INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

BY

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ABSTRACT. Inequalities for the derivative and for the maximum modulus on a larger circle of a polynomial with a given zero on the unit circle are obtained in terms of its degree and maximum modulus on the unit circle; examples are given to show that these are sharp with respect to the degree (best constants are not known). Inequalities for $L^p$ norms, in particular $L^2$ norms, are also derived. Also certain functions of exponential type are considered and similar inequalities are obtained for them. Finally, the problem of estimating $P_n(r)$ (with $0 < r < 1$) given $P_n(1) = 0$ is taken up.

1. Introduction and statement of results. If $P_n(z)$ is a polynomial of degree $n$ such that $\max_{|z|=1} |P_n(z)| = 1$ then

\( (1.1) \quad \max_{|z|=1} |P_n'(z)| \leq n, \)

\( (1.2) \quad \max_{|z| > 1} |P_n(z)| \leq R^n. \)

Inequality (1.1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [18]). Inequality (1.2) is a simple deduction from the maximum principle (see [15, p. 346], or [12, vol. 1, p. 137, Problem III 269]).

In both (1.1), (1.2) equality holds only for $P_n(z) = e^{iy} z^n$, i.e. when all the zeros of $P_n(z)$ lie at the origin. Erdos conjectured and later Lax [11] verified that if $P_n(z) \neq 0$ in $|z| < 1$ then (1.1) can be replaced by

\( (1.3) \quad \max_{|z|=1} |P_n'(z)| \leq n/2 \)

and in (1.3) equality holds if all the zeros of $P_n(z)$ lie on $|z| = 1$. Ankeny and Rivlin [2] used (1.3) to prove that if $|P_n(z)| \leq 1$ for $|z| = 1$ and $P_n(z) \neq 0$ in $|z| < 1$ then

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\[ (1.4) \max_{|z|=R>1} |P_n(z)| \leq (R^n + 1)/2 \]

which is much better than (1.2). Besides, in (1.4) equality is possible and in fact holds for \( P_n(z) = \alpha + \beta z^n \) where \(|\alpha| = |\beta| = \frac{1}{2}\).

Several years ago Professor R. P. Boas, Jr. asked one of us, namely Rahman, as to what can be said about \( \max_{|z|=1} |P'(z)| \), \( \max_{|z|=R>1} |P_n(z)| \) if we assume \( P_n(z) \) to have precisely \( k \) zeros in \(|z| \geq 1\) instead of all the zeros as Erdős did. In particular, what happens if \( P_n(1) = 0 \) or (more generally) if \(|P_n(1)| = a\)? We are thus led to consider the class \( \mathcal{P}_{n,a} \) of all polynomials \( P_n(z) \) of degree \( n \) such that \( \max_{|z|=1} |P_n(z)| = 1 \), \( \min_{|z|=1} |P_n(z)| \leq a \) where \( 0 \leq a \leq 1 \). It is clear that every polynomial \( P_n(z) \) with \( \max_{|z|=1} |P_n(z)| = 1 \) belongs to \( \mathcal{P}_{n,a} \) for some \( a \).

In view of (1.3), one might expect that if \( P_n(z) \in \mathcal{P}_{n,0} \) then \( \max_{|z|=1} |P'(z)| \leq n - c \) where \( c \) is a constant, possibly equal to \( \frac{1}{2} \). But this is far from the truth as the next two theorems show.

**Theorem 1.** If \( P_n(z) \in \mathcal{P}_{n,a} \) then for \(|z| \leq 1\)

\[ (1.5) |P'_n(z)| \leq n - (1 - a)(1 - \alpha \sin(1 - a))/4\pi n. \]

**Theorem 2.** There exists an absolute constant \( c_1 > 0 \) such that

\[ (1.6) \max_{P_n(z) \in \mathcal{P}_{n,a}} \left( \max_{|z|=1} |P'_n(z)| \right) \geq n - (1 - a)c_1/n. \]

We also prove

**Theorem 3.** If \( P_n(z) \in \mathcal{P}_{n,a} \) then for \( R > 1 \)

\[ (1.7) \max_{|z|=R} |P_n(z)| \leq R^n \left\{ 1 - \frac{1}{\left[ n/(1 - a) \right] + 1} \right\} (1 - e^{-\pi/2})(1 - R^{-1})^2. \]

**Theorem 4.** There exists an absolute constant \( c_2 > 0 \) such that for \( R > n^2 \)

\[ (1.8) \max_{P_n(z) \in \mathcal{P}_{n,a}} \left( \max_{|z|=R} |P_n(z)| \right) > R^n[1 - c_2(1 - a)/n]. \]

Theorem 1 says in particular that if \( P_n(z) \in \mathcal{P}_{n,0} \) then

\[ (1.5') \max_{|z|=1} |P'_n(z)| \leq n - (1 - \sin 1)/4\pi n. \]

But in this special case namely for polynomials \( P_n(z) \in \mathcal{P}_{n,0} \) we obtain the better estimate:
We are also able to replace

\[(1.7') \max_{P_n(z) \in \mathcal{G}_{n,0}} \left( \max_{|z|=R>1} |P_n(z)| \right) \leq R^n \left\{ 1 - \frac{1}{16n} (1 - e^{-\pi/2})(1 - R^{-1})^2 \right\} \]

obtainable from Theorem 3 by

\[(1.7") \max_{P_n(z) \in \mathcal{G}_{n,0}} \left( \max_{|z|=R>1} |P_n(z)| \right) \leq R^n \left\{ 1 - \frac{2 - \sqrt{n}}{2n} (1 - R^{-1})^2 \right\} . \]

Coming back to the original question of Boas, namely what happens if \( P_n(z) \) has \( k \) zeros in \( |z| \geq 1 \), we can prove the following theorem:

**Theorem 5.** Corresponding to every \( \epsilon > 0 \), there exist a \( \delta > 0 \) and an integer \( n_0 \) such that for all \( n > n_0 \) there is a polynomial \( P_n(z) \in \mathcal{G}_{n,0} \) which has at least \( \delta \sqrt{n} \) zeros in \( |z| \geq 1 \) and is such that \( \max_{|z|=1} |P_n'(z)| > n - \epsilon \).

Theorem 2 gives us an idea as to how large \( \max_{|z|=1} |P_n'(z)| \) can be if \( P_n(z) \) is a polynomial of degree \( n \) such that \( \max_{|z|=1} |P_n(z)| = 1 \) and \( |P_n(1)| = a \). It is natural to ask how small \( \max_{|z|=1} |P_n'(z)| \) can be under these conditions. This question turns out to be easy. Indeed, from

\[
|P(z)| \leq a + |z-1| \max_{|z| \leq 1} |P'(z)| < a + 2 \max_{|z| \leq 1} |P'(z)|
\]

that is

\[
\max_{|z| \leq 1} |P'(z)| \geq (1-a)/2.
\]

We may consider the polynomial

\[
P_n(z) = a + \frac{1 - a}{2(k+1)} (1-z)(k+(-z)^{n-1})
\]

with sufficiently large positive \( k \) to see that the bound \( (1-a)/2 \) is best possible.

In fact, \( P_n(1) = a \), \( \max_{|z|=1} |P_n(z)| = 1 \) and, for every given \( \epsilon > 0 \),

\[
\max_{|z|=1} |P'_n(z)| = (1-a) \left( \frac{1}{2} + \frac{n-1}{k+1} \right) < \frac{1-a}{2} + \epsilon
\]

if \( k > (1-a)(n-1)/\epsilon - 1 \).

Applying Theorem 3 to the polynomial \( z^n P_n(1/z) \) we deduce
\begin{align}
(1.11) \quad & \max_{P_n(z) \in \mathcal{P}_{n,a}} \left( \max_{|z|=\rho<1} |P_n(z)| \right) \leq 1 - \frac{1}{(n/(1-a) + 1)^2} \left(1 - e^{-\pi/2}(1-\rho)^2 \right).
\end{align}

For polynomials $P_n(z) \in \mathcal{P}_{n,0}$ we may use (1.7') to get
\begin{align}
(1.11') \quad & \max_{|z|=\rho<1} |P_n(z)| \leq 1 - \frac{2 - \sqrt{2}}{2n} (1 - \rho)^2.
\end{align}

Recently, but in quite a different context Halász asked how large
\begin{align}
\min_{|z|=1-(\omega/n)} |P_n(z)|
\end{align}
can be for a given $\omega$ in $(0, n]$ if $P_n(z) \in \mathcal{P}_{n,0}$. It has been shown by Rahman and Stenger [14] that for given
\begin{align}
\mu(\omega, n) = \max_{P_n(z) \in \mathcal{P}_{n,0}} \left\{ \max_{|z|=1-(\omega/n)} |P_n(z)| \right\} > 1 - \frac{\lambda}{\omega}
\end{align}
provided $\omega > A(\lambda)$. On the other hand they showed that if $\omega$ is large then for $n \geq \omega$
\begin{align}
(1.13) \quad & \mu(\omega, n) \leq 1 - \frac{1}{(e\omega)} + o(1/\omega).
\end{align}

Here we shall prove the following theorem which gives a better estimate for $\mu(\omega, n)$ than (1.13). Besides, in lieu of requiring $\max_{|z|=1} |P_n(z)| \leq 1$ we only assume $|P_n(\exp(ijn/n))| \leq 1$ for $j = 1, 2, 3, \ldots, 2n-1$ if 1 is the point on $|z| = 1$ where $P_n(z)$ vanishes.

**Theorem 6.** If $P_n(z)$ is a polynomial of degree $n$ such that $P_n(1) = 0$, $|P_n(\exp(ijn/n))| \leq 1$ for $j = 1, 2, 3, \ldots, 2n-1$ then, for $0 < \omega \leq n$,
\begin{align}
(1.14) \quad & \left| P_n \left( 1 - \frac{\omega}{n} \right) \right| \leq 1 - \left( \frac{1}{\omega} - \frac{1}{2n} \right) + \frac{1}{2n} \left(1 - \frac{\omega}{n} \right)^n.
\end{align}

Inequalities (1.1), (1.2) can be obtained by letting $p \to \infty$ in the inequalities
\begin{align}
(1.15) \quad & \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p} \leq n \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p \geq 1,
\end{align}
and
\begin{align}
(1.16) \quad & \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^p d\theta \right)^{1/p} \leq R^n \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p > 0, \quad R > 1,
\end{align}
respectively. Inequality (1.15) is due to Zygmund [20] who proved it for all trigonometric polynomials of degree $n$ and not only for those which are of the form $P_n(e^{i\theta})$. As for (1.16) it is difficult to trace its origin. We can deduce it from a well-known result of G. H. Hardy [10] according to which for every function $f(z)$ analytic in $|z| < \rho_0$ and, for every $p > 0$,
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\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right)^{1/p}
\]

is a nondecreasing function of \( \rho \) for \( 0 < \rho < \rho_0 \). If \( P_n(z) \) is a polynomial of degree \( n \), then \( f(z) = z^n P_n(1/\overline{z}) \) is again a polynomial, i.e. an entire function and by Hardy’s result
\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right)^{1/p} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad p > 0,
\]
for \( \rho = R^{-1} < 1 \). This is equivalent to (1.16).

