IRREDUCIBLE CONGRUENCES OVER $GF(2)$ (1)

BY

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ABSTRACT. In characterizing and determining the number of conjugate sets of irreducible congruences of degree $m$ belonging to $GF(p)$ relative to the group $G(p)$ of linear fractional transformations with coefficients belonging to the same field, the case $p = 2$ has been consistently excluded from considerations. In this paper we consider the special case $p = 2$ and determine the number of conjugate sets of $m$-ic congruences belonging to $GF(2)$ relative to $G(2)$.

1. Introduction. The conjugate sets of irreducible $m$-ic congruences

\[ c_m(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m = 0 \pmod{p} \]

belonging to the modular field defined by the prime $p$ under the group of linear fractional transformations

\[ T: x = (ax' + b)/(cx' + d), \quad a, b, c, d \in GF(p), \]

have been classified in terms of the irreducible factors of an absolute invariant $\pi_m(J, K)$ [2]. In this classification and in the various studies of conjugate sets that have followed, the most recent being that for which the degree $m$ is a power of an odd prime [3], the case $p = 2$ has been excluded as a possible characteristic for the base field $GF(p)$ because of the special considerations and treatments that would have been required. In this paper we consider this special case and therefore determine the number of conjugate sets of $m$-ic congruences over $GF(2)$ relative to $G(2) = G$.

For convenience we shall henceforth use $p$ rather than 2 for the characteristic 2 of our fields and we shall use the standard notation $IQ[m, p^k]$ for an irreducible monic congruence of degree $m$ over $GF(p^k)$. $GF'(p^k)$ will denote the subset of marks of $GF(p^k)$ which do not belong to any proper subfield and an $m$-ic over $GF'(p^k)$ will be regarded as a monic congruence of degree $m$ over $GF(p^k)$ with at

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least one coefficient belonging to $GF(p^k)$. We shall use $\mathcal{I}(m, p^k)|p^i$ or $f(x)|p^i$ to denote the congruence of degree $m$ whose coefficients are respectively the $p^i$th powers of those of $\mathcal{I}(m, p^k) = f(x)$. $\mathcal{I}(m, p^k)|p^i + p^w$ will mean the product of $f(x)|p^i$ and $f(x)|p^w$.

For $p = 2$ the group $G = G(2)$ is of order $p(p^2 - 1) = 6$ and may be easily recognized as the substitution group on three letters. If $T \in G$ is given by (1.2) then we shall say that $T$ is identified by the matrix $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ and that this matrix defines $T$. We will find it convenient to identify the transformations of $G$ by the six nonsingular $2 \times 2$ matrices over $GF(2)$ given as follows:

$$
\begin{align*}
I &= \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), & L &= \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \\
K &= \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right), & \overline{L} &= \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \\
K^2 &= \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right), & \overline{K} &= \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right) = LK^2.
\end{align*}
$$

Clearly the order of $K$, $o(K) = 3$; $o(L) = o(\overline{L}) = o(\overline{K}) = 2$ and the generators $K$ and $L$ of $G$ satisfy the condition $KLK = L$.

Finally, if $f(x)$ is an $\mathcal{I}(m, p^k)$ and $T \in G$ then $f(x)T = f'(x)$ will be called the transform of $f(x)$ by $T$ and is the monic polynomial congruence in $x$ obtained from $f((ax + b)/(cx + d))$. The set $\{f(x)T: T \in G\}$ is called a conjugate set relative to $G$ and the members of the set are said to be conjugate. If $f(x)T = f(x)$ then $f(x)$ is said to be self-conjugate under $T$. If $\eta$ is a root of a congruence $f(x)$ over $GF(p^k)$ and if $T = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ then $\eta T = (a\eta + b)/(c\eta + d)$ is called the transform of $\eta$ by $T$. Clearly $\eta^p T = (\eta T)^{p^i}$ for all $i \geq 0$.

Since $o(G) = 6$ then the orders of conjugate sets of $m$-ics are of the form $6/d$ where $d|m$. Thus, conjugate sets may be of order 6, 3, 2 or 1. To determine the number of conjugate sets of each possible order we consider separately the number of order 1, 2 and 3. In $\S 2$ we show that there can be no set of order 1 if $m > 2$ and the numbers $C_2$ and $C_3$ of sets of order 2 and 3 are considered in $\S 3$ and $\S 4$, respectively. The number $C_6$ of order 6 is the easily determined from the total number of irreducible $m$-ics over $GF(p)$.

