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EQUISINGULAR DEFORMATIONS OF PLANE ALGEBROID CURVES(1)

BY

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ABSTRACT. We construct a formal versal equisingular deformation of a plane algebroid curve (in characteristic zero), and show it is smoothly embedded in the whole deformation space of the singularity. Closer analysis relates equisingular deformations of the curve to locally trivial deformations of a certain (nonreduced) projective curve. Finally, we prove that algebraic π_1 of the complement of a plane algebroid curve remains constant during formal equisingular deformation.

Introduction. In a series of papers ([10], [11], [12]), Zariski has studied the concept of *equisingularity* of plane algebroid curves. Two curves are equisingular if one can simultaneously resolve their singularities; this equivalence relation is weaker than analytic equivalence, but stronger than equimultiplicity. Using topological techniques, Zariski proves that two equisingular curves over $\mathbf{C}$ have locally the same topological embedding in $\mathbf{C}^2$; in particular, the characteristic pairs of their branches are the same, whence they yield knots of the same knot type in $\mathbf{R}^3$ (cf. [4]).

Utilizing techniques developed by M. Schlessinger [6], we study infinitesimal equisingular families of curves. Our deformation theory takes place over the category $\underline{C}$ of artin local $\mathbf{C}$-algebras. Recall that if $f \in \mathbf{C}[[X, Y]]$ is reduced, and if $g_1, \cdots, g_m \in \mathbf{C}[[X, Y]]$ induce a basis of the artin ring $\mathbf{C}[[X, Y]]/(f, f_X, f_Y)$, then the formal family $f + t_1 g_1 + \cdots + t_m g_m \in \mathbf{C}[[X, Y, t_1, \cdots, t_m]]$ induces a formal versal (or semiuniversal) deformation of the singularity defined by (f). Thus, in a weak sense, the family represents the functor on $\underline{C}$ of infinitesimal deformations of the singularity.

To define equisingular deformation, we emulate Zariski's original definition. Recall that every plane algebroid singularity can be reduced to a number of ordinary double points by a finite number of quadratic transforms. We say a deformation

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$A[[X, Y]]/(\bar{f})$ $(A \in \underline{C})$ is equisingular if it is normally flat (i.e., equimultiple) along an A-section and, after blowing up the section, the transform is equisingular along A-sections lying above the first; finally, equisingular means equimultiple for an ordinary double point. Theorems 3.2 and 4.2 below imply

Theorem. *There exists a smooth closed subscheme of* Spec $\mathbf{C}[[t_1, \cdots, t_m]]$ *on which the induced family of curves yields a formal versal equisingular deformation. Infinitesimal equisingular sections are unique.*

There are some equisingular deformations that are simpler than others, namely those for which all significant equisingular sections can be simultaneously trivialized; call the functor of such deformations ES', a subfunctor of the regular equisingular functor ES. From most points of view, ES' is much easier to work with. For instance, the tangent space of ES' is easily described algebraically (Proposition 6.3), and in §8 we prove

Theorem. $ES' = ES$ *if and only if, for appropriate choice of* $g_1, \cdots, g_m$, *the subscheme of the previous theorem may be given by* $t_1 = \cdots = t_r = 0$; *that is, there is a versal equisingular deformation defined by* $f + \sum_{i=r+1}^{m} t_i g_i$.

Using this result, we show a formal versal equisingular deformation of $Y^p + X^q$ is given by $Y^p + X^q + \sum t_{ij} X^i Y^j$, where the sum is over pairs (i, j) with $i \leq q-2$, $j \leq p-2$, and $(i/q) + (i/p) \geq 1$. On the other hand, ES' is in general smaller than ES (Example 6.8); in fact, ES' is the kernel of a smooth morphism $ES \longrightarrow L$, where L may be viewed as the functor of classes of locally trivial deformations of a certain non-reduced projective curve (Theorem 5.7).

Finally, using a theorem on deformations of branched coverings from [8], we prove, in §9,

Theorem. *Let* $F(X, Y, t_1, \cdots, t_n) \in \mathbf{C}[[X, Y, t_1, \cdots, t_n]]$ *define a formal equisingular deformation of* $F(X, Y, 0, \cdots, 0) = f(X, Y)$. *Then the natural map*

$$\pi_1(\operatorname{Spec} \mathbf{C}[[X, Y]] - V(f)) \longrightarrow \pi_1(\operatorname{Spec} \mathbf{C}[[X, Y, t_1, \cdots, t_n]] - V(F))$$

is an isomorphism.

Here, π_1 is the algebraic fundamental group. Though this result is weaker than Zariski's, the proof is purely algebraic, and is established (as are all our results) for any algebraically closed field k of characteristic zero. The proof utilizes the Grothendieck existence theorem and a well-known result on the algebraicity of the ring $k[[X, Y]]/(f)$.

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0. Preliminaries.

(0.1) We shall rely on the techniques of functors of artin rings. Let $\underline{C}$ be the category of artin local k-algebras; denote by $k[\epsilon]$ the object $k[\epsilon]/\epsilon^2$. Every surjection in $\underline{C}$ may be factored by *small* extensions, i.e., ones whose kernel is one-dimensional over k.

(0.2) Let F be a covariant functor from $\underline{C}$ to *sets* with $F(k)$ = one point. If $A' \to A$ and $A'' \to A$ are morphisms in $\underline{C}$, there is a natural map

$$(\alpha): F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

Recall the conditions of Schlessinger [6]:

(H_1) (α) is a surjection when $A' \to A$ is small.

(H_2) (α) is a bijection when $A' = k[\epsilon]$, $A = k$.

(H_3) $F(k[\epsilon])$ (the tangent space) is a finite-dimensional k-vector space.

(H_4) (α) is a bijection when $A' \to A$ is small.

We say that F satisfying (H_1) and (H_2) has a *good deformation theory*; if it satisfies (H_4) as well, we say it has a *very good deformation theory*. If F has a good deformation theory and $A' \to A$ is small, then $F(k[\epsilon])$ is a k-vector space and it acts transitively on the fibres of $F(A') \to F(A)$; if F has a very good theory, the action is free as well. F is *smooth* if $F(A') \to F(A)$ is always surjective.

(0.3) Schlessinger's criterion [6] says that a functor with a good (respectively very good) deformation theory satisfying (H_3) is *versal* (resp. *pro-representable*); that is, there is a complete local k-algebra R, of residue field k, and a morphism $h_R \to F$ which is smooth and a bijection on the tangent spaces (resp. which is a bijection). We refer the reader to [8] for a discussion and examples.

1. Equimultiple liftings.

(1.1) Let $f \in k[[X, Y]] = R$ be a reduced formal power series of degree $r \geq 1$, i.e., $f \in (X, Y)^r - (X, Y)^{r+1}$. Denote by f_r the leading form of f; the linear factors of f_r define the tangent directions of $V(f) \subset \operatorname{Spec} R$. By Hensel's lemma, one knows that f irreducible implies $f_r = (aX + bY)^r$, for some $a, b \in k$; hence f irreducible implies f unitangential (has one tangent direction). Thus, any f may be written as a product of tangential components.

(1.2) We consider several important functors on $\underline{C}$:

$$H(A) = \{\text{ideals } (\bar{f}) \subset R \otimes_k A = R_A \mid \bar{f} \text{ induces } f \text{ in } R\}$$

$$= \text{set of (flat) liftings of } (f) \text{ to } R_A.$$

$$G(A) = \text{groups of } A\text{-automorphisms of } R_A \text{ inducing the identity on } R.$$

H and G have very good deformation theories and are smooth; the quotient D of H by the natural action of G is versal and smooth, and is the functor of deformation classes of the algebra $R/(f)$ [8].

(1.3) Denote by s: $R/(f) \to k$ the residue field map. Recall that a surjection of rings $S \longrightarrow T = S/I$ is *normally flat* if I^p/I^{p+1} is a locally free T-module for all $p \geq 1$. We define a functor E on $\underline{C}$ by

$E(A) = \{((\overline{f}), \overline{s}) | (\overline{f}) \in H(A)$ and

$$\overline{s} : R_A/(\overline{f}) \longrightarrow A \text{ is a normally flat } A\text{-section inducing } s\}.$$

Given $(\overline{f})$, the section $\overline{s}$ is uniquely determined by an ideal $\overline{I} \subset R_A$, with $(\overline{f}) \subset \overline{I}$; $\overline{I}$ may be written $(X - m_1, Y - m_2)$, for some $m_i \in m_A$. The section $\overline{s}$ is called *trivial* if $\overline{I} = (X, Y)$. We shall eventually show E has a very good deformation theory and is smooth.

Lemma 1.4. *Let $A' \longrightarrow A$ be small, with kernel (ϵ), and let $((\overline{f}), \overline{s}) \in E(A)$, where $\overline{s}$ is defined by the ideal $\overline{I} \subset R_A$. Suppose $(f') \in H(A')$ lifts $(\overline{f})$ and $I' \in R_{A'}$ lifts $\overline{I}$, and defines a section s': $R_{A'}/(f') \longrightarrow A'$ (i.e., $(f') \subset I'$). Then $((f'), s') \in E(A')$ if and only if $g \in I'^p$ and $\epsilon g \in (I'^{p+1}, f')$ imply $g \in (I'^{p+1}, f', m_{A'}I'^p)$, for all integers $p \geq 1$.*

Proof. This is essentially Grothendieck's infinitesimal criterion of flatness [1, IV. 5].

(1.5) There is a functorial group action $G \times E \longrightarrow E$ defined by

$$\sigma((\overline{f}), \overline{s}) = (\sigma(\overline{f}), \overline{s} \cdot \sigma^{-1}),$$

with obvious notation; one easily checks that $\sigma(\overline{f})$ is normally flat along $\overline{s} \cdot \sigma^{-1}$, and that the action is indeed a functorial group action. Note that, for every $\eta \in E(A)$, there is a $\sigma \in G(A)$ such that $\sigma(\eta)$ has a trivial section.