In both (1.15), (1.16) equality holds only if \( P_n(z) \) is a constant multiple of \( z^n \).

It was shown by de Bruijn [8] that if \( P_n(z) \neq 0 \) in \( |z| < 1 \) then (1.15) can be replaced by
\[
(1.17) \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta \right)^{1/p} \leq nC_p \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad p \geq 1,
\]
where
\[
C_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\phi}|^p \, d\phi \right)^{-1/p}.
\]
The corresponding refinement of (1.16) namely
\[
(1.18) \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^p \, d\theta \right)^{1/p} \leq K_p \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad p \geq 1, \quad R > 1,
\]
where
\[
K_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + R^ne^{in\phi}|^p \, d\phi \right)^{1/p} \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{in\phi}|^p \, d\phi \right)^{1/p}
\]
if \( P_n(z) \neq 0 \) in \( |z| < 1 \) was proved by Boas and Rahman [7].

Both inequalities (1.17), (1.18) are sharp. Equality holds for all polynomials of the form \( \lambda + \mu z^n \) with \( |\lambda| = |\mu| \).

If we let \( p \) tend to infinity in (1.17), (1.18) we get (1.3), (1.4) respectively.

In the special case \( p = 2 \) inequalities (1.17), (1.18) reduce to
\[
(1.17') \frac{1}{2\pi} \int_0^{2\pi} |P_n'(e^{i\theta})|^2 \, d\theta \leq \left( \frac{n^2}{2} \right) \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta,
\]
\[
(1.18') \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^2 \, d\theta \leq \frac{R^2 + 1}{2} \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta
\]
respectively, i.e. the hypothesis \( P_n(z) \neq 0 \) in \( |z| < 1 \) allows us to put \( n^2/2 \) in-
Instead of $n^2$ on the right-hand side of (1.17') and $(R^2n + 1)/2$ instead of $R^2n$ on the right-hand side of (1.18'). It may be asked what happens if we simply assume $P_n(z)$ to have a zero on $|z| = 1$. The answer is provided by Corollaries 1 and 2 of the following theorem.

**Theorem 7.** If $P_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree $n$ with $P_n(e^{i\theta}) = 0$ for some $\theta_0$ in $[0, 2\pi)$ and $\lambda_0, \lambda_1, \ldots, \lambda_n$ are nonnegative numbers such that $\lambda_j > \lambda_k$ for $k = 0, 1, \ldots, j - 1, j + 1, \ldots, n$, then

\begin{equation}
\sum_{k=0}^{n} \lambda_k |a_k|^2 \leq (\lambda_j - \lambda) \sum_{k=0}^{n} |a_k|^2
\end{equation}

where $\lambda$ is the unique root of the equation

\begin{equation}
\sum_{k=0}^{n} \frac{1}{\lambda_j - \lambda_k - x} = 0
\end{equation}

in the interval $(0, \Lambda = \min_{0 \leq k \leq n; \lambda_k \neq \lambda_j} (\lambda_j - \lambda_k))$.

That equation (1.20) has one and only one root in the interval $(0, \Lambda)$ follows on writing it as $f'(x)/f(x) = 0$ where

\[ f(x) = \prod_{k=0}^{n} |x - (\lambda_j - \lambda_k)|. \]

Since all the zeros of the polynomial $f(x)$ are real and $0, \Lambda$ are consecutive zeros, Rolle's theorem shows that $f'(x)$ vanishes once and only once in $(0, \Lambda)$.

Inequality (1.19) is sharp and becomes an equality if (and only if) $P_n(z)$ is a constant multiple of $\sum_{k=0}^{n} (e^{-i\theta_0})^k/(\lambda_j - \lambda_k - \lambda)$.

The following two corollaries of Theorem 7 are obtained on setting $\lambda_k = k^2$ ($0 \leq k \leq n$), $\lambda_k = R^2k$ ($0 \leq k \leq n$) respectively.

**Corollary 1.** If $P_n(z)$ is a polynomial of degree $n$ with $P_n(e^{i\theta_0}) = 0$ for some $\theta_0$ in $[0, 2\pi)$, then

\begin{equation}
\frac{1}{2\pi} \int_{0}^{2\pi} |P'(e^{i\theta})|^2 d\theta \leq (n^2 - \alpha_n) \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 d\theta
\end{equation}

where $\alpha_n$ is the unique root of the equation

\begin{equation}
\sum_{k=0}^{n} \frac{1}{n^2 - k^2 - x} = 0
\end{equation}

in the interval $(0, 2n - 1)$.

**Corollary 2.** If $P_n(z)$ is a polynomial of degree $n$ with $P_n(e^{i\theta_0}) = 0$ for some $\theta_0$ in $[0, 2\pi)$, then for $R > 1$
1 \quad \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^2 \, d\theta \leq (R^{2n} - \beta_n) \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta

where $\beta_n$ is the unique root of the equation

\sum_{k=0}^{n} \frac{1}{R^{2n} - R^{2k} - x} = 0

in the interval $(0, R^{2n} - R^{2n-2})$.

Although we do not know the explicit values of $\alpha_n, \beta_n$ we can show that (i) $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$, (ii) $\beta_n \sim R^{2n}/(n + 1)$ as $R \to \infty$ and $n$ is fixed.

(i) That $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$ may be deduced from the fact that

\begin{equation}
\sum_{k=0}^{n-1} \frac{1}{n^2 - k^2 - 2n/(\log n)} \sim \frac{\log n}{2n} \quad \text{as } n \to \infty.
\end{equation}

Indeed, if $F_n(x)$ denotes the sum $\sum_{k=0}^{n-1}[1/(n^2 - k^2 - x)]$ then by virtue of (1.25) we have for every $\epsilon > 0$

$$F_n\left(\frac{(2 + \epsilon)n}{\log n}\right) > F_n\left(\frac{2n}{\log n}\right) > \frac{\log n}{(2 + \epsilon)n}$$

and

$$F_n\left(\frac{(2 - \epsilon)n}{\log n}\right) < F_n\left(\frac{2n}{\log n}\right) < \frac{\log n}{(2 - \epsilon)n}$$

provided $n$ is sufficiently large. Thus $F_n(x) - 1/x$ changes sign between $(2 - \epsilon)n/\log n$ and $(2 + \epsilon)n/\log n$, i.e. $(2 - \epsilon)n/\log n < \alpha_n < (2 + \epsilon)n/\log n$. In other words, $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$.

In order to prove (1.25) we write $\sum_{k=0}^{n-1}[1/(n^2 - k^2 - 2n/\log n)]$ as

\[\frac{1}{2(n^2 - 2n/\log n)^{1/2}} \sum_{k=0}^{n-1} \left\{ \frac{1}{(n^2 - 2n/\log n)^{1/2} - k} + \frac{1}{(n^2 - 2n/\log n)^{1/2} + k} \right\}\]

which is equal to

\[\frac{1}{2n(1 - 2/n \log n)^{1/2}} \left\{ \sum_{k=0}^{n-1} \left( \frac{1}{n - k} + \frac{1}{n + k} \right) + r_n \right\}\]

where the positive number $r_n$ remains bounded as $n \to \infty$. In fact, the inequalities

$$\log (N + 1) < \sum_{l=1}^{N} \frac{1}{l} < 1 + \log N$$

show that (for sufficiently large $n$)
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\[
0 < \sum_{k=0}^{n-1} \frac{1}{(n^2 - 2n\log n)^{1/2} - k} - \sum_{k=0}^{n-1} \frac{1}{n - k}
\]

\[
\leq \frac{n - n(1 - 2/n \log n)^{1/2}}{1 - n + n(1 - 2/n \log n)^{1/2}} \sum_{k=0}^{n-1} \frac{1}{n - k}
\]

\[
< \frac{2}{\log n} \frac{1 + \log n}{1 - n + n(1 - 2/n \log n)^{1/2}} = O(1)
\]

and that

\[
0 < \sum_{k=0}^{n-1} \frac{1}{(n^2 - 2n\log n)^{1/2} + k} - \sum_{k=0}^{n-1} \frac{1}{n + k}
\]

\[
\leq \frac{n - n(1 - 2/n \log n)^{1/2}}{n(1 - 2/n \log n)^{1/2}} \sum_{k=0}^{n-1} \frac{1}{n + k}
\]

\[
< \frac{2}{\log n} \frac{1 + \log (2n - 1)}{n(1 - 2/n \log n)^{1/2}} = o(1).
\]

Thus

\[
\sum_{k=0}^{n-1} \frac{1}{n^2 - k^2 - 2n/\log n} = \frac{1}{2n(1 - 2/n \log n)^{1/2}} \left( \sum_{l=0}^{2n-1} \frac{1}{l + 1/n + r_n} \right)
\]

\[
\sim \frac{\log n}{2n} \text{ as } n \to \infty.
\]

This proves (1.25), and the verification of the claim that \(\alpha_n \sim 2n/(\log n)\) as \(n \to \infty\) is complete.

(ii) That, for fixed \(n\), \(\beta_n \sim R^{2n}/(n + 1)\) as \(R \to \infty\) can be deduced from the trivial fact:

\[
\sum_{k=0}^{n-1} \frac{1}{R^{2n} - R^{2k} - R^{2n}/(n + 1)} \sim \frac{n + 1}{R^{2n}} \text{ as } R \to \infty.
\]

Remark 1. Theorem 7 is easily seen to be equivalent to the following:

Theorem 7'. If \(f(s) = \sum_{k=0}^{n} a_k e^{s\lambda_k} (s = \sigma + it)\) where \(0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n\) and \(f(\sigma_0) = 0\), then

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f'(\sigma_0 + it)|^2 dt \leq (\lambda_n - \lambda) \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma_0 + it)|^2 dt
\]

where \(\lambda\) is the smallest root of the equation...
Remark 2. Instead of assuming \( P_n(e^{i\theta_0}) = 0 \) we may assume \( P_n(\rho e^{i\theta_0}) = 0 \) for some \( \rho > 0 \) and ask what becomes of (1.19).