2. Conjugate sets of order 1. If $m = 2$ then $x^2 + x + 1 = \mathcal{I}(2, p)$ is the only irreducible congruence and necessarily belongs to a set of order 1. It is, therefore, invariant under $G$.

If $m > 2$ and if $f(x) = \mathcal{I}(m, p)$ belongs to a conjugate set of order 1, then $f(x)T = f(x)$ for every $T \in G$ and it follows that $m = 6k$ for some $k \geq 1$. Thus, $f(x)$ is the product of $k$ distinct $\mathcal{I}(6, p^k)$, each of which is self-conjugate under every transformation $T \in G$ [1, p. 33]. If $s(x)$ denotes one of these factors, then
and, if $\mu$ is a root of $s(x)$, then its roots are

$$\mu, \mu^{b^k}, \mu^{b^{2k}}, \mu^{b^{3k}}, \mu^{b^{4k}}, \mu^{b^{5k}}.$$ 

Since $s(x)$ is irreducible over $GF(p^k)$, these roots are all distinct and belong to $GF(p^{6k})$.

Now this irreducible sextic $s(x)$ over $GF(p^k)$ is the product of two cubics $c_1(x)$ and $c_2(x) = \{c_1(x)\}^{b^3}$, each irreducible over $GF(p^{2k})$ and self-conjugate under $K$, the transformation of $G$ of order 3 given by (1.3). The roots of these two cubics are

$$\mu, \mu^{b^{2k}}, \mu^{b^{4k}} \text{ and } \mu^{b^k}, \mu^{b^{3k}}, \mu^{b^{5k}},$$

respectively. Now, since $s(x)$ is self-conjugate under $G$, $s(x)L = s(x)$ implies that $c_1(x)L = c_1(x)$ and we conclude that $\mu L = \mu^{b^{3k}}$ since $o(L) = 2$. We may assume that $\mu K = \mu^{b^{2k}}$. Then, since $L = LT$, we have

$$\mu L = (\mu L)L = (\mu^{b^{2k}})L = \mu^{b^{5k}}$$

and, since $o(L) = 2$, it follows that

$$\mu = \mu L^2 = (\mu L)L = (\mu^{b^{5k}})L = \mu^{b^{10k}} = \mu^{b^{4k}}$$

since $\mu^{b^{6k}} = \mu$. Thus $\mu = \mu^{b^{2k}}$, which implies that $\mu \in GF(p^{2k})$ and hence that $s(x)$ is reducible. Since $s(x)$ is irreducible we conclude that there exist no irreducible sextic over $GF(p^k)$ that is self-conjugate under $G$ and hence no conjugate set of order 1. We state these results in

**Theorem 2.1.** For $p = 2$ there exist no conjugate sets of irreducible $m$-ic congruences over $GF(p)$ of order 1 relative to $G$ if $m > 2$ and only one set of order 1 if $m = 2$.

3. Conjugate sets of order 2. Since there exist no IQ[m, p] invariant under $G$ then any IQ[m, p] = $f(x)$ that is self-conjugate under $K$ must necessarily belong to a set of order 2 and $f(x)L = f'(x)$ will be the other $m$-ic belonging to the set. That $f'(x)$ is also self-conjugate under $K$ is given by

**Theorem 3.1.** Any IQ[m, p] = $f(x)$ that is self-conjugate under $K$ belongs to a set of order 2 and $f(x)L$ is likewise self-conjugate under $K$.

**Proof.** Suppose IQ[m, p] = $f(x)$ is self-conjugate under $K$ and let $f'(x) = f(x)L$. Then $f(x) \neq f'(x)$, for otherwise $f(x)$ would belong to a set of order 1. Now $f(x)K = f(x)$ implies that
\[ f(x)K L = (f(x)K)L = f(x)L = f'(x) \]

and, since \( KLK = L \),

\[ f'(x)K = (f(x)KL)K = f(x)LK = f'(x)KLK = f'(x)L = f'(x). \]

Thus \( f'(x) \) is self-conjugate under \( K \). Since

\[ f(x)L = f(x)(LK) = (f(x)L)K = f'(x)K = f'(x) \]

and

\[ f(x)L^2 = f(x)(LK^2) = f'(x)K^2 = (f'(x)K)K = f'(x)K = f'(x), \]

we conclude that \( f(x) \) belongs to a set of order 2.

