beta \lambda) \log M(t)
\]

\[
= 2(\mu(t) - \cos \beta \lambda \log M(t)).
\]

Using (33) and (31) in (29), we obtain key inequality I.

To prove (32) we introduce another auxiliary function \( w(z) \). It is defined in the angle \( 0 \leq \theta \leq \frac{1}{4} \beta \) by

\[
(34) \quad w(re^{i\theta}) = \sup_E \int_E \log |(re^{i\omega})| \, d\omega \quad (0 < r < \infty, \ 0 \leq \theta \leq \frac{1}{4} \beta).
\]
where the sup is taken over all sets $E$ of the following form:

$$E = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3],$$

with

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq a_3 \leq b_3,$$

$$b_2 - a_2 = 2\theta, \quad (b_1 - a_1) + (b_3 - a_3) = 2\theta,$$

$$a_2 - b_1 = a_3 - b_2 = \beta - 2\theta.$$

Note that for $\theta = \beta/2$ the sets $E$ are simply intervals of length $2\beta$. Thus $w(\rho e^{i\beta/2}) = v(\rho e^{i\beta}).$

**Lemma.** (a) $w$ is subharmonic in $0 < \arg z < \frac{1}{2} \beta$ and continuous on $0 \leq \arg z \leq \frac{1}{2} \beta$.

(b) $\limsup_{r \to \frac{1}{2}\beta - \theta} \frac{u(\rho e^{i\beta/2}) - u(\rho e^{i\theta})}{2\theta/2 - \theta} \leq 2(\mu(\rho) + \log M(\rho)) \quad (0 < r < \infty)$.

Once the Lemma is proved, we obtain (32) as follows. Let $D_1 = \{z: 0 < |z| < R, 0 < \arg z < \beta/2\}$ and define $H_1(z)$ on $D_1$ by

$$H_1(re^{i\theta}) = H(r \exp\{i(\beta/2 + \theta)\}) - H(r \exp\{i(\beta/2 - \theta)\}).$$

Then $H_1$ is harmonic in $D_1$ and has the following boundary values:

$$H_1(re^{i\theta}) = 0, \quad H_1(re^{i\beta/2}) = H(re^{i\beta}) \quad (0 \leq r < R),$$

$$H_1(re^{i\theta}) = 2\pi \log M(\rho, f) \quad (0 < \theta < \frac{1}{2}\beta).$$

A look at the definition of $w$ shows

$$u(\rho e^{i\beta/2}) = u(\rho e^{i\beta}) = H(\rho e^{i\beta}) = H_1(\rho e^{i\beta/2}) \quad (0 < r < R),$$

(36)

$$u(\rho) = 0 \quad (0 < r < R), \quad u(\rho e^{i\theta}) \leq 2\pi \log M(\rho, f).$$

Thus $H_1$ majorizes $w$ on the boundary of $D_1$, hence it also majorizes $w$ inside $D_1$. Using this, together with (36), we obtain

$$\limsup_{r \to \frac{1}{2}\beta - \theta} \frac{u(\rho e^{i\beta/2}) - u(\rho e^{i\theta})}{2\theta/2 - \theta} \geq (H_1(\rho e^{i\beta/2}) = H_\theta(re^{i\beta}) + H_\theta(\rho) \quad (0 < r < R),$$

which, together with part (b) of the Lemma, proves (32).

**Proof of the Lemma.** The continuity statement follows from a routine argument which we leave to the reader.

For $r > 0$, $0 < \rho < r$, $-\pi \leq \psi \leq \pi$, define $r(\psi) > 0$ and $\alpha(\psi) \in (-\pi/2, \pi/2)$ by $r + re^{i\psi} = r(\psi)e^{i\alpha(\psi)}$. With this notation, a function $s$ defined on an open set $D$ is subharmonic in $D$ if and only if it is upper semicontinuous and, if for each $re^{i\theta} \in D$, there exists $\rho_0 > 0$ such that $s(re^{i\theta}) \leq \frac{1}{2}\pi^{-1} \int_{-\pi}^{\pi} s(r(\psi)e^{i(\theta + \alpha(\psi))})d\psi$ holds whenever $0 < \rho < \rho_0$. 