Theorem 7'. If \( \lambda_j > \lambda_k \geq 0 \) for \( k = 0, 1, 2, \ldots, j - 1, j + 1, \ldots, n \) and \( P_n(z) = \sum_{k=0}^{n} a_k z^k \) vanishes at \( z = \rho e^{i\theta_0} \) then

\[
\sum_{k=0}^{n} \lambda_k |a_k|^2 \leq (\lambda_j - \lambda^{(\rho)}) \sum_{k=0}^{n} |a_k|^2
\]

where \( \lambda^{(\rho)} \) is the unique root of the equation

\[
\sum_{k=0}^{n} \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} = 0
\]

in \((0, \Lambda = \min_{0 \leq k \leq n; k \neq j} (\lambda_j - \lambda_k))\).

Equality holds in (1.26) for

\[
P_n(z) = a \sum_{k=0}^{n} \frac{\rho_k}{\lambda_j - \lambda_k - x} (e^{-i\theta_0} z)^k.
\]

Since \( f(x) = \Pi_{k=0}^{n} (\lambda_j - \lambda_k - x)^{2k} \) vanishes at \( x = 0, x = \Lambda \) the equation

\[
\sum_{k=0}^{n} \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} = \frac{f'(x)}{f(x)} = 0
\]

must have at least one root in \((0, \Lambda)\) by the mean value theorem. That it has only one becomes obvious on writing it as

\[
\frac{\rho^{2j}}{x} = \sum_{k=0; k \neq j}^{n} \frac{\rho_k}{\lambda_j - \lambda_k - x}
\]

and noting that the left-hand side decreases whereas the right-hand side increases as \( x \) increases from 0 to \( \Lambda \).

On setting \( \lambda_k = k^2 \) \((0 \leq k \leq n)\), \( \lambda_k = R^{2k} \) \((0 \leq k \leq n)\) in Theorem 7" we obtain sharp estimates for

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta \right) \left/ \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta \right) \right.,
\]

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(R e^{i\theta})|^2 \, d\theta \right) \left/ \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta \right) \right.
\]

under the hypothesis that \( P_n(z) \) has a zero on \(|z| = \rho\). These estimates constitute generalizations of Corollaries 1, 2 respectively which deal with the case \( \rho = 1 \).
The following corollary is obtained on setting $a_j = 1$ and $a_k = 0$ for $k \neq j$ in Theorem 7.

**Corollary 3.** If $P_n(z) = \sum_{k=0}^{n} a_k z^k$ has a zero on $|z| = \rho$ then for $0 \leq j \leq n$

$$|a_j| \leq \left( \frac{\left( \sum_{k=0}^{n} \rho^2 k \right) / \left( \sum_{k=0}^{n} \rho^2 k \right)}{\left( \frac{2\pi}{2n} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}} \right)^{\frac{1}{2}}.
$$

If $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a trigonometric polynomial of degree $n$ with $S_n(\theta_0) = 0$ for some $\theta_0$ in $[0, 2\pi)$ then $e^{in\theta} S_n(\theta) = P_{2n}(e^{i\theta})$ where $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k = \sum_{k=0}^{2n} c_k z^{-n}$ is a polynomial of degree $2n$ with $P_{2n}(e^{i\theta_0}) = 0$. Corollary 3 is applicable with $2n$ in place of $n$ giving sharp upper bound for $|a_{k+n}| = |c_k|$ in terms of $(1/2n) \int_{0}^{2\pi} |S_n(\theta)|^2 d\theta)^{\frac{1}{2}}$. That is how we get

**Theorem 8.** If $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a trigonometric polynomial of degree $n$ with $S_n(\theta_0) = 0$ for some $\theta_0$ in $[0, 2\pi)$ then for $-n \leq k \leq n$

$$|c_k| \leq \left( \frac{2n}{2n+1} \right)^{\frac{1}{2}} \left( \frac{1}{2n} \int_{0}^{2\pi} |S_n(\theta)|^2 d\theta \right)^{\frac{1}{2}}
$$

where equality holds if and only if $S_n(\theta)$ is a constant multiple of

$$\sum_{k=-n; i \neq k}^{n} (e^{-i\theta_0} e^{i\theta})^k - 2n(e^{-i\theta_0} e^{i\theta}).$$

Theorem 8 is the $L^2$ analogue of a theorem of Boas [6] according to which

$$|c_0| \leq \frac{n}{n+1} \max_{0 \leq \theta < 2\pi} |S_n(\theta)|.
$$

He, in fact, showed that the maximum on the right-hand side of (1.29) may be taken only over the points $\theta = \theta_0 + 2k\pi/(n+1)$, $k = 1, 2, \cdots, n$. It follows from his argument that

$$|c_0| \leq \frac{1}{n+1} \left\{ a + n \max_{1 \leq k \leq n} \left| S_n(\theta_0 + 2k\pi/(n+1)) \right| \right\}
$$

if $|S_n(\theta_0)| = a$ where $a$ is any nonnegative number. Applying (1.30) to the trigonometric polynomial $P_n(e^{i\theta})$ as Boas [6] did we obtain

$$\frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^2 d\theta \leq \frac{1}{n+1} \left\{ a^2 + n \max_{1 \leq k \leq n} \left| P_n \left( e^{i(\theta_0 + 2k\pi/(n+1))} \right) \right|^2 \right\}
$$

if $P_n(z)$ is a polynomial of degree $n$ with $|P_n(e^{i\theta})| = a$. In (1.31) equality holds for all polynomials $P_n(z)$ for which
POLYNOMIALS WITH A PRESCRIBED ZERO

\[ |P_n(e^{i\theta})|^2 = a^2 + (\mu^2 - a^2) \left\{ 1 - \left( \frac{1}{n+1} \frac{\sin(n+1)(\theta - \theta_0)/2}{\sin \frac{1}{2}(\theta - \theta_0)} \right)^2 \right\} \]

for \(0 \leq \theta < 2\pi\). For all such polynomials \(|P_n(e^{i\theta_0})| = a\),

\[ \max_{1 \leq k \leq n} |P_n(e^{i(\theta_0 + 2k\pi/(n+1))})| = \mu^2 = \max_{|z|=1} |P_n(z)| \]

if \(a \leq \mu\).

Hence if \(a \leq \mu\) we will not get any improvement in (1.31) if maximum of \(|P_n(z)|\) is taken on \(|z| = 1\) rather than at the points

\[ z = e^{i(\theta + 2k\pi/(n+1))} \quad (k = 1, 2, \ldots, n). \]

In general

(1.32)

\[ \|P_n(e^{i\theta})\|_{p_1} \leq \|P_n(e^{i\theta})\|_{p_2} \]

if \(0 < p_1 < p_2 \leq \infty\). We know from above that

\[ \|P_n(e^{i\theta})\|_2 \leq (a^2 + n\|P_n(e^{i\theta})\|_\infty^2)/(n + 1)^{1/2} \]

if \(|P_n(e^{i\theta_0})| = a\) for some \(\theta_0\) in \([0, 2\pi)\). It is natural to ask what improvement results in (1.32) if \(|P_n(e^{i\theta_0})| = a < \|P_n(e^{i\theta})\|_\infty\) for some \(\theta_0\) in \([0, 2\pi)\). Although we cannot answer this general question we note that by applying Theorem 8 to the trigonometric polynomial \(|P_n(e^{i\theta})|\) we get

Corollary 4. If \(P_n(e^{i\theta_0}) = 0\) for some \(\theta_0\) in \([0, 2\pi)\), then

(1.33)

\[ \|P_n(e^{i\theta})\|_2 \leq (2n/(2n + 1))^{1/4} \|P_n(e^{i\theta})\|_4. \]

In (1.33) equality holds for all polynomials \(P_n(z)\) for which

\[ |P_n(e^{i\theta})|^2 = n - \sum_{k=1}^n \cos(k(\theta - \theta_0)) \]

for \(0 \leq \theta < 2\pi\).

Entire functions of exponential type. If \(P_n(z)\) is a polynomial of degree \(n\)

such that \(|P_n(z)| \leq 1\) for \(|z| = 1\) then \(f(z) = P_n(e^{iz})\) is an entire function of exponential type \(n\) and \(|f(x)| \leq 1\) for \(-\infty < x < \infty\). From (1.1) we know that

\[ |f'(x)| \leq n \quad \text{for} \quad -\infty < x < \infty \quad \text{whereas according to (1.2) } |f(x + iy)| \leq e^n|y| \quad \text{for } y < 0. \]

It was shown by S. Bernstein (see [3, Chapter 11]) that for every entire function \(f(x)\) of exponential type \(r\) satisfying \(|f(x)| \leq 1\) for \(-\infty < x < \infty\) we have \(|f'(x)| \leq r\) for \(-\infty < x < \infty\). Besides, it is a simple consequence of the Phragmén-Lindelöf principle (for references see [3, p. 82]) that for all \(y\) \(|f(x + iy)| \leq e^{|y|}\) for an arbitrary entire function \(f(x)\) of exponential type \(r\) satisfying \(|f(x)| \leq 1\) for \(-\infty < x < \infty\).
If \( P_n(z) \neq 0 \) in \( |z| < 1 \) then \( P_n(e^{iz}) \) is an entire function \( f(z) \) of exponential type of a special kind: if \( b(\theta) \) is its indicator, we have \( b(-\pi/2) = n \) and (since \( P_n(0) \neq 0 \)) \( b(\pi/2) = 0 \). Thus Boas [5] extended (1.3), (1.4) to entire functions of exponential type by proving that for all real \( x \),

\[
|f'(x)| \leq r/2
\]

and for \( y < 0 \),

\[
|f(x + iy)| \leq \frac{1}{2}(e^{r|y|} + 1)
\]

if \( f(z) \) is an entire function of exponential type \( r \) with \( |f(x)| \leq 1 \) for real \( x \), \( b(\pi/2) = 0 \) and \( f(z) \neq 0 \) for \( \text{Im} \; z > 0 \).

Here we shall prove

**Theorem 9.** Let \( f(z) \) be an entire function of exponential type \( r \) with \( |f(x)| \leq 1 \) for real \( x \),

\[
b(\pi/2) = \lim_{y \to -\infty} y^{-1} \log |f(iy)| \leq 0
\]

and \( f(0) = 0 \). Then for all real \( x \) we have

\[
|f'(x)| \leq r[1 - (4 - n)/2(r|x| + 2)^2]\]

and for \( y < 0 \) we have

\[
|f(x + iy)| \leq e^{r|y|} \left( 1 - \frac{1}{2} r |y| \frac{(1 - e^{-r|y|})(4 - n)}{(ry)^2 + (r|x| + 2)^2} \right).
\]

We shall show that (1.36) is "essentially" best possible.