Now suppose that \( f(x) = \mathbb{I}Q[m, p] \) is self-conjugate under \( K \). Then, since \( 3 \mid m \), we have \( m = 3^t k, (3, k) = 1, t \geq 1 \) and \( f(x) \) is the product of \( s = 3^{t-1} k \) distinct irreducible cubics over \( GF(p^s) \) [1, p. 33]

\[ \mathbb{I}Q[m, p] = f(x) = \prod_{i=0}^{s-1} \{c(x|x)^i \}, \]

each cubic \( \{c(x|x)^i \} \) of which is self-conjugate under \( K \). It follows therefore that there exist an \( \mathbb{I}Q[m, p] \) that is self-conjugate under \( K \) and hence a conjugate set of order 2, provided there exist an irreducible cubic \( c(x) \) over \( GF(p^s) \) that is self-conjugate under \( K \). The number of such cubics may then be used to determine the number of \( \mathbb{I}Q[m, p] \)'s that are self-conjugate under \( K \) and, hence, the number of conjugate sets of order 2.

Suppose therefore that \( c(x) \) is any cubic over \( GF(p^s) \) (reducible or irreducible) that is self-conjugate under \( K \). Then if \( \mu \) is a root of \( c(x) \) its roots are \( \mu, \mu K, \mu K^2 \); and if we set \( \alpha = \mu + \mu K + \mu K^2 \) then

\[ \alpha = \mu + \frac{\mu + 1}{\mu} + \frac{1}{\mu + 1} = \frac{\mu^3 + \mu + 1}{\mu^2 + \mu}. \]

Hence \( \mu^3 + \alpha \mu^2 + (\alpha + 1)\mu + 1 = 0 \), and we conclude that \( \mu, \mu K, \) and \( \mu K^2 \) are roots of

\[ c(x) = x^3 + \alpha x^2 + (\alpha + 1)x + 1. \]

Thus we have the following

**Theorem 3.1.** Any cubic \( c(x) \) over \( GF(p^s) \) that is self-conjugate under \( K = (1 \ 0) \) is of the form (3.2) where \( \alpha \in GF(p^s) \). Conversely, any cubic over \( GF(p^s) \) of the form (3.2) is self-conjugate under \( K \).

If \( \alpha \in GF(p^s) \) then the roots \( \mu, \mu K = (\mu + 1)/\mu, \) and \( \mu K^2 = 1/(\mu + 1) \) of \( c(x) \) necessarily belong to \( GF'(p^s) \) or \( GF'(p^{3s}) \), according as \( c(x) \) is reducible or
irreducible. If these roots are not distinct then \( \mu = \mu_K, \mu = \mu K^2 \), or \( \mu K = \mu K^2 \) each implies that \( \mu^2 + \mu + 1 = 0 \), from which we conclude that \( \mu \in GF'(p^2) \) since \( x^2 + x + 1 \) is the only irreducible quadratic over \( GF(p) \). Thus, \( s = 2 = 3^{t-1}k \) and hence \( m = 3s = 6 \). This gives

**Theorem 3.2.** The roots \( \mu, \mu K \) and \( \mu K^2 \) of the cubic \( c(x) \) over \( GF'(p^s) \) are all distinct if and only if \( s \neq 2 \).

All \( m \)-ics whose degree \( m = 3s \) is a multiple of 3 may now be considered by taking separately the three cases for \( s \), namely, \( s = 1, s = 2, \) or \( s > 2 \).

Case \( s = 3^{t-1}k = 1 \). In this case we have \( m = 3s = 3 \), and the roots of the cubic \( c(x) = x^3 + ax^2 + (a + 1)x + 1 \) over \( GF(p) \) are distinct. Since \( o(GF(p)) = 2 \) it follows that \( c(x) \) is irreducible. Moreover, since there are exactly \( (p^3 - p)/3 = (8 - 2)/3 = 2 \) irreducible cubics over \( GF(p) \) in all, they are identified by the choices \( \alpha = 0 \) and \( \alpha = 1 \) of \( GF(p) \). Therefore we have

**Theorem 3.3.** For \( p = 2 \) there exists one conjugate set of irreducible cubics over \( GF(p) \) of order 2, and this set represents the only conjugate set of cubics over \( GF(p) \).