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For fixed $re^{i\theta}$ with $0 < r < \infty$, $0 < \theta < \frac{\pi}{2}$ it is easily shown that there exists a set $E$ of the form (35) for which the supremum in (34) is attained. Let $a_j, b_j$ be the endpoints of the intervals defining one such extremal $E$ and set

$$\sigma_1 = \frac{1}{2}(b_1 - a_1), \quad \sigma_2 = \frac{1}{2}(b_2 - a_2),$$

$$\phi_1 = \frac{1}{2}(a_1 + b_1), \quad \phi_2 = \frac{1}{2}(a_2 + b_2), \quad \phi_3 = \frac{1}{2}(a_3 + b_3).$$

In terms of the function $u$ introduced in §2 we have

(37) \[ u(re^{i\theta}) = u(r, \sigma_1, \phi_1) + u(r, \theta, \phi_2) + u(r, \sigma_2, \phi_3). \]

Assume $\sigma_1 > 0$. Choose $\rho_0 \in (0, r)$ such that whenever $0 < \rho < \rho_0$ we have $0 < \theta + \alpha(\psi) < \frac{\pi}{2}$, $0 < \sigma_1 + \alpha(\psi) < \frac{\pi}{2}$ $(-\pi \leq \psi \leq \pi)$. Define

$$E(\psi) = [a_1 - a(\psi), b_1 + a(\psi)]$$

$$\cup [a_2 - a(\psi), b_2 + a(\psi)] \cup [a_3 - a(\psi), b_3 - a(\psi)].$$

Then $E(\psi)$ satisfies (35), with $\theta$ replaced by $\theta + \alpha(\psi)$. Hence

(38) \[ u(\alpha(\psi) \exp[i(\theta + \alpha(\psi))]) \geq \int_{E(\psi)} \log|/(\alpha(\psi) \exp[i \omega])| \, d\omega. \]

Now

$$\int_{E(\psi)} \log|/(\alpha(\psi) e^{i \omega})| \, d\omega = u(\alpha(\psi), \sigma_1 + a(\Psi), \phi_1)$$

$$+ u(\alpha(\psi), \theta + a(\psi), \phi_2) + u(\alpha(\psi), \sigma_2, \phi_3 - a(\psi)).$$

Substitute (39) in (38), divide by $2\pi$, and integrate from $\psi = -\pi$ to $\psi = \pi$. The subharmonicity properties of $u$ mentioned in (2) yield

(40) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\alpha(\psi) \exp[i(\theta + \alpha(\psi))]) \, d\psi \geq u(r, \sigma_1, \phi_1) + u(r, \theta, \phi_2) + u(r, \sigma_2, \phi_3). \]

(We used here the fact that $\int_{0}^{2\pi} u(\alpha(\psi), \sigma_2, \phi_3 - a(\psi)) \, d\psi = \int_{0}^{2\pi} u(\alpha(\psi), \sigma_2, \phi_3 + a(\psi)) \, d\psi.$)

Comparing (40) with (37) we see that $u$ satisfies the criterion for subharmonicity at $re^{i\theta}$. (We were assuming $\sigma_1 > 0$. If $\sigma_1 = 0$ then $\sigma_2 = \theta - \sigma_1 > 0$, and we repeat the above argument with the roles of $[a_1, b_1], [a_3, b_3]$ interchanged.) Thus part (a) of the Lemma is proved.

Recall that $l(r)$ was chosen to be an interval of length $2\beta$ such that $\nu(re^{i\theta}) = \int_{l(r)} \log|/(re^{i\omega})| \, d\omega$ and $\mu(r)$ is the inf of $\log|/(re^{i\omega})|$ over $l(r)$. Fix $r$, and let $\omega_0$ be a point of $l(r)$ such that $\mu(r) = \log|/(re^{i\omega_0})|$. (It may happen that $\mu(r) = -\infty$, but this does not affect the argument.)

Write $l(r) = [a, b]$. Note $b - a = 2\beta$. We have

(41) \[ u(re^{i\beta/2}) = \int_{a}^{b} \log|/(re^{i\omega})| \, d\omega. \]
Let $c = \frac{1}{2}(a + b)$. For the proof of (b) we consider five cases.

Case I. $\omega_0 = a$.
Case II. $\omega_0 \in (a, c)$.
Case III. $\omega_0 = c$.
Case IV. $\omega_0 \in (c, b)$.
Case V. $\omega_0 = b$.

Assume Case I. For $0 < \theta < \frac{1}{2} \beta$ define $E(\theta) = [a, a] \cup [a + \beta - 2\theta, a + \beta] \cup [a + 2\beta - 2\theta, b]$. Then $E(\theta)$ has the form (35). Thus

$$u(re^{i\theta}) \geq \int_{E(\theta)} \log |f(re^{i\omega})| \, d\omega. \tag{42}$$

Using this and (41) we see that

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_a^{a + \beta - 2\theta} + \int_{a + \beta}^{a + 2\beta - 2\theta} \log |f(re^{i\omega})| \, d\omega.$$  

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2} \beta$. The result is

$$\limsup_{\theta \to \frac{1}{2} \beta} \frac{u(re^{i\beta/2}) - u(re^{i\theta})}{\beta - 2\theta} \leq \log |f(re^{i\theta})| + \log M(r).$$

Since $\log |f(re^{i\theta})| = \mu(r)$, the inequality above is equivalent to (b).