The following result shows that E is the functor of equimultiple liftings and sections; we shall subsequently drop the term normally flat.

Proposition 1.6. *For $A \in \underline{C}$, let $m_1, \cdots, m_u$ be a k-basis for m_A, and let $(\overline{f}) \in H(A)$, where $\overline{f} = f + \Sigma\, m_i p_i$, some $p_i \in R$. Then if $\overline{t}$ is the trivial section, $((\overline{f}), \overline{t}) \in E(A)$ if and only if* degree *$p_i \geq r$, for all i. More generally, $((\overline{f}), \overline{s}) \in E(A)$ if and only if $\overline{f} \in \overline{I}^r$, where $\overline{I}$ is the ideal of $\overline{s}$.*

Proof. We proceed by induction on the length of A, the result being trivial for $A = k$. Given $A' \in \underline{C}$ and a basis $m'_1, \cdots, m'_{u+1}$ of $m_{A'}$, we may suppose that $A' \longrightarrow A'/(m'_{u+1})$ is small; rename $\epsilon = m'_{u+1}$.

If $((f'), t') \in E(A')$, then by induction we have degree $p_i \geq r$, $i = 1, 2, \cdots, u$. It remains to show $p_{u+1} \in (X, Y)^r$. Suppose $p_{u+1} \in (X, Y)^d - (X, Y)^{d+1}$, where $d < r$. Then $\epsilon p_{u+1} = f' - \Sigma_{i=1}^u m'_i p_i \in ((X, Y)^r, f') \subset ((X, Y)^{d+1}, f')$, whence (Lemma 1.4) $p_{u+1} \in ((X, Y)^{d+1}, f', m_{A'})$. Reducing mod $m_{A'}$ yields $p_{u+1} \in (X, Y)^{d+1}$, a contradiction.

Conversely, suppose degree $p_i \geq r$, for all i; since the result is true mod (ϵ), by Lemma 1.4 it suffices to show that if $g \in (X, Y)^p$ and $\epsilon g \in ((X, Y)^{p+1}, f')$, then $g \in ((X, Y)^{p+1}, f', m_{A'})$.

If $p \leq r-1$, then $g \in (X, Y)^p$, $\epsilon g \in ((X, Y)^{p+1}, f') = (X, Y)^{p+1}$ already implies that the homogeneous component of degree p in g is killed by ϵ; hence, by flatness, it belongs to $R \otimes m_{A'}$.

If $p = r$, let f'_r be the homogeneous component of degree r of f'; then $((X, Y)^{r+1}, f') = ((X, Y)^{r+1}, f'_r)$. Write $g = g_r + h$, where h denotes terms of degree $> r$. Now, $\epsilon g \in ((X, Y)^{r+1}, f')$ implies $\epsilon g_r = \alpha f'_r$; since f'_r is not a zero-divisor, we have $\alpha = a\epsilon$. But $\epsilon(g_r - af'_r) = 0$ implies $g_r - af'_r \in R \otimes m_{A'}$, so $g \in ((X, Y)^{r+1}, f', m_{A'})$.

If $p > r$, assume $g \in (X, Y)^p$ and $\epsilon g = Af' + h$, where $\deg h > p$. With f'_r as above, we note it is not a zero-divisor, whence the leading term A_{p-r} of A is of degree $p - r$. If g_p is the p-component of g, $\epsilon g_p = A_{p-r}f'_r$ implies as above $A_{p-r} = \epsilon a_{p-r}$, so $g_p - a_{p-r}f'_r \in R \otimes m_{A'}$. Since we may suppose $a_{p-r} \in (X, Y)^{p-r}$, we have $g - a_{p-r}f' \in ((X, Y)^{p+1}, m_{A'})$, as desired. This completes the proof of the first claim.

The final remark follows easily by choosing a $\sigma \in G(A)$ such that $\sigma(\overline{T}) = (X, Y)$.

Proposition 1.7. *E has a very good deformation theory and is smooth. Further, the tangent space is*

$$\{((f + \epsilon g), (X - a\epsilon, Y - b\epsilon)) | g + af_X + bf_Y \in (X, Y)^r\}.$$

Proof. The proof follows easily by trivializing the sections involved, using the fact that H and G have very good deformation theories and are smooth, and applying Proposition 1.6. For instance, suppose $\beta = ((f + \epsilon g), (X - a\epsilon, Y - b\epsilon))$ is in $E(k[\epsilon])$, with $a, b \in k$. Let $\sigma \in G(k[\epsilon])$ send X to $X + a\epsilon$, Y to $Y + b\epsilon$. Then $\sigma(\beta) = (\sigma(f + \epsilon g), (X, Y)) = ((f + \epsilon(af_X + bf_Y + g)), (X, Y))$, so Proposition 1.6 yields the result.

(1.8) Let $\mu: E \longrightarrow H$ be the forgetful morphism which neglects the section. The injectivity of μ is equivalent to equimultiple sections being uniquely determined by the lifting.

Proposition 1.9. *$\mu: E \longrightarrow H$ is injective if and only if f is not unitangential, i.e., the leading form f_r of f has distinct factors.*

Proof. By [8, 1.1.4], it suffices to consider μ_ϵ. If f is unitangential, we may assume after change of coordinates that $f_r = Y^r$. Then one easily checks that if $\overline{s}$ is defined by $(X - \epsilon, Y)$, then $((f + \epsilon f_X), \overline{T})$ and $((f + \epsilon f_X), \overline{s})$ are in $E(k[\epsilon])$. Thus, μ_ϵ is not injective.

If f is not unitangential, we can change coordinates and write $f_r = Y^m g$, where Y does not divide g. If $((f), (X - a\epsilon, Y - b\epsilon))$ is in the kernel of μ_ϵ, then $af_X + bf_Y \in (X, Y)^r$, whence $aY^m g_X + b(mY^{m-1}g + Y^m g_Y) = 0$. Thus, Y divides bg, so $b = a = 0$, and μ_ϵ is injective.

Proposition 1.10. *Equimultiple liftings preserve tangential components. That is, if* $(f) = (g_1)\cdots(g_t)$ *is a factorization into tangential components, there exists a natural inclusion*

$$E_{(f)} \to \prod_{i=1}^{t} E_{(g_i)},$$

with image all tuples with the same equimultiple section.

Proof. Let $((\bar f), \bar s) \in E_{(f)}(A)$; it suffices to assume $\bar s$ is trivial. We show we can write $(\bar f)$ uniquely as $(\bar g_1)\cdots(\bar g_t)$, where the $(\bar g_i)$ are equimultiple liftings of the (g_i) over the trivial section. We first write $(f) = (g_1)(G)$, and factor $(\bar f) = (\bar g_1)(\bar G)$. This may be done by an induction on the length of A using a standard formal argument and the key fact that $(X, Y)^r \subset (g_1, G)$. The uniqueness follows because g_1 and G are relatively prime in R. Factoring $(\bar G)$ in this fashion yields the desired morphism. The final remark follows from Proposition 1.6.

Remarks. (1.11) Equimultiple liftings do not, in general, preserve irreducible components (i.e., branches); for instance, $(Y(Y + X^3) + \epsilon X^2)$ is equimultiple along the trivial section, but does not factor $(Y + \epsilon\alpha)(Y + X^3 + \epsilon\beta)$. For, that would imply $(\alpha + \beta)Y + \alpha X^3 = X^2$, whence Y divides $X^2(1 - \alpha X)$, an impossibility.

(1.12) If $f \in k[[X_1, \cdots, X_n]]$ is reduced, then one can define the functor E as above, and all the results in (1.2) through (1.8) carry over verbatim. Note the leading form f_r will not in general factor into linear terms. However, we generalize Proposition 1.9 as follows: μ: $E \to H$ is not injective if and only if f_r is, after a linear change of coordinates, a form in $(n - 1)$ variables.

2. Higher order equimultiple functors.

(2.1) We now wish to consider liftings that are not only equimultiple, but equimultiple in the "first infinitesimal neighborhoods" as well. Suppose that $((\bar f), \bar s) \in E(A)$. Let $\bar p$: $B_{\bar s} \to \operatorname{Spec} R_A$ be the blowing-up of Spec R_A along the sheaf of ideals $\bar I$ defined by the section $\bar s$. Let $\bar I_{\bar s} = \bar p^{-1}(V(\bar I))$ denote the exceptional fibre, a divisor in $B_{\bar s}$, and let the total transform of $V(\bar f)$ be $\bar p^{-1}(V(\bar f)) = T(\bar f)$.

Lemma 2.2. *The divisor* $T(\bar f) - r\bar I_{\bar s}$ *is effective and flat over* A, *called the proper transform* $P(\bar f)$.

Proof. If $\bar I = (X + m_1, Y + m_2)$, then $B_{\bar s}$ is the union of the affines Spec $R_A[X + m_1/Y + m_2]$ and Spec $R_A[Y + m_2/X + m_1]$. In the first affine, $\bar I_{\bar s}$ has local equation $Y + m_2$, while $(\bar f)$ defines $T(\bar f)$. By Proposition 1.6, $\bar f \in \bar I^r$, whence $\bar f/(Y + m_2)^r \in R_A[X + m_1/Y + m_2]$. The result now follows easily.