Theorem 6 can be reformulated as follows:

**Theorem 6'.** If the entire function \( f(z) \) of exponential type \( n \) is periodic on the real axis with period \( 2\pi \) (and hence bounded for real \( x \)), such that \( b(\pi/2) \leq 0 \), \( f(0) = 0 \) and \( |f(j\pi/n)| \leq 1 \) for \( j = 1, 2, \ldots, 2n - 1 \), then, for \( 0 < \omega < n \),

\[
|f(-i \log(1 - \omega/n))| \leq 1 - (1/\omega - 1/2n)(1 - \omega/n)^n.
\]

Indeed, an entire function \( f(z) \) of exponential type \( r \) is periodic on the real axis with period \( 2\pi \) if and only if \( f(z) = \sum_{k=-n}^{n} a_k e^{ikz} \; (n \leq r) \). If, in addition, \( b(\pi/2) \leq 0 \), then

\[
f(z) = \sum_{k=0}^{n} a_k e^{ikz} = P_n(e^{iz})
\]

where \( P_n(z) \) satisfies the hypotheses of Theorem 6. Hence (1.14') holds.

We observe that the requirement of periodicity in Theorem 6' can be dropped with little change in the conclusion if \( f(z) \) is bounded on the real axis.
Theorem 10. // $f(z)$ is an entire function of exponential type $r$ such that $|f(x)|$ is bounded on the real axis, $b(n/2) \leq 0$, $f(0) = 0$ and $|f(jn/r)| \leq 1$ for $j = \pm 1, \pm 2, \ldots$, then, for $0 < \omega < r$, we have

\[
|f(-i \log(1 - \omega/r))| \leq 1 + \frac{1 - (1 - \omega/r)^r}{\log(1 - \omega/r)^r}
\]

\[
< 1 - (1/\omega - 1/2r) + (1/\omega - 1/2r)(1 - \omega/r)^r + \omega|1 - (1 - \omega/r)^r l/2r^2
\]

2. Lemmas. We now prove or simply quote certain results which we shall need later. First, an interpolation formula:

Lemma 1. // If $P_n(x)$ is a polynomial of degree $n$ then for $R > 1$

\[
P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k A_k P_n(e^{i(\phi + k\pi/n)})
\]

where

\[
A_k = (R^n - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j \frac{kn}{n}.
\]

The coefficients $A_k$ are positive and

\[
\frac{1}{2n} \sum_{k=1}^{2n} A_k = R^n - 1.
\]

Proof. Let $t(\theta) = P_n(e^{i\theta}) = \sum_{\nu=0}^{n} (a_\nu \cos \nu \theta + b_\nu \sin \nu \theta)$. As

\[
a_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta) \cos \nu \theta \, d\theta,
\]

\[
b_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta) \sin \nu \theta \, d\theta
\]

for $\nu = 1, 2, \ldots, n$, we have

\[
P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \sum_{\nu=1}^{n} \left( R^n - 1)(a_\nu \cos \nu \phi + b_\nu \sin \nu \phi) \right)
\]

Since $t(\theta + \phi)$ is a trigonometric polynomial of degree $n$ in $\theta$ we may add to the sum terms in $e^{i\nu \theta}$ ($|\nu| > n$) without changing the value of the integral. Thus, we can write

\[
P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta + \phi) \left[ (R^n - 1) \cos n\theta + \sum_{\nu=1}^{n-1} (R^n - 1)\cos \nu \theta - \cos (2n - \nu)\theta \right] \, d\theta
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta + \phi)(\cos n\theta) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\} \, d\theta.
\]

Once again, we can replace $\cos n\theta$ by $b(n\theta)$, where $b(\theta)$ is simply required to be
continuous, periodic (with period $2\pi$) and have a Fourier series of the form $b(\theta) \sim \cos \theta + c_2 \cos 2\theta + d_2 \sin 2\theta + \cdots$. If $0 < \rho < 1$, we may choose

$$b(\theta) = \frac{1}{4\rho} \left( \frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos \theta + \rho^2} \right)$$

the series being uniformly convergent. We will then have

$$P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \lim_{\rho \to 1} \frac{1}{4\pi \rho} \int_{-\pi}^{\pi} \theta(\theta + \phi) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\}$$

$$\times \left( \frac{1 - \rho^2}{1 - 2\rho \cos n\theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos n\theta + \rho^2} \right) d\theta.$$ 

Now, by a well-known property of Poisson's kernel, we have, for every continuous periodic function $F(\theta)$ with period $2\pi$,

$$\lim_{\rho \to 1} \frac{1}{4\pi \rho} \int_{-\pi}^{\pi} F(\theta) \left( \frac{1 - \rho^2}{1 - 2\rho \cos n\theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos n\theta + \rho^2} \right) d\theta$$

$$= \frac{1}{2\pi} \sum_{k=1}^{2n} (-1)^k F(k\pi/n).$$

Applying this to the function

$$F(\theta) = \theta(\theta + \phi) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\}$$

we get the interpolation formula (2.1). The relation (2.2) follows from (2.1) if we set $P_n(z) = z^n$ and $\phi = 0$. The idea of the above proof comes from [19]. The fact that the coefficients $A_k$ are positive is clear from Lemma 2 below.

**Lemma 2.** For $R > 1$

$$A_k = (R^n - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\frac{k\pi}{n} \geq R^{n-2}(R - 1)^2.$$ 

**Proof.** If $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ and $\lambda_j - 2\lambda_j + \lambda_{j+1} \geq 0$ for $j = 1, 2, \cdots, n - 1$ then (see [17, p. 75])

$$\lambda_0 + 2 \sum_{j=1}^{n} \lambda_j \cos j\theta \geq 0$$

for all real $\theta$. This result may be applied with $\lambda_j = R^{n-j} - 1$ for $1 \leq j \leq n$ and $\lambda_0 = 2 R^{n-1} - R^{n-2} - 1$. Thus
\[ \chi(\theta) = (2R^n - 1 - R^{n-2} - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \geq 0 \]

for all real \( \theta \) and \( A_k \equiv R^{n-2}(R - 1)^2 + \chi(k\pi/n) \geq R^{n-2}(R - 1)^2 \).

For the proof of Theorems 1 and 3 we shall need the following

**Lemma 3.** Let \( P_n(z) \) be a polynomial of degree \( n \). If \( \max_{|z|=1} |P_n(z)| = 1 \) and \( |P_n(1)| = a \), then, for \( |\theta| \leq (1-a)/n \),

\[ |P_n(e^{i\theta})| \leq (1+a)/2. \]

**Proof.** Without loss of generality we may suppose that \( P_n(z) \neq 0 \) in \(|z| < 1\). In fact, if \( z_1, z_2, \ldots, z_k \) are the zeros of \( P_n(z) \) in \(|z| < 1\) then

\[ P^*(z) = P_n\left( \frac{1-z\bar{z}}{z-z\bar{z}} \right) \]

is a polynomial of degree \( n \) which does not vanish in \(|z| < 1\) and

\[ |P^*_n(e^{i\theta})| = |P_n(e^{i\theta})| \quad \text{for} \quad 0 < \theta < 2\pi. \]

Since

\[ P_n(e^{i\theta}) = P_n(1) + \int_1^e^{i\theta} P_n'(z) \, dz \]

we have

\[ |P_n(e^{i\theta})| \leq a + |e^{i\theta} - 1| \max_{|z|=1} |P_n'(z)|. \]

Hence, if \( \max_{|z|=1} |P_n(z)| \leq 1 \) and \( P_n(z) \neq 0 \) in \(|z| < 1\) (as we may assume), (1.3) implies

\[ |P_n(e^{i\theta})| \leq a + n|\sin(\theta/2)| \leq a + n|\theta|/2 \leq (1+a)/2 \]

provided that \( |\theta| \leq (1-a)/n \). This concludes the proof of Lemma 3.

Whereas the bound in (2.4) is not attained for any \( \theta \neq 0 \) (unless \( a = 1 \)) the following lemma gives sharp estimate for \( |P_n(e^{i\theta})| \) for every \( \theta \) in \([-\pi/n, \pi/n]\).

**Lemma 4.** If \( P_n(z) \) is a polynomial of degree \( n \) such that

\[ \max_{0 \leq k \leq n-1} |P_n(e^{i(2k+1)/n})| \leq 1 \quad \text{and} \quad P_n(1) = 0, \]

then for \( |\theta| \leq \pi/n \),

\[ |P_n(e^{i\theta})| \leq |\sin(n\theta/2)|. \]

**Proof.** This result is, essentially, due to Boas. Consider the trigonometric polynomial of degree \( n \)

\[ S_n(\theta) = e^{-in\theta} P_n(e^{2i\theta}). \]

Since \( \max_{0 \leq k \leq 2n-1} |S_n((2k + 1)\pi/(2n))| \leq 1 \) and \( S_n(0) = 0 \) it follows from a theorem of Boas (see [6, p. 43]) that \( |S_n(\theta)| \leq |\sin n\theta| \) for \( |\theta| \leq \pi/(2n) \). In terms of \( P_n(z) \) this says that, for \( |\theta| \leq \pi/n \),

\[ |P_n(e^{i\theta})| \leq |\sin(n\theta/2)| \]

which is the desired estimate.
Lemma 5 below will be used in the proofs of Theorems 2 and 4.

**Lemma 5.** The integral

\[ I = \int_0^{\pi/2} - n \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta}{\sin^2 \theta} \right) d\theta \]

remains bounded as \( n \) tends to infinity over the positive integers.

**Proof.** Break the range of integration \([0, \pi/2]\) into two parts, namely \([0, \pi/(2n)]\), \([\pi/(2n), \pi/2]\). Let the integral over \([0, \pi/(2n)]\) be denoted by \( I_1 \) and that over \([\pi/(2n), \pi/2]\) by \( I_2 \).

In the range \( \pi/(2n) \leq t \leq \pi/2 \) we have

\[ 0 < |\sin(n+1)t|/(n+1) \sin t| \leq \pi/[2(n+1)]t < 1 \]

so that

\[ 0 \leq - n \int_{\pi/(2n)}^{\pi/2} \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) dt \]

\[ \leq - n \int_{\pi/(2n)}^{\pi/2} \log \left( 1 - \frac{n^2}{2(n+1)^2} \right) dt \]

\[ = n \int_{\pi/(2n)}^{\pi/2} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\pi}{2(n+1)} \right)^{2k} t^{-2k} \right\} dt \]

\[ = n \frac{\pi}{2} \sum_{k=1}^{\infty} \left( \frac{1}{n+1} \right)^{2k} \frac{1}{k(1-2k)} (1-n^{2k-1}) < \frac{n}{2} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)} \left( \frac{n}{n+1} \right)^{2k}, \]

i.e., for all positive \( n \), \(|I_2| < (\pi/2) \sum_{k=1}^{\infty} (1/k(2k-1))\).