Case \( s = 3^{t-1}k = 2 \). In this case \( m = 3s = 6 \) and we have \( \mu = \mu K = \mu K^2 \in GF'(p^2) \). This implies that \( \alpha = \mu + \mu + \mu = \mu \) and identifies the reducible cubic \( c(x) = x^3 + \alpha x^2 + (\alpha + 1)x + 1 \) over \( GF'(p^2) \). Moreover, \( \mu^p = \mu + 1 \), the only other mark of \( GF'(p^2) \), likewise determines a reducible cubic, namely,

\[
c'(x) = [c(x)]^p = x^3 + \mu p x^2 + (\mu p + 1)x + 1
\]

which is also self-conjugate under \( K \). Since \( o(GF'(p^2)) = 2 \), there are no irreducible cubics over \( GF'(p^2) \) that are self-conjugate under \( K \) and hence no irreducible sextics over \( GF(p) \) that are self-conjugate under \( K \). Thus,

**Theorem 3.4.** For \( p = 2 \) there exist no conjugate sets of irreducible sextics over \( GF(p) \) of order 2 relative to \( G = G(p) \).

Case \( s = 3^{t-1}k > 2 \). For any choice of \( \alpha \in GF'(p^s) \) the cubic \( c(x) \) given by (3.2) may or may not be irreducible. If \( S = o(GF'(p^s)) \) then there are \( S \) distinct choices for \( \alpha \) and hence this many cubics \( c(x) \) over \( GF'(p^s) \) that are self-conjugate under \( K \). The number of irreducible ones may be obtained by deciding the number of choices for \( \alpha \in GF'(p^s) \) that determine reducible ones.

Since \( \alpha \in GF'(p^s) \), the roots \( \mu, \mu K, \mu K^2 \) each belong to \( GF'(p^s) \) or \( GF'(p^{3s}) \). Suppose \( \mu \in GF'(p^s) \). Then, \( \mu, \mu K, \mu K^2 \) are distinct and, since each belongs to \( GF'(p^s) \), they are roots of irreducible \( s \)-ics over \( GF(p) \). Now their sum \( \alpha = \mu + \mu K + \mu K^2 \) is clearly a mark of \( GF(p^s) \) and may or may not belong to \( GF'(p^s) \). If \( \alpha \notin GF'(p^s) \) then \( \alpha \in GF(p^s) \) where \( GF(p^s) \) is a proper subfield of \( GF(p^s) \). Then
the cubic \( c(x) \) defined by \( \alpha \) is a cubic over \( GF'(p^r) \) whose roots \( \mu, \mu K, \mu K^2 \) belong to \( GF'(p^s) \). Therefore, \( c(x) \) is an irreducible cubic over \( GF'(p^r) \) and, since \( \mu \in GF'(p^s) \), we have \( r = s/3 \). Thus, the marks of \( GF'(p^s) \) that are roots of irreducible cubics over \( GF'(p^s/3) \) that are self-conjugate under \( K \) determine a mark \( \alpha \) of \( GF'(p^r) \) where \( r = s/3 \). Now if \( L_r \) is the number of irreducible cubics over \( GF'(p^r) \) that are self-conjugate under \( K \) then the roots \( \mu, \mu K, \mu K^2 \) of any such cubic each belong to \( GF'(p^s) \) and their sum \( \alpha \in GF'(p^r) \). It follows that there are \( 3L_r \) distinct marks \( \mu \in GF'(p^s) \) that identify an \( \alpha \) in \( GF'(p^r) \). Then the other marks \( \mu \) of \( GF'(p^s) \) determine an \( \alpha \) belonging to \( GF'(p^r) \). Since \( S = o(GF'(p^s)) \) there are \( (S - 3L_r)/3 \) distinct choices for \( \alpha \) in \( GF'(p^s) \) each determined by the set \( \{ \mu, \mu K, \mu K^2 \} \). These \( \alpha \)'s determine cubics \( c(x) \) over \( GF'(p^s) \) whose roots belong to \( GF'(p^s) \) and therefore are reducible cubics. Now any mark \( \alpha \) of \( GF'(p^s) \) not among this collection of \( (S - 3L_r)/3 \) distinct marks will determine a cubic \( c(x) \) over \( GF'(p^s) \) of the form (3.2) that is irreducible. There are therefore