(2.3) We define the *reduced total transform* $R(\bar f)$ to be the divisor $T(\bar f) - (r - 1)\bar I_{\bar s}$. Looking at the closed fibre, $R(f)$ has singularities precisely at $P(f) \cap I_s$; the number of such points is t, the number of tangent directions of (f). $P(f)$

is the union of the proper transforms of the tangential components. We define

$$E^2(A) = \{((\bar{f}), \bar{s}; \bar{s}_1, \cdots, \bar{s}_t) | ((\bar{f}), \bar{s}) \in E(A) \text{ and the } \bar{s}_i : \operatorname{Spec} A \to B_{\bar{s}} \text{ are } A\text{-sections inducing } \bar{s} \ (\bar{s} = \bar{p} \cdot \bar{s}_i) \text{ and inducing equimultiple sections of } R(\bar{f}) \text{ in } \hat{\mathcal{O}}_{p_i}, \text{ for } p_i \text{ a singular point of } R(\bar{f})\}.$$

The supports of the s_i's are called the *first infinitesimal neighborhoods* of the singularity of (f). As before, there is a natural group action $G \times E^2 \to E^2$ compatible with the forgetful morphism $E^2 \to E$. If $\sigma \in G(A)$, $\alpha \in E^2(A)$, then $\sigma(\alpha)$ consists of the lifting $\sigma(\bar{f})$, an equimultiple section $\bar{u} = \bar{s} \cdot \sigma^{-1}$, and equimultiple sections $\bar{s}_i \cdot \sigma^{-1}$ in $B_{\bar{u}}$, where we view σ also as an isomorphism $B_{\bar{u}} \xrightarrow{\sim} B_{\bar{s}}$.

Lemma 2.4. *Suppose (f) is unitangential. If $((\bar{f}), \bar{s}; \bar{s}_1) \in E^2(A)$, then there is a $\sigma \in G(A)$ such that $\sigma((\bar{f}), \bar{s}; \bar{s}_1) = (\sigma(\bar{f}), \bar{t}; \bar{t}_1)$, where $\bar{t}$ and $\bar{t}_1$ are trivial sections.*

Proof. First, there is a $\sigma \in G(A)$ such that $\bar{s} \cdot \sigma^{-1}$ is trivial; so, we may assume $\bar{s} = \bar{t}$. Suppose $f_r = Y^r$, and let $k[[X, Y, T]]/(XT - Y)$ denote the completion of the local ring of the one point of B_t at which $R(f)$ has a singularity. By assumption, $R(\bar{f})$ is equimultiple along some section of $A[[X, T]]$ defined by $(X, T - m)$, for some $m \in m_A$, since it lies over the trivial section $\bar{t}$ of $A[[X, Y]]$. But if $\sigma \in G(A)$ sends X to X and Y to $Y + mX$, then σ is the desired automorphism.

Proposition 2.5. *E^2 has a very good deformation theory and is smooth.*

Proof. Let $A' \to A$ be small, $A'' \to A$ arbitrary, $A^* = A' \times_A A''$, and consider

$$(\alpha): E^2(A^*) \to E^2(A') \times_{E^2(A)} E^2(A'').$$

If $((f'), s'; s_1', \cdots, s_t') \in E^2(A')$, $((f''), s''; s_1'', \cdots, s_t'') \in E^2(A'')$ have the same image in $E^2(A)$, then (Proposition 1.7) there is a unique $((f^*), s^*) \in E(A^*)$ lying over the liftings and first equimultiple sections. There are natural isomorphisms $B_{s^*} \xrightarrow{\sim} B_{s'} \times_{B_{\bar{s}}} B_{s''}$ and $R(f^*) \xrightarrow{\sim} R(f') \times_{R(\bar{f})} R(f'')$. Therefore, there are unique sections s_i^* of B_{s^*} lifting the s_i' and s_i''. A local argument (using Proposition 1.7) shows $((f^*), s^*; s_1^*, \cdots, s_t^*)$ is in $E^2(A^*)$, whence (α) is surjective. The injectivity is similar.

For smoothness, let $A' \to A$ be small, and consider $((\bar{f}), \bar{s}; \bar{s}_1, \cdots, \bar{s}_t)$ in $E^2(A)$. By Proposition 1.9, we may write $(\bar{f}) = (\bar{f}_1) \cdots (\bar{f}_t)$. But Lemma 2.4 implies we may lift each $((\bar{f}_i), \bar{s}; \bar{s}_i)$ to $((f_i'), s'; s_i') \in E^2(A')$ (with the same section s' for all i), whence (Proposition 1.10) $((f_1' \cdots f_t'), s'; s_1', \cdots, s_t')$ is in $E^2(A')$ and lifts the original element.

Remark 2.6. The equimultiple sections in $B_{\overline{s}}$ need not be unique, as seen from the example $((Y^2 + X^3 + \epsilon XY), \overline{t})$.

(2.7) It is now clear how to define inductively the nth equimultiple functor E^n on $\underline{C}$. Suppose E^j has been defined for $2 \leq j \leq n-1$, with natural maps $E^j \longrightarrow E^{j-1}$. For an element α of $E^j(A)$, let $\alpha^* = ((\overline{f}), \overline{s}; \overline{s}_1, \cdots, \overline{s}_t)$ denote the image of α in $E^2(A)$; also, let B_α denote the Spec A-scheme obtained by blowing up Spec R_A successively along the A-sections of α. Let $(\overline{f}_i)$ be the ideal of $R(\overline{f}) \subset B_{\overline{s}}$ at the singular point $\overline{p}_i$ defined by $\overline{s}_i$. There are ordered collections α_i of sections in α such that $((\overline{f}_i), \alpha_i) \in E^{j-1}_{(\overline{f}_i)}(A)$; this makes sense, because for instance A-sections of $B_{\overline{s};\overline{s}_i}$ inducing $\overline{s}_i$ on $B_{\overline{s}}$ yield A-sections of the blowing-up of Spec $\hat{\mathcal{O}}_{\overline{p}_i}$ along $\overline{s}_i$.

An element of $E^n(A)$ is then, by definition, $(\alpha; \overline{u}_1, \cdots, \overline{u}_m)$, where

(a) $\alpha \in E^{n-1}(A)$,

(b) $\overline{u}_i$ is an A-section of B_α,

(c) every $\overline{u}_i$ lies over a section $\overline{s}_j$ of $B_{\overline{s}}$ in α^*; letting $\overline{u}_{j,1}, \cdots, \overline{u}_{j,q}$ denote all such sections, we have, for all j,

$$(\overline{f}_j, \alpha_j; \overline{u}_{j,1}, \cdots, \overline{u}_{j,q}) \in E^{n-1}_{(\overline{f}_j)}(A).$$

The supports of the sections in the data of an element of $E^n(A)$ are called the *(infinitesimal) neighborhoods* of $(\overline{f})$. We may further speak of the *reduced total transform* of $(\overline{f})$ in B_α; as above, it is a lifting of a reduced divisor supported on the total transform of (f).

Proposition 2.8. *E^n has a very good deformation theory for all n.*

Proof. Using induction on n, one proceeds exactly as in Proposition 2.5. The point is that because E^{n-1} satisfies (H_4), all sections of $E^n(A' \times_A A'')$ are equimultiple and uniquely determined by $E^n(A')$ and $E^n(A'')$, except possibly in the last blowing-up. But one then applies Proposition 2.5.

Remarks. (2.9) The usual functorial group action $G \times E^n \longrightarrow E^n$ is defined for every n, and commutes with the forgetful maps $E^n \longrightarrow E^{n-1}$.

(2.10) Presumably all functors E^n are smooth. However, Lemma 2.4 does not generalize; we may not be able to trivialize simultaneously all sections of an element of $E^n(A)$, even if f is irreducible.

(2.11) Suppose (f) is a curve for which $R(f) \subset B_s$ has an ordinary double point at a section s_i. Then if $((\overline{f}), \overline{s}) \in E(A)$, then there exists a unique A-section $\overline{s}_i$ of $B_{\overline{s}}$ lifting s_i along which $R(\overline{f})$ is equimultiple (by 1.9).

(2.12) If (f) defines an ordinary multiple point (i.e., f_r has r distinct factors), then one computes that $E^2 \xrightarrow{\sim} E$. (Use the previous remark.)

(2.13) Our definition of E^n requires that all sections be supported on *singular* points of $R(f)$ (i.e., points on the proper transform).

3. The equisingular functor ES.

(3.1) It is a well-known fact in the theory of curves that after blowing up Spec $k[[X, Y]]$ and its transforms sufficiently many times, the reduced total transform of (f) has only ordinary double points as singularities.

Theorem 3.2. *There exists an integer N such that the natural maps $E^{N+i} \to E^{N+i-1}$ are bijective, for all $i > 0$. For such an N, the natural map $E^N \to H$ is injective, unless f is a regular parameter.*

Proof. Let N be the smallest positive integer so that in any sequence of the form $\{(f), (f_1), (f_2), \cdots\}$, where (f_{i+1}) is the ideal in the complete local ring at a singular point of the reduced total transfom of a blowing-up of (f_i) [10, p. 512], we have that (f_j) defines an ordinary double point for $j \geq N-1$. It suffices to show $E^{N+1} \to E^N$ is a-bijection. The question being one of existence and uniqueness of equimultiple sections, it suffices to show that for an ordinary double point, $E^2 \xrightarrow{\sim} E$. But this fact was mentioned in Remark 2.12.

To show $E^N \to H$ is injective, it suffices to check on tangent spaces. Further, by induction on N, we have only to check that if $((f), \overline{s}) \in E(k[\epsilon])$ is in the image of $E^N(k[\epsilon]) \to E(k[\epsilon])$, then $\overline{s}$ is the trivial section. The result is obvious by Proposition 1.9 if f is not unitangential.

Thus, suppose f has tangent $Y = 0$ and multiplicity $r > 1$; since we are only concerned with (f), the Weierstrass preparation theorem allows us to assume $f(X, Y) = Y^r + \Sigma_{i=0}^{r-1} d_i(X)Y^i$, with $d_i \in k[[X]]$. After a further coordinate change, $d_{r-1}(X) = 0$, so we write $f(X, Y) = Y^r + \Sigma_{i\in A} d_i(X)Y^i$, where $A \subset \{0, 1, \cdots, r-2\}$, and $d_i(X)$ has leading form $c_iX^{e_i}$, with $e_i > r - i$.