On the other hand,

\[ \frac{\sin(n+1)t}{\sin t} = e^{-it} + e^{-i(n-2)t} + \cdots + e^i(n-2)t + e^{int} \]

so that

\[ \frac{\sin(n+1)t}{\sin t} = 1 + 2 \sum_{k=1}^{n/2} \cos 2kt \]

if \( n \) is even and

\[ \frac{\sin(n+1)t}{\sin t} = 2 \sum_{k=0}^{(n-1)/2} \cos (2k+1)t \]

in case \( n \) is odd. For \( 0 \leq x \leq 1 \),

\[ \cos x = 1 - x^2/2 + x^4/24 - x^6/720 + \cdots \leq 1 - x^2/2 + x^4/24 \leq 1 - 11x^2/24. \]
Hence, for $0 \leq t \leq \pi/(2n)$, we have
\[
\cos 2kt \leq 1 - 11 k^2 t^2/6 \quad (1 \leq k \leq j < n/n)
\]
and
\[
\cos (2k+1)t \leq 1 - 11 (2k+1)^2 t^2/24 \quad (0 \leq k \leq j < (2n-n)/2n).
\]
Thus, for even $n$ and $j < n/n$,
\[
\frac{\sin(n+1)t}{\sin t} \leq 1 + 2 \left( j - \frac{11}{6} t^2 \sum_{k=1}^{j} k^2 \right) + 2(n/2 - j) = (n+1) - \frac{11}{8} t^2 (j+1)(2j+1),
\]
whereas, for odd $n$ and $j < (2n-n)/2n$,
\[
\frac{\sin(n+1)t}{\sin t} \leq 2 \left( j+1 - \frac{11}{24} t^2 \sum_{k=0}^{j} (2k+1)^2 \right) + 2((n-1)/2 - j) = (n+1) - \frac{11}{12} t^2 (2j/3 + 1)(j+1)(2j+1).
\]
In case $n$ is even we may choose $j = \lfloor n/n \rfloor > (n/n) - 1$ to conclude that for $0 < t \leq \pi/(2n)$:
\[
\begin{align*}
0 &< \frac{\sin(n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{9} t^2 \frac{(n-n)n(n-n/2)}{(n+1)n^3} < 1 \quad \text{if } n \geq 4.
\end{align*}
\]
Therefore for even $n \geq 4$
\[
0 \leq l_1 \leq n \int_{0}^{\pi/(2n)} \log \left\{ 1 - \left( 1 - \frac{11}{9} t^2 \frac{(n-n)n(n-n/2)}{(n+1)n^3} \right)^2 \right\} dt = - \int_{0}^{\pi/2} \log \left\{ 1 - \left( \frac{11}{9} s^2 \frac{(n-n)n(n-n/2)}{n(n+1)n^3} \right)^2 \right\} ds = O(1).
\]
For odd $n$, we may choose $j = \lfloor (n-n)/n \rfloor$ to obtain, corresponding to (2.6),
\[
0 < \frac{\sin(n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{36} t^2 \frac{(2n-n)(n-n)(2n-3n)}{(n+1)n^3} < 1 \quad \text{if } n \geq 5,
\]
and then argue as before.
For the proof of Theorem 2 we shall also need

**Lemma 6.** The integral
\[
-n \int_{\pi/4}^{\pi/2} \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t}
\]
remains bounded as $n$ tends to infinity over the odd positive integers.

**Proof.** Write the integral as
\[
\left( \int_{\pi/4}^{\pi/2-\pi/(2n)} + \int_{\pi/2-\pi/(2n)}^{\pi/2} \right) - n \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t}
\]

For the second integral, we have, since \( n \) is odd

\[
0 \leq \int_{\pi/(2n)}^{\pi/(2n)} - n \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\cos^2 t} \right) \frac{dt}{2 \sin^2 t}
\]

\[
= \frac{\pi^2}{8} \int_{-\pi/(2n)}^{\pi/(2n)} - n \log \left( 1 - \frac{1}{(n+1)^2} \frac{t^2}{\cos^2 t} \right) dt
\]

\[
= \frac{\pi^2}{8} n \int_{0}^{\pi/(2n)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{t^{2(k-1)}}{\cos^{2k} t} dt = O(1).
\]

The first integral can be estimated as follows.

\[
0 \leq \int_{\pi/4}^{\pi/2-\pi/(2n)} - n \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t}
\]

\[
\leq \int_{\pi/4}^{\pi/2} - n \log \left( 1 - \frac{\pi^2}{(n+1)^2} \frac{1}{4t^2} \right) \frac{dt}{2 \sin^2(t - \pi/2)}
\]

\[
= \frac{\pi^2}{8} \int_{-\pi/4}^{\pi/4} - n \log \left( 1 - \frac{\pi^2}{(n+1)^2} \frac{1}{4(t + \pi/2)^2} \right) dt
\]

\[
= \frac{\pi^2}{8} n \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\pi}{2(n+1)} \right)^{2k} \frac{1}{t^{2k}(t + \pi/2)^{2k}}
\]

\[
< \frac{\pi}{8} n \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\pi}{2(n+1)} \right)^{2k} = O(1).
\]

The next lemma is, in fact, the well-known interpolation formula of M. Riesz [16] expressing the derivative of a trigonometric polynomial in terms of the values of the polynomial at \( 2n \) different points.

**Lemma 7.** If \( S(\theta) \) is a trigonometric polynomial of degree \( n \) then

\[
(2.8) \quad S'(\theta) = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^k}{1 - \cos(2k+1)/2n} \left( \frac{\pi}{n} + \frac{2k + 1}{2n} \right) S \left( \theta + \frac{2k + 1}{2n} \right).
\]

Besides,
There is a corresponding interpolation formula for entire functions of exponential type. We shall need it for the proof of the first inequality appearing in the statement of Theorem 9 and it reads as follows (see [3, p. 210]).

Lemma 8. If \( f(z) \) is an entire function of exponential type \( \tau \) bounded on the real axis, then for all real \( x \)

\[
(2.10) \quad f'(x) = r(4/\pi^2) \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n + 1)^2} f\left(x + \frac{2n + 1}{2r} \right).
\]

Further,

\[
(2.11) \quad \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n + 1)^2} = 1.
\]

The next interpolation formula will be needed for the proof of the second inequality appearing in the statement of Theorem 9 as well as for the proof of Theorem 10.

Lemma 9. If \( f(z) \) is an entire function of exponential type \( \tau \) bounded on the real axis, such that \( b(\pi/2) \leq 0 \), then

\[
(2.12) \quad f(x + iy) = ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2} f\left(x + \frac{km}{r} \right).
\]

Proof. Suppose first that \( f(z) = \int_0^1 e^{i\pi t} \phi(t) dt \) with \( \phi \in L^2(0, r) \). Setting

\[
g(t) = \begin{cases} e^{-yt} & \text{if } 0 \leq t \leq r, \\ e^{yt} & \text{if } -r \leq t < 0, \\ 0 & \text{if } -r \leq t \leq 0, \end{cases}
\]

and \( \psi(t) = \begin{cases} e^{ixt} \phi(t) & \text{if } 0 \leq t \leq r, \\ 0 & \text{if } -r \leq t \leq 0, \end{cases} \)

we can write

\[
(2.13) \quad f(x + iy) = \int_{-r}^{r} g(t)\psi(t) \, dt.
\]

Now let \( \sum_{k=-\infty}^{\infty} c_k e^{ik(\pi/r)t} \) be the Fourier series of the continuous function of bounded variation \( g(t) \) on the interval \([-r, r]\). Then

\[
c_k = \frac{1}{2\pi} \int_{-r}^{r} g(t)e^{-ik(\pi/r)t} \, dt = ry \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2}
\]

and

\[
g(t) = ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2} e^{ik(\pi/r)t}
\]

for \(-r \leq t \leq r\). Since the series is uniformly convergent, we may substitute it for \( g(t) \) in (2.13) and integrate term by term to obtain
\[ f(x + iy) = ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (kn)^2} \int_{0}^{\infty} e^{i(x + kn/r)t} \phi(t) \, dt = ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (kn)^2} f(x + kn/r), \]

which proves the desired interpolation formula in the special case under consideration. In the general case, consider the function

\[ f_\delta(z) = f(z) e^{i\delta z} \frac{\sin \delta z}{\delta z} \]

with \( \delta > 0 \). It is clear that \( f_\delta(z) \) is an entire function of exponential type \( r + 2\delta \) such that

\[ \limsup_{y \to \infty} y^{-1} \log |f_\delta(iy)| \leq 0. \]

Besides, it belongs to \( L^2 \) on the real axis. Hence by the Paley-Wiener theorem (see [3, p. 103]), it has the form \( \int_{0}^{\infty} e^{i\delta t} \phi_\delta(t) \, dt \) with \( \phi_\delta \in L^2[0, r + 2\delta] \). By the above argument,

\[ f_\delta(x + iy) = (r + 2\delta)y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-(r+2\delta)y}}{(r+2\delta)(r+2\delta)y^2 + (kn)^2} f_\delta(x + k \frac{n}{r + 2\delta}) \]

and we obtain (2.12) on letting \( \delta \) tend to zero.

This kind of reasoning has been used before for proving certain other interpolation formulas (see for instance [4], [13]).

For the proof of Theorem 9 we shall also need:

**Lemma 10.** Let \( f(z) \) be an entire function of exponential type \( r \). If \( f(0) = 0 \),

\[ \sup_{-\infty < x < \infty} |f(x)| \leq 1 \text{ and } h_{f\pi/2} = \limsup_{y \to -\infty} y^{-1} \log |f(iy)| \leq 0 \]

then for all real \( x \)

\[ |f(x)| \leq (r/2)|x|. \tag{2.14} \]

**Proof.** An entire function of order less than 1 which is bounded on the real axis is necessarily a constant. If \( f(0) = 0 \) then it is identically zero and (2.14) is trivially true.