\[ L_{3r} = L_s = S - [(S - 3L_r)/3] = L_r + 2S/3 \]

distinct irreducible cubics over \( GF'(p^s) \) that are self-conjugate under \( K \) for each \( r \geq 1 \). Thus we have

**Theorem 3.5.** For \( p = 2 \), if \( L_r \) is the number of irreducible cubics over \( GF'(p^r) \) of the form (3.2), \( r = 1, 2, 3, \ldots, \), then there are

\[ L_{3r} = L_r + 2 \cdot o(GF'(p^{3r}))/3 \]

distinct irreducible cubics over \( GF'(p^{3r}) \) of the form (3.2).

The numbers \( L_r \) and hence \( L_s, s = 3r = 3t-1k > 2, (3, k) = 1 \), may now be determined by using the recursion formula (3.3) of the above theorem. First, we consider \( L_k \) where \( k = 1 \) and \( k = 2 \). Clearly, \( L_1 = 2 = p \) since each irreducible cubic over \( GF(p) \) is of the form (3.2). Then by (3.3) we have \( L_{3, 1} = L_1 + 2(p^3 - p)/3 = 2(p^3 + 1)/3 \). Hence

\[ L_{3, 1} = L_{3, 1} + 2 \cdot o(GF'(p^{3^2}))/3 = 2(p^{3^2} + 1)/3. \]

In general, we have

**Lemma 3.1.** For \( p = 2 \) and \( s = 3t-1, t \geq 1 \), the number \( L_s \) of irreducible cubics over \( GF'(p^s) \) of the form (3.2) (i.e. self-conjugate under \( K \)) is given by

\[ L_s = 2(p^s + 1)/3. \]

Now for \( k = 2 \) we have \( L_k = L_2 = 0 \) since there exist no irreducible cubics over \( GF'(p^2) \) of the form (3.2). Thus

\[ L_{3, 2} = L_2 + 2 \cdot o(GF'(p^{3*2}))/3 = 2(p^{3*2} - p^3 - p^2 + 1)/3 \]

from which it follows that
In general, we have

**Lemma 3.2.** For \( p = 2 \) and \( s = 3^{t-1} \cdot 2, \ t > 1 \), the number \( L_s \) of irreducible cubics over \( GF'\left(p^s\right) \) of the form (3.2) is given by

\[
L_s = 2 \left[ p^s - p^{s/2} - p^2 + 1 \right] / 3.
\]

Finally, we determine the number \( L_k \) of irreducible cubics over \( GF'\left(p^k\right) \) that are self-conjugate under \( K \) where \( k > 2 \) and \( (3, k) = 1 \). Any such cubic is of the form (3.2) where \( \alpha \in GF'\left(p^k\right) \), and its roots \( \mu, \mu K, \mu K^2 \) are distinct and belong to \( GF'\left(p^{3k}\right) \). If we choose \( \mu \in GF'\left(p^k\right) \) and set \( \alpha = \mu + \mu K + \mu K^2 \) then \( \alpha \in GF'\left(p^k\right) \) or \( \alpha \) belongs to a proper subfield of \( GF(p^k) \), say \( GF(p^{k'}) \). If \( \alpha \in GF'\left(p^{k'}\right) \) then \( k' \mid k \), and the cubic defined by \( \alpha \) would be an irreducible cubic over \( GF'\left(p^{k'}\right) \), from which it follows that \( 3k' = k \) and hence that \( 3 \mid k \). Since \( (3, k) = 1 \) we conclude that \( \alpha \) cannot belong to a proper subfield of \( GF(p^k) \), and hence that \( \alpha \in GF'\left(p^k\right) \). Now, if we set \( K_0 = o(GF'\left(p^k\right)) \) then there are \( K_0/3 \) distinct values for \( \alpha \in GF'\left(p^k\right) \) corresponding to the \( K_0/3 \) distinct subsets \{\mu, \mu K, \mu K^2\} of \( GF'\left(p^k\right) \), each of which identifies a reducible cubic over \( GF'\left(p^k\right) \) of the form (3.2). Therefore, there are \( K_0 - K_0/3 = 2K_0/3 \) choices for \( \alpha \) that determine irreducible ones. Thus we have