With $((f), \overline{s})$ as above, $\overline{s}$ defined by $(X - a\epsilon, Y)$, we prove $a = 0$. Let $\sigma \in G(k[\epsilon])$ send X to $X + a\epsilon$, Y to Y, whence $(\sigma(f), \overline{t})$ is in the image of $E^N(k[\epsilon])$. Note

$$\sigma(f) = Y^r + \sum_{i\in A} (d_i(X) + a\epsilon d_i'(X))Y^i.$$

Since $(\sigma(f), \overline{t})$ is in the image of $E^2(k[\epsilon])$, the reduced total transform (with coordinate $T_1 = Y/X$) in $B_{\overline{t}}$ is given by

$$\overline{f}_1(X, T_1) = X(T_1^r + \sum (d_i(X) + a\epsilon d_i'(X))X^{i-r}T_1^i),$$

and it is equimultiple along a section of the form $(X, T_1 - b\epsilon)$. The multiplicity q_1 of f_1 is $\min_{i\epsilon A}\{r+1, e_i + 2i - r + 1\}$. We distinguish two cases.

If $q_1 < r + 1$, let j be the largest integer in A such that $e_j + 2j - r + 1 = q_1$. Then, by Proposition 1.7,

$$a \sum d_i'(X)X^{i-r+1}T_1^i + b\frac{\partial}{\partial T_1}(f_1) \in (X, T_1)^{q_1},$$

whence $ac_je_j = 0$. Since $e_j > r - j > 0$, $a = 0$, and we are done in this case.

If $q_1 = r + 1$, then, as above, the series

$$b(rXT_1^{r-1} + \sum ic_iX^{i+e_i-r+1}T_1^{i-1}) + a\sum c_ie_iX^{e_i+i-r}T_1^i$$

is in $(X, T_1)^{r+1}$. Since every $i \leq r - 2$, the term $brXT_1^{r-1} = 0$, whence $b = 0$, so that $a = 0$ or $e_i + 2i - r + 1 > r + 1$, for all i. In the second case, $(\overline{f}_1)$ is equimultiple along the trivial section only; since $((\overline{f}), \overline{t}; \overline{t})$ is in the image of $E^3(k[\epsilon])$, we blow up again, and have (with $T_2 = T_1/X$)

$$\overline{f}_2(X, T_2) = X(T_2^r + \sum (d_i(X) + a\epsilon d_i'(X))X^{2i-2r}T_2^i)$$

is equimultiple along a section $(X, T_2 - b\epsilon)$. Proceeding as above, we find that either $a = 0$ or else $b = 0$ and $r + 1 < e_i + 3i - 2r + 1$, for all $i \in A$. Repeating this procedure n times, we have either $a = 0$ or the reduced total transform is equimultiple along the trivial section and $r + 1 < e_i + (n+1)i - nr + 1$, for all $i \in A$. The second case implies $r < (e_i/n + 1)) + i$; but $A \neq \emptyset$ and $i < r$, so this is a contradiction for n sufficiently large. Therefore, $a = 0$, and the theorem is proved.

(3.3) We can now make the following definition. With N as in Theorem 3.2, we call E^N the *functor of equisingular liftings*, and denote it by ES. Since the natural map $ES \longrightarrow H$ is an inclusion, we need not mention the sections; although they are part of the data, they are uniquely determined once the lifting is given. Recall (Proposition 1.9) that the functor of equimultiple liftings does not have a good deformation theory—it must be equipped with sections. Note also that in considering equisingular liftings, one need not blow up every section as in each E^i; one only must blow up sections supported on nonordinary double points, and we can consider any resolution of the singularity $B \longrightarrow \operatorname{Spec} R$.

We shall frequently say "$(\overline{f})$ is equisingular along $\overline{s}$" to mean $((\overline{f}), \overline{s})$ is in the image of $E^N \longrightarrow E$.

Proposition 3.4. *Suppose* $(f) = (g_1)(g_2)$. *Then there exists a natural injection* $ES_{(f)} \longrightarrow ES_{(g_1)} \times ES_{(g_2)}$. *In particular, equisingular liftings preserve branches.*

Proof. Let $(\overline{f}) \in ES_{(f)}(A)$; we claim $(\overline{f})$ factors uniquely as a product $(\overline{g}_1)(\overline{g}_2)$ of equisingular liftings of g_1 and g_2. Let $\overline{p}\colon \overline{B} \longrightarrow \operatorname{Spec} R_A$ be a resolution of the singularity of $(\overline{f})$; that is, the reduced total transform $(\overline{f})$ has only ordinary double points as singularities, all trivially deformed. Let $\overline{I}$ be the ideal sheaf of $\overline{X} \subset \overline{B}$, the proper transform of $(\overline{f})$. It is well known that $\overline{p}_*(\mathcal{O}_{\overline{B}}) = \Gamma(\mathcal{O}_{\overline{B}}) = R_A$; $R^1\overline{p}_*\mathcal{O}_{\overline{B}} = 0$; and $\overline{p}_*\overline{p}^*(\overline{f}) = (\overline{f})$. We shall show below that $\overline{p}_*\overline{I} = (\overline{f})$; assume it is true.

Reducing mod m_A, we have the resolution $p\colon B \longrightarrow \operatorname{Spec} R$ and the sheaf of ideals I of X on B. Since $(f) = (g_1)(g_2)$, we may write $I = I_1 \cap I_2$, where I_i is the ideal sheaf of the proper transform X_i of (g_i). But $X_1 \cap X_2 = \emptyset$, so we may write $\bar{I} = \bar{I}_1 \cap \bar{I}_2$ on $\bar{B}$, where $\bar{I}_i$ is a uniquely determined lifting of I_i. Thus,

$$\bar{p}_*\bar{I} = (\bar{f}) = \bar{p}_*(\bar{I}_1 \cap \bar{I}_2) = \bar{p}_*(\bar{I}_1) \cap \bar{p}_*(\bar{I}_2).$$

We claim $\bar{p}_*(\bar{I}_i)$ is a flat R_A-module, hence a principal ideal; but using the infinitesimal criterion of flatness, one sees that it suffices to show $\bar{p}_*(\bar{I}_i)$ is A-flat. On $\bar{B}$, we have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \bar{I}_1 \cap \bar{I}_2 & \to & \mathcal{O}_{\bar{B}} & \to & \mathcal{O}_{\bar{X}_1} \oplus \mathcal{O}_{\bar{X}_2} & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \bar{I}_i & \longrightarrow & \mathcal{O}_{\bar{B}} & \to & \mathcal{O}_{\bar{X}_i} & \longrightarrow & 0. \end{array}$$

Taking $\bar{p}_*$ yields

$$\begin{array}{ccccccccccc} 0 & \to & \Gamma(\bar{I}_1 \cap \bar{I}_2) & \to & \Gamma(\mathcal{O}_{\bar{B}}) & \to & \Gamma(\mathcal{O}_{\bar{X}_1}) \oplus \Gamma(\mathcal{O}_{\bar{X}_2}) & \to & R^1\bar{p}_*(\bar{I}_1 \cap \bar{I}_2) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(\bar{I}_i) & \longrightarrow & \Gamma(\mathcal{O}_{\bar{B}}) & \longrightarrow & \Gamma(\mathcal{O}_{\bar{X}_i}) & \longrightarrow & R^1\bar{p}_*\bar{I}_i & \longrightarrow & 0 \end{array}$$

Since the third vertical map is a projection of $\Gamma(\mathcal{O}_{\bar{B}})$-modules and hence admits a section, a diagram chase yields an isomorphism $R^1\bar{p}_*(\bar{I}_1 \cap \bar{I}_2) \xrightarrow{\sim} R^1\bar{p}_*\bar{I}_1 \oplus R^1\bar{p}_*\bar{I}_2$. But since the two middle terms of either row are A-flat (X_i is affine), the first module in a row is A-flat if and only if the fourth is. But $\Gamma(\bar{I}_1 \cap \bar{I}_2) = (\bar{f})$ is flat, hence so is $R^1\bar{p}_*(\bar{I}_1 \cap \bar{I}_2)$ and each direct summand $R^1\bar{p}_*\bar{I}_i$. Therefore, $\bar{p}_*\bar{I}_i$ is A-flat, hence is a principal ideal $(\bar{g}_i) \subset R_A$ lifting (g_i). By construction, $(\bar{g}_i) \in ES_{(g_i)}(A)$. We leave the rest of the details to the reader.

It remains to show $\bar{p}_*\bar{I} = (\bar{f})$. We immediately reduce to the case where $\bar{p}: \bar{B} \longrightarrow \operatorname{Spec} R_A$ is one blowing-up along the trivial section. If $\bar{J}$ is the ideal sheaf of the exceptional fibre $\bar{E}$, then we have $\bar{p}^*(\bar{f}) = \bar{J}^r \otimes \bar{I}$ (r = degree of f). We claim

$$\Gamma(\bar{J}^i \otimes \bar{I}) = \Gamma(\bar{J}^{i+1} \otimes \bar{I}), \qquad i = 0, 1, \cdots, r-1.$$

Due to the exact sequence $0 \to \bar{J}^{i+1} \otimes \bar{I} \to \bar{J}^i \otimes \bar{I} \to \bar{J}^i \otimes \bar{I} \otimes \mathcal{O}_{\bar{E}} \to 0$, it suffices to show $\Gamma(\bar{J}^i \otimes \bar{I} \otimes \mathcal{O}_{\bar{E}}) = 0$. However, since $\bar{E}$ is isomorphic to $\mathbf{P}^1 \times \operatorname{Spec} A$ and $H^1(\mathcal{O}_{\mathbf{P}^1}) = 0$, we have that $\bar{J}^i \otimes \bar{I} \otimes \mathcal{O}_{\bar{E}}$ is a product defomation of the invertible sheaf $J^i \otimes I \otimes \mathcal{O}_E$ on $E \subset B$ [1, III. 7]. Now $I \otimes \mathcal{O}_E \cong \mathcal{O}_E(-r)$, while $J \otimes \mathcal{O}_E = \mathcal{O}_E(1)$ = conormal sheaf of E in B; thus, $J^i \otimes I \otimes \mathcal{O}_E \cong \mathcal{O}_E(-r+i)$. The result now follows because $H^0(E, \mathcal{O}_E(-r+i)) = 0$, for all $i \leq r-1$.