So let \( f(z) \) be an entire function of order 1 type \( r \). If \( h_{f\pi/2} = c \) let

\[ F(z) = e^{-i((r-c)/2)z} z^{r-1} f(z). \]

The function \( F(z) \) is an entire function of order 1 type \( \frac{1}{2}(r + c) \) and \( h_{F\pi/2} = h_{f\pi/2} = \frac{1}{2}(r + c) \). Let \( x_0 \) be a point of the real axis where \( |F(x_0)| = \max_{-\infty < x < \infty} |F(x)| \). Such a point exists since \( |F(x)| \) tends to zero as \( |x| \to \infty \).
Choose γ such that $e^{i\gamma} F(x_0)$ is positive. If $e^{i\gamma} F(z) = \sum_{n=0}^{\infty} a_n z^n$ then the function

$$G(z) = \sum_{n=0}^{\infty} (\text{Re} a_n) z^n$$

is an entire function of exponential type $\frac{1}{2}(r + c)$. Besides it is real for real $x$ and

$$|G(x)| = |\text{Re} \{e^{i\gamma} F(x)\}| \leq |x|^{-1}.$$

The function $H(x) = z G(z)$ is therefore an entire function of exponential type $\frac{1}{2}(r + c)$ which is real for real $x$ and whose modulus is bounded by 1 on the real axis. According to a theorem of Duffin and Schaeffer (see [9, p. 555]) we have for all real $x$

$$((r + c)/2)^2 H^2(x) + |H'(x)|^2 \leq ((r + c)/2)^2,$$

i.e.

$$((r + c)/2)^2 x^2 G^2(x) + \{G(x) + xG'(x)\}^2 \leq ((r + c)/2)^2.$$

At the point $x_0$ where $G(x)$ attains its maximum we have $G'(x_0) = 0$ and therefore

$$((r + c)/2)^2 x_0^2 G^2(x_0) + G^2(x_0) \leq ((r + c)/2)^2$$

or

$$G(x_0) \leq ((r + c)/2)/1 + ((r + c)/2)^2 x_0^2 \leq (r + c)/2 \leq r/2.$$

Since

$$G(x_0) = \max_{-\infty < x < \infty} |G(x)| = \max_{-\infty < x < \infty} |e^{i\gamma} F(x)| = \max_{-\infty < x < \infty} |x|^{-1}|f(x)|$$

we get $|f(x)| \leq (r/2)|x|$ for all real $x$. This proves Lemma 10.

### 3. Proofs of results announced in §1.

Proof of Theorem 1. Let $P_n(z) = \sum_{n=0}^{\infty} a_n z^n$. There is no loss of generality in supposing that $|P_n(1)| = a$. If $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ then

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i\phi}) e^{ik\phi} d\phi \quad (0 \leq k \leq n).$$

Hence

$$e^{i\theta} P_n(e^{i\theta}) = \sum_{k=1}^{n} k a_k e^{ik\theta} = \sum_{k=1}^{n} \frac{k}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i\phi}) e^{ik(\theta - \phi)} d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i(\theta + t)}) e^{-it} \sum_{k=1}^{n} k e^{-i(k-1)t} dt.$$

Since $P_n(e^{i(\theta + t)}) e^{-it} = a_0 e^{-it} + a_1 e^{it} + a_2 e^{2it} + \cdots + a_n e^{nit}$, we may
add to the sum \( \sum_{k=1}^{n} k e^{-i(k-1)t} \) terms in \( e^{-ikt} \) with \( k \geq n \) without changing the value of the integral provided \( n > 1 \). In particular, noting the identity
\[
\left( \sum_{k=1}^{n} z^{k-1} \right)^{2} = \sum_{k=1}^{n} k z^{k-1} + \text{higher powers of } z
\]
we can write (also if \( n = 1 \))
\[
e^{i\theta} P'(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i(\theta + t)}) e^{-it} \left( e^{it} \right) dt + e^{-it} + e^{-it} + \ldots + e^{-i(n-1)t} \right) dt.
\]
Consequently,
\[
|P'(e^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P_n(e^{i(\theta + t)}) \right| \left( \frac{\sin nt/2}{\sin t/2} \right) dt.
\]
Since
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\sin nt/2}{\sin t/2} \right)^2 dt = n
\]
we find, using (2.4) together with \( |P_n(z)| \leq 1 \) (\( |z| < 1 \)) that
\[
|P'(e^{i\theta})| \leq \frac{1}{2\pi} \left( \int_{|\theta + t| \leq (1-a)/n} - \int_{|\theta + t| \leq (1-a)/n} \right) \left| P_n(e^{i(\theta + t)}) \right| \left( \frac{\sin nt/2}{\sin t/2} \right)^2 dt
\]
\[
\leq n - \frac{1-a}{4\pi} \int_{|\theta + t| \leq (1-a)/n} \left| e^{i(\theta + t)} \right| \left( \frac{\sin nt/2}{\sin t/2} \right)^2 dt
\]
\[
\leq n - \frac{1-a}{4\pi} \int_{-\theta - (1-a)/n}^{\theta + (1-a)/n} \sin^2 t/2 dt
\]
\[
= n - \frac{1-a}{4\pi} \left( (1-a) - \cos n\theta \sin (1-a) \right)
\]
\[
\leq n - \frac{1-a}{4\pi} \left( (1-a) - \sin (1-a) \right).
\]
This proves Theorem 1.

Proof of Theorem 2. Since the trigonometric polynomial
\[
1 - \frac{1}{(n+1)^2} \left( \sum_{k=0}^{n} e^{ik\theta} \right)^2 = 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2 \theta/2}
\]
is nonnegative for all real \( \theta \), it follows from a classical result of Fejér and Riesz (see [1, p. 152]) that the equation
\[
|P_n(e^{i\theta})|^2 + \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2 \theta/2} = 1
\]
defines a family of polynomials belonging to \( \mathcal{I}_{n,0} \). Let \( \hat{P}_n(z) \) be the one which does not vanish in \( |z| < 1 \) and assumes a positive value at the origin. For \( |z| < 1 \)
In fact, given \( z_0 \) (\( |z_0| < 1 \))

\[
\log \hat{p}_n(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |\hat{p}_n(e^{it})| \, dt.
\]

for \( |z_0| < \rho < 1 \) by Poisson-Schwarz formula. Since

\[
1 \geq |\rho e^{it} - 1| \geq 2\sqrt{\rho} |\sin t/2| \geq 2\sqrt{\rho} |t| \geq \sqrt{2}|t|/\pi
\]

for \( \rho \geq \frac{1}{2} \) and hence

\[
|\log |\rho e^{it} - 1|| < \log (\sqrt{2}) + |\log |t||,
\]

Lebesgue’s dominated convergence theorem allows us to let \( \rho \) tend to \( 1 \) under the integral sign giving (3.1). Besides, the function \( \log |\hat{p}_n(e^{it})| \) being integrable over \([0, 2\pi]\), we can differentiate (3.1) under the integral sign and get

(3.2)

\[
\hat{p}_n'(z) = \hat{p}_n(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2} \log |\hat{p}_n(e^{it})|^2 \, dt
\]

for \( |z| < 1 \). But we are interested in \( \hat{p}_n'(-1) \) and will like (3.2) to hold for \( z = -1 \) as well. We observe that for odd \( n \) it is indeed the case. For odd \( n \), the function

\[
\log |\hat{p}_n(e^{it})|^2 = \log \left\{ 1 - \frac{1}{(n + 1)^2 \sin^2 t/2} \right\}
\]

is analytic in \( |z - \pi| < \rho_0 \) for some \( \rho_0 > 0 \) and has a double zero at \( z = \pi \). Let \( \delta = \frac{1}{2} \min (\rho_0, 1) \). Then for \( |t - \pi| \leq \delta \):

\[
\log |\hat{p}_n(e^{it})|^2 = (t - \pi)^2 \alpha(t)
\]

where \( \alpha(t) \) is continuous on \( |t - \pi| \leq \delta \). Hence, if

\[
f_{\rho}(t) = \frac{e^{it}}{(e^{it} + \rho)^2} \log \left\{ 1 - \frac{1}{(n + 1)^2 \sin^2 (n + 1)t/2} \right\}
\]

then, for \( 0 < \rho < 1, |t - \pi| \leq \delta \),

\[
|f_{\rho}(t)| = \frac{(t - \pi)^2 \alpha(t)}{|e^{it} + \rho|^2} \leq \frac{(t - \pi)^2 \alpha(t)}{4\rho \sin^2 ((t - \pi)/2)} = \frac{1}{\rho} \beta(t)
\]

where the function \( \beta(t) \) is integrable over \( |t - \pi| \leq \delta \). For \( 0 < \rho < 1, \delta < |t - \pi| \leq \pi \), we have

\[
|f_{\rho}(t)| \leq \frac{-1}{1 + 2\rho \cos (\pi - \delta) + \rho^2} \log \left\{ 1 - \frac{1}{(n + 1)^2 \sin^2 (n + 1)t/2} \right\};
\]

Once again, Lebesgue’s dominated convergence theorem shows that
Formula (3.3) can be rewritten as
\[ |\hat{P}'(-1)| = -\hat{P}'_n(-1) = -\frac{1}{\pi} \int_{\pi/2}^{\pi/2} \log \left( 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\phi}{\sin^2\phi} \right) \frac{d\phi}{1 + \cos 2\phi}. \]

Therefore, Lemmas 5 and 6 show that there exists a positive constant \( c \) such that
\[ |\hat{P}'(-1)| \leq c/n. \]
If \( n \) is even and \( m = n/2 \) is odd, \( g_n(z) = (\hat{P}_m(z))^2 \) is a polynomial of degree \( n \) such that \( g_n(1) = 0 \) and \( \max_{|z|=1} |g_n(z)| = g_n(-1) = 1. \) Besides,
\[ |g_n'''(-1)| = 2|\hat{P}_m'''(-1)| \leq 4c/n. \]
If \( n \) is even and \( m = n/2 \) is also even, the polynomial \( b_n(z) = \hat{P}_{m-1}(z)\hat{P}_{m+1}(z) \in \mathcal{G}_{n,0} \) (with \( b(-1) = 1 \)) and
\[ |b_n'''(-1)| = -b_n'''(-1) \leq 4c/n(n^2 - 4). \]
Thus, there exist a positive constant \( c_1 \) and, for each \( n \), a polynomial \( P_n(z) \in \mathcal{G}_{n,0} \) with \( P_n(1) = 0, P_n(-1) = 1, P_n'(-1) < 0 \) and
\[ |P_n'(-1)| \leq c_1/n. \]
Now, let \( G_n(z) = (-1)^n z^n P_n(1/z) \) which belongs to \( \mathcal{G}_{n,0} \) with \( G_n(1) = 0, G_n(-1) = 1. \) Besides, using (3.4) we see that
\[ |G_n'''(-1)| = -G_n'''(-1) \geq n - c_1/n. \]
Finally, given \( a \in [0, 1] \), set \( H_n(z) = (-1)^n a z^n + (1 - a)G_n(z) \). Then \( H_n(z) \in \mathcal{G}_{n,a} \) (with \( H_n(1) = a \) and \( H_n(-1) = 1 \)) and
\[ |H_n'''(-1)| = |na - (1 - a)G_n'''(-1)| = na + (1 - a)|G_n'''(-1)| \geq n - c_1(1 - a)/n. \]
This completes the proof of Theorem 2.