**Lemma 3.3.** If \( k > 2, \ (3, k) = 1 \) and \( K_0 = o(GF'\left(p^k\right)) \) then \( L_k = 2K_0/3 \).

This lemma along with the recursion formula (3.3) gives

**Lemma 3.4.** If \( k > 2, \ (3, k) = 1 \) and \( K_1 = o(GF'\left(p^{3k}\right)) \) then \( L_{3k} = 2(K_0 + K_1) / 3 \).

In general, we have

**Lemma 3.5.** For \( p = 2, \) if \( s = 3^{t-1} \cdot k \) where \( k > 2, \ t \geq 1, \ (3, k) = 1, \) and \( i \) if \( K_i = o(GF'\left(p^{3^i k}\right)) \) then

\[
L_s = 2 \left[ \sum_{i=0}^{t-1} K_i \right] / 3.
\]

**Lemmas 3.1, 3.2 and 3.5 now give us the following important**
Theorem 3.6. If $p = 2$ and $s = 3^{t-1}k > 2$ where $t \geq 1$ and $(3, k) = 1$, then the number $L_s$ of distinct irreducible cubics over $\text{GF}(p^s)$ that are self-conjugate under $K$ is given by:

(a) $L_s = 2(p^s + 1)/3$,

(b) $L_s = 2(p^s - p^{s/2} - p^2 + 1)/3$, or

(c) $L_s = 2(\sum_{i=0}^{t-1} K_i)/3$, where $K_i = o(\text{GF}(p^{3^i}k))$,

according as (a) $k = 1$, (b) $k = 2$, or (c) $k > 2$.

We may now determine the number of conjugate sets of irreducible $m$-ics of order 2 for $m = 3^tk$, where $t \geq 1$, $(3, k) = 1$ and $s = 3^{t-1}k$. First let us note that if $a \in \text{GF}(p^s)$ and determines the irreducible cubic $c(x) = x^3 + ax^2 + (a + 1)x + 1$, then $a + 1 \in \text{GF}(p^s)$ and determines the cubic $c'(x) = x^3 + (a + 1)x^2 + ax + 1$ which is likewise irreducible over $\text{GF}(p^s)$ and self-conjugate under $K$. Moreover, these two cubics are conjugate since one may easily show that $c(x) L = c'(x)$, where $L = \left( \begin{array}{cc} 1 & 1 \\ \mu & 1 \end{array} \right)$. Now if $\mu$ is a root of $c(x)$ then its roots are $\mu, \mu K = \mu^{p^s}, \mu K^2 = \mu^{p^{2s}}$ and $\mu + 1, (\mu + 1)K = \mu^{p^s} + 1, (\mu + 1)K^2 = \mu^{p^{2s}} + 1$ are the roots of $c'(x)$.

We conclude from this that no $p^i$th power of $c(x)$ can give us $c'(x)$. Hence it follows that

$$IQ[m, p] = \prod_{i=0}^{s-1} [c(x)]^{p^i}$$

and

$$IQ'[m, p] = \prod_{i=0}^{s-1} [c'(x)]^{p^i} = IQ[m, p] L$$

are distinct and constitute the irreducible $m$-ics over $\text{GF}(p)$ belonging to a set of order 2.

Since the $s$ cubic factors of both $IQ[m, p]$ and $IQ'[m, p]$ are each irreducible cubics over $\text{GF}(p^s)$ and of the form (3.2) it follows that there exist $L_s/2s$ distinct conjugate sets of $m$-ics of order 2 where $m = 3^tk$, $t > 0$, $s = 3^{t-1}k$, $(3, k) = 1$. Thus, since $m = 3s$ we have the following

Theorem 3.7. If $p = 2$ and $m = 3^tk$, where $t > 0$ and $(3, k) = 1$, then the number $C_2$ of conjugate sets of $m$-ic congruences over $\text{GF}(p)$ of order 2 is given by:

(a) $C_2 = (\frac{p^{m/3}}{3} + 1)/m$,

(b) $C_2 = (\frac{p^{m/3} - p^{m/6} - p^2 + 1}{3}/m$, or

(c) $C_2 = (\sum_{i=0}^{t-1} K_i)/m$, where $K_i = o(\text{GF}(p^{3^i}k))$,

according as (a) $k = 1$, (b) $k = 2$ and $m \neq 6$, or (c) $k > 2$. If $m = 6$ then $C_2 = 0$.