Remarks. (3.5) The above result guarantees that equisingular liftings induce "tangentially stable pairings" of the branches (part of Zariski's definition in [10]).

(3.6) It is not true that the product of two equisingular liftings is equisingular, unless one knows all sections in all neighborhoods of the liftings are disjoint or coincide. For example, $Y(Y + X^2 + \epsilon X)$ is not equisingular. However, we have the

Proposition 3.7. *Let* $(f) = (f_1)\cdots(f_t)$ *be a factorization into tangential components. Then the inclusion* $ES_{(f)} \longrightarrow \Pi\, ES_{(f_i)}$ *has as image all products of equisingular liftings with the same equisingular section in* Spec R_A, $A \in \underline{C}$.

4. The smoothness of ES.

(4.1) Let (f) be a reduced curve, and suppose $p: B \longrightarrow \operatorname{Spec} R$ resolves the singularity (i.e., $R(f)$ has only nodes as singularities). As usual, we require that all blown-up sections lie on the proper transforms of (f) (2.13). Let $E = p^{-1}\{m_R\}$ be the reduced exceptional subscheme; it is a tree of $\mathbf{P}^1$'s. By [EGA, II. 4.6.13], there is an effective divisor Z of B, with support E, such that $\mathcal{O}(-Z)$ is ample for p; thus, we may assume $H^1(T_B \otimes \mathcal{O}(-Z)) = 0$, where we denote by T_B the dual of $\Omega^1_{B/k}$. Let L_Z be the functor on $\underline{C}$ of classes of locally trivial deformations of Z over Spec k. From [8], it follows that L_Z is versal and smooth (since $H^2(T_Z) = 0$), with tangent space $H^1(T_Z)$; further, local triviality is equivalent to formal triviality [8, Proposition 2.1.5]. In this section, we prove the following

Theorem 4.2. *There is a natural smooth morphism* $\gamma: ES \longrightarrow L_Z$. *In particular,* ES *is smooth.*

(4.3) We start with a description of L_E. Let $E_1, \cdots, E_s$ be the components of E, $T_i = E_i \cap (\bigcup_{j\neq i} E_j)$, and t_i = number of elements in T_i.

Proposition 4.4. *There is a natural isomorphism* $T_E \xrightarrow{\sim} \bigoplus p_{i*}\mathcal{O}_{E_i}(2 - t_i)$, *where* $p_i: E_i \longrightarrow E$ *is the inclusion. In particular,*

$$\dim H^1(T_E) = \sum_i \max(0,\, t_i - 3).$$

Proof. The natural map $T_E \longrightarrow p_{i*} T_{E_i}$ is easily seen to land in $p_{i*}(T_{E_i} \otimes I_i)$, where I_i is the sheaf of ideals of the subscheme $T_i \subset E_i$ (since derivations of E must vanish on the inverse images in E_i of the singular points of E). A local argument shows the induced map $T_E \longrightarrow \bigoplus p_{i*}(T_{E_i} \otimes I_i)$ is an isomorphism; for, if E is given formally at a singular point by $\mathcal{O} = k[[X, Y]]/XY$, then the map

$$\left\{a\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y} \in \mathcal{O}\frac{\partial}{\partial X} \oplus \mathcal{O}\frac{\partial}{\partial Y}\,\middle|\, aY + bX = 0\right\} \longrightarrow Xk[[X]]\frac{\partial}{\partial X} \oplus Yk[[Y]]\frac{\partial}{\partial Y}$$

is an isomorphism. But $T_{E_i} = \mathcal{O}_{E_i}(2)$, $I_i = \mathcal{O}_{E_i}(-t_i)$, whence the result.

(4.5) Unfortunately, T_Z does not in general admit as simple a description. On the other hand, if $N_Z^0 = \operatorname{Coker}(T_Z \longrightarrow T_B \otimes \mathcal{O}_Z)$ (the "locally trivial" normal sheaf, cf. [8, 3.2.3]), the natural map $N_Z^0 \longrightarrow N_E^0$ is an isomorphism (a formal check).

On the other hand, an argument as in Proposition 4.4 shows $N_E^0 \simeq \bigoplus p_{i*} N_{E_i}$, where N_{E_i} is the normal bundle of E_i in B. Since $H^0(N_{E_i}) = 0$, there is an exact sequence

$$0 \to H^1(T_Z) \to H^1(T_B \otimes \mathcal{O}_Z) \to H^1(N_E^0) \to 0.$$

(4.6) We want to introduce an auxiliary functor L. Note B was obtained from Spec R by a sequence of blowings-up $B = B_M \to B_{M-1} \to \cdots \to \operatorname{Spec} R$, where $B_{i+1} \to B_i$ is centered at a section σ_i of the exceptional fibre of $p_i: B_i \to \operatorname{Spec} R$. Although B is regular and excellent, it is not smooth over k in the usual sense (nor even formally smooth in the sense of Lichtenbaum-Schlessinger [3]), since the local rings have mixed "algebraic" and "continuous" structure over k; thus, it is not clear that all deformations of B are locally trivial in the Zariski topology.

If Spec S = Spec $k[X, Y]$, then it is clear there is a sequence of blowings-up $C = C_M \to C_{M-1} \to \cdots \to \operatorname{Spec} S$ inducing the resolution $B \to \operatorname{Spec} R$ via flat base change. Of course, C is now a nonsingular variety over k. Denote also by E the exceptional subscheme of $r: C \to \operatorname{Spec} S$, and by σ_i the section of C_i to be blown up. Let $L(A)$ be the set of classes of deformations $\overline{C} \to \operatorname{Spec} A$ which may be obtained by a sequence of blowings-up $\overline{C} \to \overline{C}_{M-1} \to \cdots \to \operatorname{Spec} S_A$, where $\overline{C}_{i+1} \to \overline{C}_i$ is centered at a normally flat A-section $\overline{\sigma}_i$ of the exceptional subscheme of $\overline{C}_i \to \operatorname{Spec} S_A$ lifting σ_i and inducing $\overline{\sigma}_{i-1}$.

Lemma 4.7. *L is versal and smooth, and is the functor of those deformation classes of C to which E admits a locally trivial lifting.*

Proof. Since T_C is coherent, $H^1(T_C) = R^1 r_* T_C$ is a finite $k[X, Y]$-module (r is proper); since it is supported on $V(X, Y)$, $H^1(T_C)$ is finite over k, whence the functor of deformation classes of C is versal [6]. Since L is contained in this functor, it suffices to show L satisfies (H_1). One can do this once we show that if $A' \to A$ is small, two Spec A'-morphisms $C' \to \operatorname{Spec} S_{A'}$, inducing the same map over Spec S_A, differ by an automorphism of Spec $S_{A'}$. But this follows from [1, III. 5.4], because $H^0(C, r^* T_{\operatorname{Spec} S}) = T_{\operatorname{Spec} S} = \operatorname{Der}_k(S, S)$, since $H^0(C, \mathcal{O}_C) = S$. The smoothness of L is obvious.

If $f: X \to Y$ is a blowing-up of a (closed) point P of a nonsingular variety Y of dimension ≥ 2, then one establishes easily the exact sequence

$$0 \to f_* T_X \to T_Y \to N_P \to 0,$$

where N_P is the normal sheaf of P in Y. There is a morphism of the functor of deformation classes of Y and liftings of P into the functor of deformation classes of X; the exact sequence

$$H^0(N_P) \to H^1(T_X) \to H^1(T_Y) \to 0$$

shows the morphism is surjective on the tangent spaces, whence the usual argument shows it is surjective. We conclude that every deformation class $[\overline{C}]$ of C is obtained from Spec S_A by a sequence of blowings-up.

Any element of $L(A)$ certainly induces a locally trivial deformation of E. For the converse, one uses induction and reduces to proving the following: if $(g) \subset k[[X, Y]]$ is (X) or (XY), $(\overline{g}) \subset R_A$ a lifting, and $\overline{s}$ an A-section of R_A, then the total transform $T(\overline{g}) \subset B_{\overline{s}}$ (2.1) is a locally trivial deformation of $T(g)$ if and only if $((\overline{g}), \overline{s}) \in E_{(g)}(A)$. The proof is an easy computation.

(4.8) From obstruction theory and the map $T_C \to T_C \otimes \mathcal{O}_E \to N_E^0$, one has that

$$L(k[\epsilon]) = \operatorname{Ker}(H^1(T_C) \to H^1(N_E^0)).$$

On the other hand, Z was chosen so that $H^1(T_C) \xrightarrow{\sim} H^1(T_C \otimes \mathcal{O}_Z)$, whence, as in (4.5), we have $L(k[\epsilon]) = H^1(T_Z)$. In fact, we have

Proposition 4.9. *There is a natural isomorphism* $\beta\colon L \xrightarrow{\sim} L_Z$.

Proof. Since L is smooth and β_ϵ is surjective (4.8), it follows that β is surjective [8, 1.1.4].