Proof of Theorem 3. Let \( P_n(z) \in \mathcal{G}_{n,a} \). Without loss of generality we may suppose \( P_n(1) = a \). Let \( m = \lceil n/(1-a) \rceil + 1 \); and consider the polynomial \( P^\ast(z) = (P_n(z))^m \). By Lemma 1, we have for \( R > 1 \)
\[ P^\ast(R e^{i\phi}) = P^\ast(e^{i\phi}) + \frac{1}{2mn} \sum_{k=1}^{2mn} (-1)^k A_k P^\ast(e^{i(\phi + kn/mn)}). \]
where
\[ \frac{1}{2mn} \sum_{k=1}^{2mn} A_k = R^{mn} - 1, \]
and according to Lemma 2 $A_k > R^{mn}(1 - R^{-1})^2$. By Lemma 3

$$|P^*(e^{i\phi})| \leq ((1 + a)/2)^m < ((1 + a)/2)^{m/(1-a)} < e^{-\pi/2} \quad \text{for } |\phi| < (1 - a)/n.$$  

At least two of the points $\phi_k = \phi + kn/mn$, $k = 1, 2, \ldots, 2mn$ (say $\phi_{j_1}, \phi_{j_2}$) lie in the angle $|\phi| \leq \pi/mn < (1 - a)/n$. Hence from above

$$|P^*(e^{i\phi})| \leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k |P^*(e^{i(\phi + kn/mn)})|$$

$$\leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k - \frac{1}{2mn} (1 - e^{-\pi/2})(A_{j_1} + A_{j_2})$$

$$\leq R^{mn}(1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2)$$

and

$$|P^*(e^{i\phi})| = |P^*(e^{i\phi})|^{1/m} \leq R^n[1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2]^{1/m}$$

$$\leq R^n[1 - (m^2n)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2]$$

by virtue of the inequality $(1 - x)^a < 1 - ax$ valid for $0 < a < 1$ and $0 < x < 1$.

Since $m = \lceil\pi/(1 - a)\rceil + 1$ we have, in fact, proved (1.7).

**Proof of Theorem 4.** We again consider the polynomial

$$\hat{P}_n(z) = \sum_{k=0}^{n} \hat{a}_k z^k \in \mathcal{P}_{n,0}$$

defined by the relation

$$|P^*_n(e^{i\phi})|^2 = 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2\theta}$$

not vanishing in $|z| < 1$ and assuming a positive value at the origin. From (3.1)

$$0 < -\log \hat{a}_0 = \frac{1}{\pi} \int_0^{\pi/2} -\log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta}{\sin^2\theta}\right) d\theta.$$  

Hence according to Lemma 5 there exists a constant $c_2$ such that $-\log \hat{a}_0 < (c_2 - 1)/n$, i.e. $\hat{a}_0 > \exp[-(c_2 - 1)/n] > 1 - (c_2 - 1)/n$. Now if

$$\hat{Q}_n(z) = z^n \hat{P}_n(1/z) = \sum_{k=0}^{n} \hat{a}_k z^{-n-k}$$

then

$$|\hat{Q}_n(-R)| \geq \hat{a}_0 R^n - \sum_{k=1}^{n} |a_k| R^{n-k}$$

$$\geq \hat{a}_0 R^n - n R^{n-1} (\text{since } |\hat{Q}_n(z)| \leq 1 \text{ for } |z| = 1)$$

$$> R^n|1 - (c_2 - 1)/n - n|R > R^n(1 - c_2/n)$$
if \( R > n^2 \). The polynomial \( \hat{Q}_n(z) \) is real on the real axis and for odd \( n \), 
\[ \hat{Q}_n(-R) < 0 \text{ if } R > 1 \text{ (note that } \hat{Q}_n(z) \text{ does not vanish for } |z| > 1 \text{ and } Q_n(-R) \text{ tends to } -\infty \text{ as } R \to \infty). \]
Hence, if \( n \) is odd and
\[ V_n(z) = (1 - a)[r/(1 - a)]z^n + \hat{Q}_n(z) \]
then \( |V_n(z)| \leq 1 \) for \( |z| \leq 1 \), \( V_n(-1) = -1 \) and \( V_n(1) = a \). Thus \( V_n(z) \in \mathcal{G}_{n,a} \) and for \( R > n^2 \):
\[
|V_n(-R)| = \left| 1 - a \right| \left\{ a/(1 - a) R^n + \hat{Q}_n(-R) \right\} > aR^n + (1 - a)R^n(1 - c^2/n) = R^n[1 - c^2(1 - a)/n].
\]

If \( n \) is even, we may consider
\[ W_n(z) = (1 - a)[r/(1 - a)]z^n + z\hat{Q}_{n-1}(z) \]
instead of \( V_n(z) \). Clearly \( W_n(z) \in \mathcal{G}_{n,a} \) and for \( R > n^2 \):
\[
|W_n(-R)| > R^n[1 - c^2(1 - a)/n].
\]

**Proof of (1.5').** Let \( P_n(z) \in \mathcal{G}_{n,0} \), where we may suppose \( P_n(1) = 0 \). Since \( P_n(e^{i\theta}) \) is a trigonometric polynomial of degree \( n \) we have by Lemma 7
\[
|P_n'(e^{i\theta})| \leq \frac{1}{2n} \sum_{k=1}^{2n} \frac{1}{1 - \cos((2k + 1)\pi/2n)} |P_n(e^{i(\theta + (2k + 1)\pi/2n)})|.
\]
One of the points \( e^{i(\theta + (2k + 1)\pi/2n)} \) \((1 \leq k \leq 2n)\) is 1 if \( \theta \) is an odd multiple of \( \pi/(2n) \) and hence for such values of \( \theta \)
\[
|P_n'(e^{i\theta})| \leq n - \frac{1}{2n} \frac{1}{1 - \cos((2j + 1)\pi/2n)} \leq n - \frac{1}{4n}.
\]
If \( \theta \) is not an odd multiple of \( \pi/(2n) \) then precisely two of the points \( \theta + (2k + 1)\pi/2n \) (say \( \theta + n(2j_1 + 1)/2n \), \( \theta + n(2j_2 + 1)/2n \)) lie in the interval \( (-\pi/n, \pi/n) \) and by Lemma 4
\[
|P_n'(e^{i\theta})| \leq n - \frac{1}{2n} \left\{ \frac{1 - |\sin n(\theta + (2j_1 + 1)n/2n)|}{1 - \cos((2j_1 + 1)n/2n)} + \frac{1 - |\cos n(\theta + (2j_2 + 1)n/2n)|}{1 - \cos((2j_2 + 1)n/2n)} \right\} \leq n - \frac{2 - \sqrt{2}}{4n}
\]
which completes the proof of (1.5').
Proof of (1.7°). Let \( P_n(z) \in \mathcal{P}_{n,0} \). Clearly, we may suppose \( P_n(1) = 0 \). In this case two and possibly three of the interpolation points in the formula
\[
P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2\pi} \sum_{k=1}^{2n} (-1)^k A_k P_n e^{i(k\pi/n)}
\]
lie in \( |\theta| \leq \pi/n \) where by Lemma 4 \( |P_n(e^{i\theta})| \leq |\sin(n\theta/2)| \). If three of the points (say \( \phi + j_1 \pi/n, \phi + j_2 \pi/n, \phi + j_3 \pi/n \)) fall in the interval \([-\pi/n, \pi/n]\) then they have to be \(-\pi/n, 0, \pi/n\) and
\[
|P_n(Re^{i\phi})| \leq R^n - \frac{1}{2n} [A_{j_1} (1 - |\sin(-\pi/2)|) + A_{j_2} (1 - |\sin 0|) + A_{j_3} (1 - |\sin:\pi/2|)]
\]
\[
\leq R^n[1 - (1 - R^{-1})^2/2n].
\]
If there are only two points (say \( \phi + j_1 \pi/n, \phi + j_2 \pi/n \)) in \([-\pi/n, \pi/n]\), then
\[
|\sin n(\phi + j_2 \pi/n)/2| = |\cos n(\phi + j_1 \pi/n)/2|
\]
and hence
\[
|P_n(Re^{i\phi})| \leq R^n - \frac{1}{2n} [A_{j_1} (1 - |\sin n(\phi + j_1 \pi/n)/2|) + A_{j_2} (1 - |\cos n(\phi + j_1 \pi/n)/2|)]
\]
\[
\leq R^n[1 - (2 - \sqrt{2})(1 - R^{-1})^2/2n]
\]
which completes the proof of (1.7°).

Proof of Theorem 5. In the proof of Theorem 2 it was shown that for each positive integer \( n \) there exists a polynomial \( G_n(z) \in \mathcal{P}_{n,0} \) with \( G_n(1) = 0 \), \( G_n(-1) = 1 \) and \( |G_n'(-1)| = -G_n'(-1) \geq n - c_1/n \) where \( c_1 \) is an absolute constant.

Given \( \epsilon > 0 \), and a positive integer \( n \) let \( m = \lceil \sqrt{n} \rceil \), \( k = \lceil \epsilon \sqrt{n}/2c_1 \rceil \), \( j = n - \lceil \sqrt{n} \rceil \lceil \epsilon \sqrt{n}/2c_1 \rceil \). Then \( P_n(z) = G_m(z) k G_j(z) \) has a zero of multiplicity \( k + 1 > \epsilon \sqrt{n}/2c_1 = \delta \sqrt{n} \) at \( z = 1 \). Moreover
\[
|P_n'(1)| = |kG_m'(-1) + G_j'(-1)|
\]
\[
\geq k(m - c_1/m) + (j - c_1/j) = n - c_1(k/m + 1/j) > n - \epsilon
\]
if \( n \) is large enough, say \( n > n_0 \).

Proof of Theorem 6. If \( Q_n(z) = z^n P_n(1/z) \) then by the hypothesis of Theorem 6
\[
Q_n(1) = P_n(1) = 0, \quad |Q_n(\exp ij\pi/n)| = |P_n(\exp ij\pi/n)| \leq 1
\]
for \( j = 1, 2, \ldots, 2n - 1 \).