Now assume Case II. Let $I_1(\theta)$ be the interval with center $\omega_0$ and length $\beta - 2\theta$, and let $I_2(\theta)$ be the interval of length $\beta - 2\theta$ whose left endpoint lies $2\theta$ units to the right of the right endpoint of $I_1$. For $\theta$ sufficiently close to $\frac{1}{2} \beta$ we have $I_1 \cup I_2 \subseteq [a, b]$. Let $E(\theta)$ be the complement of $I_1 \cup I_2$ in $[a, b]$. Then $E(\theta)$ has the form (35). Thus (42) holds, and we have, for $\theta$ sufficiently close to $\frac{1}{2} \beta$,

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_{I_1(\theta)} + \int_{I_2(\theta)} \log |f(re^{i\omega})| \, d\omega.$$}

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2} \beta$. The first term on the right tends to $\mu(r)$ and the second one is dominated by $\log M(r)$. This proves (b) for Case II.

For Case III we let $E(\theta)$ consist of two intervals of length $2\theta$ and one degenerate interval. The right endpoint of the first interval is $\frac{1}{2} \beta - \theta$ units to the left of $c$, and the left endpoint of the second interval is $\frac{1}{2} \beta - \theta$ units to the right of $c$. Then $E(\theta)$ has the form (35), and we deduce this time

$$u(re^{i\beta/2}) - u(re^{i\theta}) \leq \int_a^{a + \frac{1}{2} (\beta - \theta)} + \int_{a + \frac{1}{2} (\beta - \theta)}^{b} \log |f(re^{i\omega})| \, d\omega.$$  

Divide by $\beta - 2\theta$ and let $\theta \to \frac{1}{2} \beta$. The first term on the right tends to $\mu(r)$ and the sum of the other two is dominated by $\log M(r)$. This proves (b) for Case III.
Cases IV and V are handled in a fashion similar to II and I, respectively. This completes the proof of the Lemma.

6. Proof of key inequality II. We established (18) under the hypothesis that the function $B(t)$ whose Poisson integral is $h$ satisfies $B(0) = 0$. However, the formula is still valid if we only assume

\begin{align*}
B(t) &= B^*(t) + A_1 \log t + A_2, \\
\end{align*}

where $B^*$ satisfies all the hypotheses of the $B$ in §3, and $A_1, A_2$ are constants. To see this, simply observe that in this case, with obvious notation, $b(re^{i\theta}) = b^*(re^{i\theta}) + \pi^{-1}A_1 \log r + \pi^{-1}A_2 \theta$, and (18) can be established for each of the harmonic functions on the right.

As in §5 we set $B(t) = v(t^\gamma e^{i\beta})$, where $\gamma = \beta/\pi$. We are assuming $\log M(r) = O(r^\lambda)$. Since $v(te^{i\beta}) \leq 2\pi \log M(t)$, it follows that $B(t) = O(t^{\lambda/\pi}) (t \to \infty)$. Since $\beta\lambda/\pi < 1$, $B(t)$ satisfies the growth condition of §3. We can write

\begin{align*}
\log |f(z)| &= \log |f_1(z)| + A_1 \log |z| + A_2
\end{align*}

with $f$ entire and $f_1(0) = 1$. It follows from this that $B(t)$ can be written in the form (43). Let $b$ be the Poisson integral of $B(t)$, as in §2, and define $H(z)$ by $H(z) = b(z^{1/\gamma}) (0 < \arg z < \beta)$. Using (18), we obtain

\begin{align*}
H_\theta(r^\gamma) = \int_0^{r^\gamma} [H_\theta(t^{1/\gamma}) + H_\theta(t^{1/\gamma} e^{i\beta})]Q(r, t) dt.
\end{align*}

To prove our key inequality (8) all we need to do is show that (31) and (32) are true for the $H$ being considered in this section. (In this case these inequalities are to hold for $0 < r < \infty$.)

Consider first (31). The function $H$ and $v$ are harmonic and subharmonic, respectively, in the angle $0 < \arg z < \beta$, and they are equal on the boundary, with the possible exception of $z = 0$, where well-defined boundary values need not exist. However, by considering the decomposition (43) one can easily deduce that in fact $H(re^{i\theta}) - v(re^{i\theta})$ tends to zero uniformly in $\theta$ as $r \to 0$. Since $v$ and $H$ are both $O(r^{\lambda})$ in the angle as $r \to \infty$, and since $\beta\lambda < \pi$, we once again can conclude that $v(z) \leq H(z)$ inside the angle. This is exactly what we needed to prove (31).

To prove (32) we define $H_1$ just as before, except now its domain is the full angle $0 < \arg z < \frac{1}{2}\beta$. Arguing as above, we conclude that $H_1$ majorizes the function $w$ inside this angle, and the deduction of (32) proceeds as in §5.

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