To show β is injective, suppose $C_i' \to \operatorname{Spec} A'$ induce elements of $L(A')$ with the same image in $L_Z(A')$; we may suppose they are isomorphic over Spec A, where $A' \to A$ is small. Let $Z_i' \subset C_i'$ be the induced subschemes, and consider the diagram

$$\begin{tikzcd}
Z_1' \arrow[d, phantom, "\cap"] \arrow[dr, dashed] & \\
C_1' \arrow[r, dashed] \arrow[dr] & C_2' \arrow[d] \\
& \operatorname{Spec} A'
\end{tikzcd}$$

The obstruction to lifting the isomorphism over A to an A'-homomorphism $C_1' \to C_2'$ lies in $H^1(T_C)$ [1, III. 5.4], while the obstruction to lifting to a map $Z_1' \to C_2'$ is easily seen to lie in $H^1(T_C \otimes \mathcal{O}_Z)$. But by assumption, $H^1(T_C) \xrightarrow{\sim} H^1(T_C \otimes \mathcal{O}_Z)$; thus, if $Z_1' \xrightarrow{\sim} Z_2'$, the obstruction vanishes, so C_1' and C_2' are necessarily A'-isomorphic.

Proof of Theorem 4.2. Suppose $Z = \sum r_i E_i$. If $(\overline{f}) \in ES(A)$, one blows up successively the equisingular sections of $(\overline{f})$ and its transforms to get a map $\overline{p}$: $\overline{B} \to \operatorname{Spec} R_A$; such a process is well defined, since all sections are unique ((2.13) and Theorem 3.2). Let $\gamma_A((\overline{f}))$ be the deformation class of $\overline{Z} = \sum r_i \overline{E}_i$, where $\overline{E}_i$ is the unique lifting of E_i to $\overline{B}$. As in (4.6), $\overline{p}$ is induced by $\overline{C} \to \operatorname{Spec} S_A$.

If $[Z'] \in L_Z(A')$ lifts $\overline{Z}$, where $A' \longrightarrow A$ is small, then by Proposition 4.9, there is a $C' \longrightarrow \operatorname{Spec} S_{A'}$ lifting $\overline{C}$ and containing Z'; suppose C' induces p': $B' \longrightarrow \operatorname{Spec} R_{A'}$. The total transform $T(\overline{f}) \subset \overline{B}$ is defined by the free sheaf of $\mathcal{O}_{\overline{B}}$-modules, $\overline{p}^*(\overline{f})$, which consists of the product of the ideal sheaves of $n_i\overline{E}_i$ (some n_i) and of the proper transform of $\overline{f}$. Having lifted the $\overline{E}_i$ to $E_i' \subset B'$, one can easily lift $T(\overline{f})$ in a locally trivial fashion to a relative A'-divisor on B', defined by the sheaf of $\mathcal{O}_{B'}$-ideals K'. But K' is a *free* $\mathcal{O}_{B'}$-module, since it lifts the free $\mathcal{O}_B$-module $p^*(f)$ and since $H^1(B, \mathcal{O}_B) = 0$ [1, III. 7.1]. Thus, $p'_*(K') \subset R_{A'}$ is a free $R_{A'}$-module, hence is a principal ideal (f') lifting $(\overline{f})$. Further, the natural map $p'^*p'_*K' \longrightarrow K'$ of free $\mathcal{O}_{B'}$-modules is necessarily an isomorphism.

We must show $(f') \in ES(A')$. Let $E_1 \subset E$ correspond to the blowing-up of the origin in Spec R; then the multiplicity of E_1 in $p^*(f)$ is r, the multiplicity of (f). By construction, f' vanishes r times on E_1'; equivalently (Proposition 1.6), (f') is equimultiple along the section defined by $\Gamma(\mathcal{O}_{B'}) = R_{A'} \longrightarrow \Gamma(\mathcal{O}_{E_1'}) \simeq A'$. Next, let p_1': $B_1' \longrightarrow \operatorname{Spec} R_{A'}$ be the blowing-up of this section; then p' factors $B' \xrightarrow{q'} B_1' \xrightarrow{p_1'} \operatorname{Spec} R_{A'}$. If $E_2 \subset E$ comes from blowing up a point of B_1, then the morphisms $\mathcal{O}_{B_1'} \xrightarrow{\sim} q'_*\mathcal{O}_{B'} \longrightarrow q'_*\mathcal{O}_{E_2'} \simeq A'$ gives rise to the section of B_1' to be blown up. Thus, equimultiplicity of the transform of (f') along this section is equivalent to f' vanishing sufficiently many times on $E_2' \subset B'$. But this is guaranteed by construction. In this fashion, we find $(f') \in ES(A')$, so γ is smooth.

Remarks. (4.10) The proof shows that if Z is any divisor for which $H^1(T_C) \xrightarrow{\sim} H^1(T_C \otimes \mathcal{O}_Z)$ ("the deformations of C are determined by those of Z"), then $L \xrightarrow{\sim} L_Z$.

(4.11) One can show that $ES \longrightarrow L_E$ is smooth; L_E is pro-representable and has tangent space described by Proposition 4.4.

(4.12) For a direct proof of the smoothness of ES, see [9, p. 134].

5. The functor ES'.

(5.1) Let p: $B \longrightarrow \operatorname{Spec} R$ be a *minimum resolution* of the singularity of (f); that is, one blows up only nonnodes of the reduced total transforms.

Lemma 5.2. *If* $(\overline{f}) \in ES(A)$ *is equisingular along the trivial section in all neighborhoods of* $B \longrightarrow \operatorname{Spec} R$ *except at nodes, then it is automatically equisingular along the trivial section at nodes of the exceptional fibre.*

Proof. We may restrict ourselves to the following case. Suppose $\mathcal{O}_i$ is the local ring of a point P in B_i, $(\overline{f})$ is equisingular along the trivial section in $\mathcal{O}_i \otimes A$, while $Q \in B_{i+1}$ (the blowing-up of P) is a point not on the proper transform of (f), but at which $R(f)$ has a node. Let $\mathcal{O}_{i+1}$ be the local ring of Q. We may assume $(\overline{f}) = (Y)(\overline{g})$ in $\mathcal{O}_i \otimes A$, where $Y = 0$ is the local equation of an exceptional line (as all previous blown-up sections are trivial). Therefore, the reduced total transform of $(\overline{f})$ in $\mathcal{O}_{i+1} \otimes A$ is given by (XT) (where $Y = XT$), hence is equisingular along the trivial section.

Convention (5.3). We shall say that those $(\overline{f})$ as in Lemma 5.2 are *equisingular via trivial sections*, recalling that this does not apply to nodes of the proper transforms. This is not to be confused with "$(\overline{f})$ equisingular along the trivial section" (see (3.3)).

(5.4) Consider the subfunctor of ES defined by

$$ES'(A) = \{(\overline{f}) \in ES(A) | \sigma(\overline{f}) \text{ is equisingular via trivial sections, some } \sigma \in G(A)\}.$$

Thus, ES' consists of the "easiest" equisingular liftings; one does not worry about equisingular sections.

Proposition 5.5. *ES' has a very good deformation theory and is smooth.*

Proof. If $(\overline{f})$ and $\sigma(\overline{f})$ are equisingular via trivial sections, then we may view σ as an element of $G'(A)$, the group of infinitesimal Spec A-automorphisms of $B \times \operatorname{Spec} A$. The result follows directly once we show G' is smooth. For this, one imitates the proof of [8, 1.3.1], in showing that one can "exponentiate in the Lie algebra $G'(k[\epsilon])$". In particular, if $d \in G'(k[\epsilon])$, $m \in m_A$, one can define $e^{md} \in G'(A)$, and prove that every element of $G'(A)$ is a composite of such Spec A-automorphisms.

Remark 5.6. One could have started by defining ES' as a subfunctor of H, and proving directly as above that ES' has a very good deformation theory and is smooth; one can therefore avoid Theorems 3.2 and 4.2 for this simpler functor. However, as we shall see below, ES' is generally smaller than ES. Denote henceforth by L the functor L_Z, where Z is as in (4.1).

Theorem 5.7. *ES' is the kernel of the smooth morphism γ: $ES \longrightarrow L$. That is, $ES'(A) = \{(\overline{f}) \in ES(A) | \gamma_A((\overline{f}))$ is the trivial element of $L(A)\}$.*

Proof. If $(\overline{f})$ is equisingular via trivial sections, then the induced deformation $\overline{B}$ is trivial, whence so is the element of $L(A)$; therefore, $ES'(A)$ is in the kernel of $ES(A) \longrightarrow L(A)$. Conversely, if $(\overline{f}) \in ES(A)$ induces a trivial element of $L(A)$, then by Proposition 4.9 it induces a trivial deformation of B. So, if $\overline{B}_M \longrightarrow \overline{B}_{M-1} \longrightarrow \cdots \longrightarrow \operatorname{Spec} R_A$ is a resolution attached to $(\overline{f})$, then after automorphism $\sigma \in G(A)$, it becomes $B_M \times \operatorname{Spec} A \longrightarrow \cdots \longrightarrow \operatorname{Spec} R \times \operatorname{Spec} A$ (cf. Lemma 4.7). Therefore, $\sigma(\overline{f})$ is equisingular via trivial sections, whence $(\overline{f}) \in ES'(A)$.

Corollary 5.8. *$ES' = ES$ if and only if L is trivial.*

6. The tangent space of ES.

Proposition 6.1. *The tangent space of ES, $I = \{g \in k[[X, Y]] | (f + \epsilon g) \in ES(k[\epsilon])\}$, is an ideal.*

Proof. I is certainly a k-module because ES has a good deformation theory. Thus, we must show $g \in I$ implies $\alpha g \in I$, any $\alpha \in R = k[[X, Y]]$. Since, for $\sigma \in G(k[\epsilon])$, there are $p, q \in R$ such that $\sigma(f + \epsilon g) = (f + \epsilon(g + pf_X + qf_Y))$, it follows that I contains the ideal (f, f_X, f_Y); further, we may suppose $(f + \epsilon g)$ is equisingular along the trivial section, and prove $(f + \epsilon\alpha g) \in ES(k[\epsilon])$.