Hence applying Lemma 1 to \( Q_n(z) \) we get for \( R > 1 \)
Thus

\[
|Q_n(R)| \leq \frac{1}{2n} \sum_{k=1}^{2n} A_k - \frac{1}{2n} A_{2n} = R^n - 1 - \frac{1}{2n} A_{2n}
\]

\[
= R^n - 1 - \frac{1}{2n} \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \right\}
\]

\[
= R^n - 1 - \frac{1}{2n} \left\{ (R^n - 1) \frac{R + 1}{R - 1} - 2n \right\}.
\]

Thus

\[
|P_n(1/R)| \leq \left\{ 1 - R^{-n} \right\} \left( 1 - \frac{1}{2n} \frac{1 + R^{-1}}{1 - R^{-1}} \right) + R^{-n}.
\]

If \(0 < \omega \leq n\) then \((1 - \omega/n)^{-1} > 1\) and hence, in particular,

\[
|P_n(1 - \omega/n)| \leq 1 - (1/\omega - 1/2n) + (1/\omega - 1/2n)(1 - \omega/n)^n.
\]

The above reasoning also shows that if \(|P_n(z)| = a < 1\) and \(|P_n(\exp i\pi/n)| < 1\) for \(j = 1, 2, \ldots, 2n - 1\) then for \(0 < \omega \leq n\)

\[
|P_n(1 - \omega/n)| \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n.
\]

Besides, the formula obtained by equating the real parts on the two sides of (2.1) may be applied to the polynomial \(Q_n(z) = z^n P_n(1/z)\) to prove in exactly the same way as above that:

(i) if \(P_n(z)\) is a polynomial of degree \(n\) such that \(|\text{Re} P_n(1)| = a < 1\), \(|\text{Re} P_n(\exp i\pi/n)| < 1\) for \(j = 1, 2, \ldots, 2n - 1\) then for \(0 < \omega \leq n\)

\[
|\text{Re} P_n(1 - \omega/n)| \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n.
\]

(ii) if \(P_n(z)\) is a polynomial of degree \(n\) such that \(\text{Re} P_n(1) = a < 1\), \(\text{Re} P_n(\exp i\pi/n) \leq 1\) for \(j = 1, 2, \ldots, 2n - 1\) then for \(0 < \omega \leq n\)

\[
\text{Re} P_n(1 - \omega/n) \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n.
\]

Proof of Theorem 7. There is no loss of generality in supposing that the point of the unit circle where the polynomial vanishes is 1, i.e.

\[
\sum_{k=0}^{n} a_k = 0.
\]

We write the left-hand side of (1.19) as

\[
\sum_{k=0}^{n} \lambda_k |a_k|^2 = \lambda_j \sum_{k=0}^{n} |a_k|^2 - \sum_{k=0; k \neq j}^{n} (\lambda_j - \lambda_k) |a_k|^2
\]

\[
= \lambda_j \sum_{k=0}^{n} |a_k|^2 - \sum_{k=0; k \neq j}^{n} (\lambda_j - \lambda_k - \lambda) |a_k|^2 - \lambda \sum_{k=0; k \neq j}^{n} |a_k|^2,
\]
where for the moment $\lambda$ is a constant such that
\[
0 < \lambda < A = \min_{0 < k < n; k \neq j} (\lambda_j - \lambda_k).
\]

From (3.8) and Schwarz's inequality we obtain
\[
|a_j|^2 = \left| \sum_{k=0; k \neq j}^n a_k \right|^2 \leq \left( \sum_{k=0; k \neq j}^n |a_k| \right)^2
\]
\[
= \left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{1/2} |a_k| (\lambda_j - \lambda_k - \lambda)^{-1/2} \right\}^2
\]
\[
\leq \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)|a_k|^2 \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1},
\]
so that
\[
- \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)|a_k|^2 \leq - |a_j|^2 \left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1}.
\]

Now if $\lambda$ happens to be the root of the equation (1.20) lying in $(0, \Lambda)$, then
\[
\left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1} = \lambda
\]
and we get
\[
\sum_{k=0}^n \lambda_k |a_k|^2 \leq \lambda \sum_{k=0}^n |a_k|^2 - \lambda |a_j|^2 - \lambda \sum_{k=0; k \neq j}^n |a_k|^2 = (\lambda_j - \lambda) \sum_{k=0}^n |a_k|^2
\]
which is the desired inequality.

Observe that in (1.19) equality is possible only if
\[
\left| \sum_{k=0; k \neq j}^n a_k \right| = \sum_{k=0; k \neq j}^n |a_k|
\]
and equality holds in Schwarz's inequality, i.e. $a_k = a(\lambda_j - \lambda_k - \lambda)^{-1}$ if $k \neq j$ whereas $a_j = - a/\lambda$. Thus the extremal polynomial has the form
\[
P_n(z) = a \sum_{k=0}^n \frac{1}{\lambda_j - \lambda_k - \lambda} (e^{-i\theta_0 z})^k.
\]

The proof of Theorem 7" is analogous and we omit it.

Proof of Theorem 9. For an entire function $f(z)$ satisfying the hypotheses of the theorem we have by virtue of Lemma 8
\[ |f'(x)| \leq r(4/\pi^2) \sum_{k=-\infty}^{\infty} \frac{1}{(2n+1)^2} |f(x + (2n+1)\pi/2r)| \]

for all real \( x \). According to (2.14) \( |f(x)| < (r/2)|x| \) in \((-2/r, 2/r)\). If only one of the interpolation points (say \( x + (2m+1)\pi/2r \)) falls in this interval it must, in fact, lie in the subinterval \([- (\pi - 2)/r, (\pi - 2)/r]\) and hence by (2.14) in conjunction with (2.11)

\[ |f'(x)| \leq r \left\{ 1 - \frac{4}{\pi^2} \left( 1 - \frac{r}{2} \left| x + \frac{2m+1}{2r} \right| \right) \right\} \leq r \left\{ 1 - \frac{4}{\pi^2} \frac{2 - \pi/2}{(2m+1)^2} \right\}. \]

Since

\[ |(2m+1)\pi/2r| - |x| \leq |x + (2m+1)\pi/2r| \leq (\pi - 2)/r \]

we have

\[ \frac{1}{(2m+1)^2} \geq \frac{n^2}{4(r|x| + \pi - 2)^2} \]

and hence

\[ |f'(x)| \leq r \left\{ 1 - \frac{4}{\pi^2} \left( 1 - \frac{r}{2} \left| x + \frac{2m+1}{2r} \right| \right) \right\}. \]

If, on the other hand, two points \( x + (2m+1)\pi/2r, x + (2m+3)\pi/2r \) lie in \((-2/r, 2/r)\), then

\[ |f'(x)| \leq r \left\{ 1 - \frac{4}{\pi^2} \left( 1 - \frac{r}{2} \left| x + \frac{2m+1}{2r} \right| + \frac{1 - r|x + (2m+3)\pi/2r|/2}{(2m+3)^2} \right) \right\}. \]

Since \( |x + (2m+1)\pi/2r| < 2/r \) and \( |x + (2m+3)\pi/2r| < 2/r \), we have

\[ 1/(2m+1)^2 > \frac{n^2}{4(r|x| + 2)^2}, \quad 1/(2m+3)^2 > \frac{n^2}{4(r|x| + 2)^2} \]

and hence

\[ |f'(x)| < r \left\{ 1 - \frac{4}{\pi^2} \left( 1 - \frac{r}{2} \left| x + \frac{2m+1}{2r} \right| \right) \right\}. \]

which completes the proof of (1.36).

To prove inequality (1.37), we use formula (2.12) which we may apply to the constant function 1 to first conclude that

\[ (3.9) \quad ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (kn)^2} = 1. \]

Now let \( y > 0 \). If only one of the interpolation points, say \( x + n\pi/r \), belongs to \([-2/r, 2/r]\), we have in fact \( |x + n\pi/r| \leq (\pi - 2)/r \) so that \((nn)^2 \leq (\pi - 2 + r|x|)^2\) and

\[ |f(x + iy)| \leq ry \sum_{k=-\infty, k \neq n}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (kn)^2} + ry \frac{1 - (-1)^n e^{-ry}}{(ry)^2 + (nn)^2} \cdot r|x + n\pi/r|/2 \]

\[ = 1 - \frac{ry(1 - (-1)^n e^{-ry})}{(ry)^2 + (nn)^2} \left( 1 - r|x + n\pi/r|/2 \right) \]

\[ \leq 1 - \frac{ry(1 - (-1)^n e^{-ry})}{(ry)^2 + (nn)^2} \left( 2 - \pi/2 \right) \leq 1 - \frac{ry(2 - \pi/2)(1 - e^{-ry})}{(ry)^2 + (\pi - 2 + r|x|)^2}. \]
If two interpolation points \( x + \pi \tau/2 \) and \( x + (\pi + 1)\tau/2 \) are in \([-2/\tau, 2/\tau]\), we similarly obtain

\[
|f(x + iy)| \leq 1 - \tau y \frac{(2 - \pi/2)(1 - e^{-\tau y})}{(\tau y)^2 + (2 + \tau |x|)^2}
\]

for \( y > 0 \). Thus, in any case

\[
|f(x + iy)| \leq 1 - \tau y \frac{(2 - \pi/2)(1 - e^{-\tau y})}{(\tau y)^2 + (2 + \tau |x|)^2}
\]

if \( y > 0 \).

Finally, applying this inequality to the function \( g(z) = f(z)e^{iz} \) which satisfies the same hypothesis as the function \( f(z) \) we get

\[
|f(x + iy)| \leq e^{r|y|} \left\{ 1 - r|y| \frac{(1 - e^{-\tau |y|})(2 - \pi/2)}{(\tau y)^2 + (2 + \tau |x|)^2} \right\}
\]

for \( y < 0 \).

We observe that inequality (1.36) is best possible in the sense that there exists an absolute constant \( A \) such that given \( \tau \) and a positive number \( B \), we can find a function \( f \) satisfying the hypothesis of Theorem 9 and a point \( x > B \) where

\[
|f'(x)| > \tau - A/\tau x^2.
\]

In the proof of Theorem 2 it was shown that for each positive integer \( n \) there exists a polynomial \( P_n(z) \in \mathbb{P}_{n,0} \) such that \( P_n(1) = 0 \), \( P_n(-1) = 1 \), \( P'_n(-1) < 0 \) and \( |P'_n(-1)| > n - c_1/n \) where \( c_1 \) is an absolute constant. If \( P_m(z) \) is such a polynomial of degree \( m = [rB/\pi] + 1 \) then

\[
f(z) = P_m(\exp(izm/\tau))
\]
satisfies all the hypotheses of Theorem 9 and

\[
|f'(mn/\tau)| = \frac{r}{m} |P'_m(-1)| > r - \frac{c_1\pi^2}{r(mn/\tau)^2} = \tau - \frac{A}{r(mn/\tau)^2}
\]

where, clearly, \( x = mn/\tau > B \).

We do not make such a remark concerning inequality (1.37).

Proof of Theorem 10. By Lemma 9 we have

\[
f(\tau y) = \tau y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(\tau y)^2 + (kn)^2} (kn/\tau),
\]

which in conjunction with (3.9) gives us \( |f(\tau y)| \leq 1 - (1 - e^{-\tau y})/\tau y \). If \( 0 < \omega < \tau \) we may set \( y = -\log(1 - \omega/\tau) \) to get the desired inequality

\[
|f(-i \log(1 - \omega/\tau))| \leq 1 + \frac{1 - (1 - \omega/\tau)^r}{\log(1 - \omega/\tau)^r},
\]
In very much the same way, inequalities (3.5), (3.6) and (3.7) can also be extended to entire functions of exponential type.

REFERENCES


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