4. Conjugate sets of order 3. Any conjugate set of order 3 must contain an $IQ[m, p]$ that is self-conjugate under $L = \left( \begin{array}{cc} 1 & 1 \\ \mu & 1 \end{array} \right)$. Moreover, $2|m$, and hence $m$ must be of the form $m = 2^n n$ where $(2, n) = 1$ and $t \geq 1$. In such a case the $IQ[m, p]$ is the product of $2^{t-1}n$ distinct irreducible quadratics $[q(x)]^{p^i}$, $i = 0, 1, 2, \ldots$, $(2^{t-1}n - 1)$, each of which is self-conjugate under $L$ and hence of the form

$$q(x) = x^2 + x + \alpha, \quad \alpha \in \text{GF}(p^{2^{t-1}n}).$$
Now let \( y \) be a root of an \( IQ[2, p^{2t-1}] = Q(x) \) and without loss of generality suppose \( Q(x) \) is of the form

\[
Q(x) = x^2 + x + \beta, \quad \beta \in GF'(p^{2t-1}).
\]

Then the roots of \( Q(x) \) are \( y \) and \( y^{p^{2t-1}} \) and we have \( y + y^{p^{2t-1}} = 1 \) and \( y \cdot y^{p^{2t-1}} = \beta. \) Since \((2, n) = 1\) then any mark \( \eta \) of \( GF'(p^{2t+n}) \) is uniquely expressible in the form

\[
(4.3) \quad \eta = \phi_1 + \phi_2 y, \quad \phi_1, \phi_2 \in GF'(p^{2t-1}),
\]

and if \( \eta \in GF'(p^{2t+n}) \) then \( \eta \) may be regarded as a root of an irreducible quadratic over \( GF'(p^{2t-1+n}). \)

Since \((2, n) = 1\) implies that \( n = 2w + 1 \) for some \( w \) and hence that \( 2^{t-1}n = 2^t w + 2t-1 \)

\[
\eta^{p^{2t-1+n}} = y^{p^{2t}w + 2t-1} = [y^{p^{2t}w}]^{p^{2t-1}} = y^{p^{2t-1}}
\]

and we have

\[
\eta^{p^{2t-1+n}} = (\phi_1 + \phi_2)^{p^{2t-1}} = \phi_1 + \phi_2 y^{p^{2t-1}} = \phi_1 + \phi_2 y^{p^{2t-1}}
\]

since \( \phi_1, \phi_2 \in GF(p^{2t-1+n}). \)

Now if, in particular, \( \eta = \phi_1 + \phi_2 y \) is a root of an irreducible quadratic \( q(x) \) of the form (4.1) then

\[
\eta^{p^{2t-1}} = 1
\]

implies that \((\phi_1 + \phi_2 y) + (\phi_1 + \phi_2 y^{p^{2t-1}}) = \phi_2 (y + y^{p^{2t-1}}) = \phi_2 \cdot 1 = 1\) and hence that \( \nu = \phi_1 + y. \) Conversely, if \( \nu \) is any mark of \( GF'(p^{2t+n}) \) of the form \( \eta = \phi + y \) then \( \eta \) and hence \( \eta^{p^{2t-1+n}} \) are roots of an irreducible quadratic \( q(x) \) over \( GF'(p^{2t-1+n}) \) and, since

\[
\eta^{p^{2t-1+n}} = (\phi + y) + (\phi + y^{p^{2t-1}}) = 1,
\]

then \( q(x) \) is of the form (4.1) and hence self-conjugate under \( L. \) Thus we have