We use induction on M, the number of blowings-up needed to resolve the singularity of (f). For $M = 1$, (f) is an ordinary multiple point, and $I = (f, f_X, f_Y, (X, Y)^r)$, since in this case equimultiplicity is the same as equisingularity. If $(f) = (f_1) \cdots (f_t)$ is a product of tangential components, then $(f + \epsilon g) = \Pi\,(f_i + \epsilon g_i)$ (Proposition 1.10), whence $(f + \epsilon\alpha g) = \Pi\,(f_i + \epsilon\alpha g_i)$; thus, it suffices to prove the result when (f) is unitangential. But after another automorphism (Lemma 2.4), we may assume $(f + \epsilon g)$ and its first transform are equisingular along the trivial section. A direct computation now shows that $(f + \epsilon\alpha g)$ and its first transform are equimultiple along the trivial section; but induction in the first neighborhood implies $(f + \epsilon\alpha g) \in ES(k[\epsilon])$.

(6.2) To get a better hold of I, we let $B \longrightarrow \operatorname{Spec} R$ be a minimal resolution of (f) (5.1). Let $\{\mathcal{O}_\alpha\}_{\alpha\in T}$ be the set of $\mathcal{O}_0 = R$ and all local rings of the B_i in the resolution at which the reduced total transform has a singularity. Thus, there are t such local rings $\mathcal{O}_1, \cdots, \mathcal{O}_t$ in B_1, where t is the number of tangential components of (f). Let $S \subset T$ be the index set corresponding to the nonnodes. Let $m_\alpha \subset \mathcal{O}_\alpha$ be the maximal ideal, and let v_α be the m_α-adic valuation on $\mathcal{O}_\alpha$. Of course, $R \subset \mathcal{O}_\alpha$ is birational.

Proposition 6.3. $(f + \epsilon g) \in ES'(k[\epsilon])$ *if and only if* g *is in the ideal* $I' = (f, f_X, f_Y, \bigcap_{\alpha\in S} m_\alpha^{v\alpha(f)})$. *In particular,* $I' \subset I$.

Proof. If $(f + \epsilon g) \in ES'(k[\epsilon])$, there is a $\sigma \in G(k[\epsilon])$ such that $\sigma(f + \epsilon g)$ is equisingular via trivial sections. Since $(f, f_X, f_Y) \subset I'$, it suffices to assume $(f + \epsilon g)$ is equisingular via trivial sections. The result then follows by definition and Proposition 1.6.

Conversely, suppose $g \in I'$; writing $g = af + bf_X + cf_Y + g_1$, where $g_1 \in \bigcap_{\alpha\in S} m_\alpha^{v\alpha(f)}$, one has $(f + \epsilon g) \in ES'(k[\epsilon])$ if and only if $(f + \epsilon g_1) \in ES'(k[\epsilon])$. Thus, it suffices to show $g \in \bigcap_{\alpha\in S} m_\alpha^{v\alpha(f)}$ implies $(f + \epsilon g)$ is equisingular via trivial sections. If $M = 1$, (f) is an ordinary multiple point, and this case is clear. In the general case, certainly $(f + \epsilon g)$ is equimultiple along the trivial section. If $Y + \beta_i X$, $i = 1, \cdots, t$, are linear forms defining the tangent directions of (f), then

$$\mathcal{O}_i = (k[[X, Y]][T]/Y - TX)_{(X, T+\beta_i)}.$$

But the reduced total transform of $(f + \epsilon g)$ in $\mathcal{O}_i[\epsilon]$ is given by $(f + \epsilon g)/X^{r-1} = \bar{f}_i + \epsilon\bar{g}_i$. For any $\mathcal{O}_\alpha$ $(\alpha \in S)$ dominating $\mathcal{O}_i$, we have $v_\alpha(f) = v_\alpha(\bar{f}_i) + v_\alpha(X^{r-1})$

and $v_\alpha(g) = v_\alpha(\overline{g}_i) + v_\alpha(X^{r-1})$. Thus, $v_\alpha(g) \geq v_\alpha(f)$ (i.e., $g \in m_\alpha^{v_\alpha(f)}$) if and only if $v_\alpha(\overline{g}_i) \geq v_\alpha(\overline{f}_i)$. Since taking completions does not change the values of v_α, the inductive assumption is fulfilled for $(\overline{f}_i + \epsilon\overline{g}_i)$, unless $(\overline{f}_i)$ defines an ordinary double point. Since this case is covered by Remark 2.11, we are done.

Remarks. (6.4) Of course, $I \subset (f, f_X, f_Y, (X, Y)^r)$, since equisingular liftings are equimultiple.

(6.5) If B_i is some blowing-up in a minimal resolution of (f), and if the corresponding functor L of B_i (4.6) is trivial, then one easily shows

$$I \subset \left(f, f_X, f_Y, \bigcap_{\alpha \in S_i} m_\alpha^{v_\alpha(f)}\right),$$

where $S_i \subset S$ corresponds to the local rings $\mathcal{O}_\alpha$ contained in the local rings of B_i.

Proposition 6.6. *In case $f(X, Y) = Y^p + X^q$ $(p \leq q)$, then $I = I'$ is the ideal generated by X^{q-1}, Y^{p-1}, and the monomials X^iY^j, where $pi + qj \geq pq$. In particular, (f) has nontrivial equisingular liftings if and only if $p = 3$, $q > 5$ or $p > 3$.*

Proof. We proceed by induction on pq, the case $pq = 1$ being trivial; also, (2.12) proves the result for $p = q$.

Suppose $p < q$, and $(Y^p + X^q + \epsilon \sum a_{ij}X^iY^j) \in ES(k[\epsilon])$; we may as well suppose $i \leq q - 2$, $j \leq p - 2$, and the lifting is equisingular along the trivial section. Letting $Y = TX$, the reduced total transform is given by

$$X(T^p + X^{q-p} + \epsilon \sum a_{ij}X^{i+j-p}T^j).$$

By Proposition 3.4, $(T^p + X^{q-p} + \epsilon \sum a_{ij}X^{i+j-p}T^j)$ is equisingular; since $p(q-p) < pq$, we may use induction. An easy computation shows $pi + qj \geq pq$, for all i, j such that $a_{ij} \neq 0$.

Conversely, it suffices to show $(Y^p + X^q + \epsilon X^iY^j) \in ES(k[\epsilon])$, for $pi + qj \geq pq$. This may be done using Zariski's discriminant criterion for the formal family over $k[[t]]$ defined by $Y^p + X^q + tX^iY^j$ [10, Theorem 7]. However, we show directly (by induction) that $(Y^p + X^q + \epsilon X^iY^j)$ is equisingular via trivial sections if $pi + qj \geq pq$. Since $i + j \geq p$, the lifting is equimultiple along the trivial section; letting $Y = TX$, induction shows the proper transform $(T^p + X^{q-p} + \epsilon X^{i+j-p}T^j)$ is equisingular via trivial sections. To show this for $X(T^p + X^{q-p} + \epsilon X^{i+j-p}T^j)$, we must prove that the equisingular sections of each component are compatible in all neighborhoods. However, since these sections are almost all trivial, one need only consider the following case: there is an ordinary double point of the reduced total transform of $T^p + X^{q-p}$ at which the transform of $X(T^p + X^{q-p})$ does not have a node. An elementary argument shows this occurs only if $q = p + 1$. But a direct computation then shows that $pi + qj \geq pq$ implies $X(T^p + X + \epsilon X^{i+j-p}T^j)$ is equisingular via trivial sections. This completes the proof.

Corollary 6.7. *In case* $f(X, Y) = Y^p + X^q$, *then* $ES' = ES$.

Proof. $ES' \longrightarrow ES$ is surjective on the tangent spaces, and ES' is smooth, whence $ES' \longrightarrow ES$ is surjective.

Example 6.8. Consider the reduced curve defined by $f(X, Y) = (X^4 - Y^4)^2 - X^{10}$. This curve has four tangential components, each of which consists of two nonsingular branches with a first-order contact (i.e., the branches become transversal after one blowing-up). If $B_1 \longrightarrow \operatorname{Spec} R$ is one blowing-up of the origin, the singularities of $R(f)$ are four ordinary triple points. Thus, $(f + \epsilon g)$ is equisingular via trivial sections precisely when g is of the form $g_8 + g_9 + g'$, where $g_8 = c(X^4 - Y^4)^2$ $(c \in k)$, $g_9 \in (X^2, XY, Y^2)\cdot(f_X, f_Y)$, and $g' \in (X, Y)^{10}$ (verification is straightforward). Therefore, $I' = (f, f_X, f_Y, (X, Y)^{10})$.

However, one computes that

$$(f + \epsilon g) = ((X^4 - Y^4)^2 - X^{10} + \epsilon X(X^4 - Y^4)(X^3 + X^2Y + XY^2 + Y^3))$$

is equimultiple along the trivial section, and $R(f + \epsilon g)$ is equimultiple along four sections of $B_1[\epsilon]$, only three of which are trivial. Further, $g \notin I'$ (essentially because $X^2Y^2(X^4 - Y^4) \notin I'$), whence $I' < I$. In fact,

$$I = (I', X^2Y^2(X^4 - Y^4)).$$

This extra equisingular lifting arises from the one-dimensional vector space $L_E(k[\epsilon])$ (the exceptional fibre E of a minimal resolution consists of four lines intersecting a given line in four points) via the smooth morphism $ES \longrightarrow L$ (Theorem 5.7) and $L \longrightarrow L_E$ (in this case a bijection).

Remark 6.9. Via (5.7) and (6.3), a thorough knowledge of $ES(k[\epsilon])$ requires knowledge of $H^1(T_Z)$, where Z is a certain nonreduced curve. We hope to return to this question in a future paper.

7. Equisingular deformation classes.

(7.1) The notions of equimultiplicity and equisingularity may be considered in terms of the local ring of the singularity, and not just in terms of the embedding in Spec R. As in (1.2), we may speak of the functor D of classes of deformations of $P = R/(f)$; assume P is not regular.