**Theorem 4.1.** If \( p = 2 \) then a necessary and sufficient condition that \( \eta = \phi_1 + \phi_2 y \) be a root of an irreducible quadratic \( q(x) \) over \( GF'(p^{2t-1+n}) \) that is self-conjugate under \( L \) is that \( \eta \in GF'(p^{2t+n}) \) and be of the form \( \eta = \phi + y \) where \( \phi \in GF(p^{2t-1+n}). \)
Now if $\eta = \phi + \gamma \in GF'(p^{2^t})$ is a root of $q(x) = x^2 + x + \beta$ then $q(x)L = q(x)$ implies that the roots of $q(x)$ are

$$\eta = \phi + \gamma \quad \text{and} \quad \eta p^{2^t-1} = \eta L = \eta + 1 = (\phi + 1) + \gamma.$$ 

Thus, if $\phi \in GF(p^{2^t-1})$ defines the root $\eta$ of the $lQ[2, p^{2^t-1}] = q(x)$ and hence $q(x)$ itself then $\phi + 1$ will likewise define $q(x)$.

The number of conjugate sets of order 3 may now be obtained by determining the number of irreducible quadratics over $GF'(p^{2^t-1})$ that are self-conjugate under $L$. To do this we must therefore determine the number of distinct marks $\phi \in GF(p^{2^t-1})$ such that $\eta = \phi + \gamma$ belongs to $GF'(p^{2^t})$. Since $\gamma \in GF'(p^{2^t})$ and since $GF(p^{2^t-1}) \cap GF(p^{2^t}) = GF(p^{2^t-1})$ and $(2, n) = 1$ it readily follows that $\eta = \phi + \gamma \in GF'(p^{2^t})$ if and only if $\phi$ is a root of any irreducible $n$-ic over $GF'(p^{2^t-1})$. (2) Thus, if $n = q_1^{r_1} \cdot q_2^{r_2} \cdots q_r^{r_h}$ is the standard form for $n$, then there are

$$nN = \sum_{n \neq 1} p^{2^t-1} - \sum_{n \neq 1} p^{2^t-1}/q_i + \sum_{n \neq 1} p^{2^t-1}/(q_i q_j) - \cdots$$

(4.4) distinct choices for $\phi$ such that $\eta = \phi + \gamma \in GF'(p^{2^t})$, where the sums $\Sigma$ are taken for all combinations of the distinct prime factors of $n$ in the numbers indicated [1, p. 18]. [Note that if $n = 1$ then this number given by (4.4) is $p^{2^t-1}$.] 

Now, since two distinct choices of $\phi$ identify one quadratic and since $2^{t-1}n$ of these go together to determine one irreducible $m$-ic ($m = 2^t n$) over $GF(p)$ that is self-conjugate under $L$ and hence one set of order 3 then the number of sets of order 3 is easily determined. We state this result in the following

**Theorem 4.2.** If $p = 2$ and $m = 2^t n$, where $t \geq 1$, $(2, n) = 1$ and if $n = q_1^{r_1} \cdot q_2^{r_2} \cdots q_r^{r_h} > 2$ then there exist

$$C_3 = \left[ p^{2^t-1} - \sum_{n \neq 1} p^{2^t-1}/q_i + \sum_{n \neq 1} p^{2^t-1}/q_i q_j - \cdots \right] / m$$

distinct conjugate sets of irreducible $m$-ic congruences over $GF(p)$ of order 3.

If $n = 1$ then the number of conjugate sets of order 3 is $C_3 = [p^{2^t-1}]/m = 2^{t-1} - t$.

The number of conjugate sets of order 6, say $C_6$, is now easily determined and is given by the following

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(2) In fact, if $\phi$ is a root of an irreducible $n$-ic over $GF(p)$ then since $\gamma \in GF'(p^{2^t})$ it is clear that $\eta = \phi + \gamma$ belongs to $GF'(p^{2^t})$.
Theorem 4.3. If \( p = 2 \) then the number of conjugate sets of order 6 is given by
\[
C_6 = \frac{N_{m,p} - 2C_2 - 3C_3}{6}
\]
where \( C_2 \) and \( C_3 \) are the numbers of sets of order 2 and 3 respectively and \( N_{m,p} \) is the total number of irreducible \( m \)-ics over \( GF(p) \) (see [1, p. 18]).

BIBLIOGRAPHY

1. L. E. Dickson, Linear groups, Teubner, Leipzig, 1901.

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