If $\overline{P}$ is a deformation of P to A, and if $\overline{s}\colon \overline{P} \longrightarrow A$ is a normally flat A-section inducing the canonical map $s\colon P \longrightarrow k$, then we say $(\overline{P}, \overline{s})$ is an *equimultiple deformation* of (P, s). Two such pairs $(\overline{P}_1, \overline{s}_1)$ and $(\overline{P}_2, \overline{s}_2)$ are said to be *equivalent* if there is an A-isomorphism $\phi\colon \overline{P}_1 \longrightarrow \overline{P}_2$, inducing the identity over k, such that $\overline{s}_2 \cdot \phi = \overline{s}_1$. We then define the functor of *equimultiple deformation classes plus sections* by

$$\overline{E}(A) = \text{set of equivalence classes of } (\overline{P}, \overline{s}).$$

Proposition 7.2. *$\overline{E}$ is the quotient functor of E by G, hence is versal and smooth. The natural morphism $\overline{E} \longrightarrow D$ is injective if and only if (f) is not unitangential.*

Proof. The group action was defined in 1.5; that $\overline{E}$ is the quotient is straightforward, so one applies [8, Proposition 1.1.6]. The finite dimensionality of $\overline{E}(k[\epsilon])$ follows from that of $D(k[\epsilon])$, since there is a two-dimensional space of $k[\epsilon]$-sections of $R[\epsilon]$ inducing the trivial section over k. The second statement follows easily from Proposition 1.9.

(7.3) One can define higher order functors $\overline{E}^n$ of equimultiple deformation classes plus sections; again, they are the quotients of the actions $G \times E^n \longrightarrow E^n$ and are versal.

In particular, for N large enough, $\overline{E}^N \simeq \overline{E}^{N+1} \simeq \cdots$ is the quotient of ES by G, the functor $\overline{ES}$ of *equisingular deformation classes* (plus sections). By Theorem 3.2, $ES \hookrightarrow H$ is injective, hence so is the map of G-quotients $\overline{ES} \hookrightarrow D$, and we can again neglect the (uniquely determined) sections. The fundamental morphism $ES \longrightarrow L$ factors through $\overline{ES}$, whence there is a smooth morphism $\overline{ES} \longrightarrow L$. We also have a smooth versal subfunctor $\overline{ES}'$ of $\overline{ES}$. Gathering up our previous results, we have

Theorem 7.4. *The functor $\overline{ES}$ of equisingular deformation classes is a versal and smooth subfunctor of the deformation functor D of the singularity. The tangent space is given by $I/(f, f_X, f_Y)$. There is a smooth morphism $\overline{ES} \longrightarrow L$, with "kernel" $\overline{ES}'$, where L is the locally trivial deformation functor of a projective curve. If $(f) = (f_1) \cdots (f_s)$ is a factorization into irreducible factors, there is a natural map*

$$\overline{ES}_{(f)} \longrightarrow \prod \overline{ES}_{(f_i)}.$$

8. **Formal versal equisingular deformations.**

(8.1) An important property of the deformation functor of a complete intersection with isolated singularity is the fact that a formal versal deformation may be written down from the tangent space [7]. In particular, if (f) defines a (reduced) algebroid curve, and if $g_1, \cdots, g_m \in k[[X, Y]]$ have residues forming a basis of $k[[X, Y]]/(f, f_X, f_Y)$, then the homomorphism

$$k[[t_1, \cdots, t_m]] \longrightarrow k[[X, Y, t_1, \cdots, t_m]]/(f + \sum t_i g_i)$$

is a formal versal deformation of the singularity.

It would be nice to be able to write down a formal versal equisingular deformation, and not just the generic first-order one (given by the tangent space). The next result indicates that ES' is the "maximum" subfunctor with the property that a versal family involves only linear terms in the t_i's.

Theorem 8.2. *Let $(f) \subset R$ define a reduced curve. Then the following are equivalent*:

(a) $ES' = ES$;

(a$'$) $I' = I$;

(b) L *is trivial for a minimal resolution of* (f);

(c) *there are elements* $g_1, \cdots, g_s \in I$ *such that the formal family*

$$k[[t_1, \cdots, t_s]] \to k[[X, Y, t_1, \cdots, t_s]] \Big/ \left(f + \sum_{i=1}^{s} t_i g_i\right)$$

defines a versal deformation for $\overline{ES}$.

Proof. The equivalence of (a), (a$'$), and (b) has already been observed.

Suppose then $ES' = ES$, and let $g_1, \cdots, g_s \in I$ induce a basis of $\overline{ES}(k[\epsilon])$; we may also suppose that $(f + \epsilon g_i)$ is equisingular via trivial sections, for all i. Let $C = k[[t_1, \cdots, t_s]]$, and define a morphism $\phi\colon h_C \to \overline{ES}$ by associating to each local map $\alpha\colon C \to A$ the deformation class defined by $(f + \Sigma\, \alpha(t_i)g_i)$. An easy induction on the number of blowings-up needed to resolve the singularity shows that this lifting is equisingular via trivial sections, whence ϕ is well defined. Since ϕ_ϵ is a bijection, ϕ will be versal once we show it is smooth. One easily reduces to the proof of the following: if $A' \to A$ is small of kernel (η), and if $(f + \Sigma\, m_i' g_i + \eta g) \in ES'(A')$ is equisingular via trivial sections, then so is $(f + \epsilon g)$ in $ES'(k[\epsilon])$. Again, the straightforward proof is by induction on the number of blowings-up. Thus, (a) implies (c).

To see that (c) implies (a), it suffices to show that if $(f + tg) \in ES(k[t]/t^n)$, for all n, then $(f + \epsilon g)$ is equisingular via trivial sections. This follows from the usual induction by the next

Lemma 8.3. *Suppose $r > 1$ and (f) does not define a node; let $g \in k[[X, Y]]$. If $(f + tg) \in k[[X, Y, t]]$ is equimultiple along the $k[[t]]$-section defined by $I = (X - a(t), Y - b(t))$ (i.e., one gets equimultiple liftings and sections on all truncations $k[[t]]/t^n$), then $g \in (X, Y)^r$ and either*

(i) $a(t) = b(t) = 0$ *or*

(ii) f *is a product of two regular parameters with the same tangent direction.*

If $(f + tg)$ induces equisingular liftings, then $a(t) = b(t) = 0$.

Proof. Proposition 1.6 implies $(f + tg) \in I^r$ in $k[[X, Y, t]]$. Differentiation with respect to t yields $g \in I^{r-1}$, whence $f, g \in I^{r-1}$. Let

$$P = \{h \in k[[X, Y]] \mid h(a(t), b(t)) \equiv 0\};$$

then $P = R \cap I$ is prime. If $P = (X, Y)$, then $a(t) = b(t) = 0$, and $g \in (X, Y)^r$.

The only other possibility is $P = (h)$, where h is irreducible; write $f = f_1 h$, $g = g_1 h$. Since h does not divide f_1, we have $h \in I^{r-1}$; if $r \geq 3$, differentiation yields $h_X, h_Y \in P$ have lower degree. Thus, $r = 2$. If f_1 is a constant, then $(f + tg) = (h) = (f)$, whence $h_X, h_Y \in I \cap R$, a contradiction. Thus, f_1 and h are regular parameters with the same tangent direction (since $V(f)$ is not a node). Assuming (after change of coordinates) that $h = Y$, we have that $Y(f_1 + tg_1) \in (X - a(t), Y - b(t))^2$, whence $b(t) = 0$ and $f_1 + tg_1 \in (X - a(t), Y)$. But if $f_1(a(t), 0) + tg_1(a(t), 0) = 0$, then g_1 has no constant term, since the linear term of f_1 is a multiple of Y. Consequently, $g = g_1 h \in (X, Y)^2$.

If (f) is as in case (ii), we claim (f) has no nontrivial equisingular liftings; then $(f + tg)$ equisingular would imply, for all n, there is an element $\sigma_n \in G(k[t]/t^n)$ such that $(\sigma_n(f)) = (f + tg)$ in $R \otimes k[t]/t^n$. A small computation would then show $g \in (f)$, whence $(f + tg) = (f)$; but Theorem 3.2 says (f) is equisingular along only the trivial section in $R \otimes k[t]/t^n$, whence $a(t) = b(t) = 0$.

Now, after a change of coordinates we may suppose $f = Y(Y + X^n)$, some $n \geq 2$. One checks any first-order lifting is equivalent to one of the form $(f + \epsilon \sum_{i=0}^{2n-2} a_i X^i)$; if one such is equisingular (with $a_i \neq 0$), then so is $(f + \epsilon X^{2n-2})$, since I is an ideal. However, an easy induction on n shows this is false.

Corollary 8.4. *Let $g_1, \cdots, g_s \in I'$ be equisingular via trivial sections and induce a basis of I' $\operatorname{mod}(f, f_X, f_Y)$. Then $(f + \sum t_i g_i) \subset k[[X, Y, t_1, \cdots, t_s]]$ defines a versal formal deformation for $\overline{ES}'$.*

Corollary 8.5. *A formal versal equisingular deformation of $Y^p + X^q$ $(p \leq q)$ is defined by the formal family $(Y^p + X^q + \sum t_{ij} X^i Y^j)$ over $k[[t_{ij}]]$, where we consider pairs (i, j) with $pi + qj \geq pq$, $i \leq q - 2$, $j \leq p - 2$.*

Proof. Follows from Proposition 6.6 and the Theorem.

(8.6) The formal lifting $Y(Y - X^3 + tX^2)$ is equimultiple along $(X - t, Y)$ as well as along (X, Y); thus, the phenomenon in (ii) of Lemma 8.3 does occur.

(8.7) In the general case, choose $g_1, \cdots, g_s \in I$ inducing a basis of $\overline{ES}(k[\epsilon])$, and then let $g_{s+1}, \cdots, g_m \in k[[X, Y]]$ be such that $g_1, \cdots, g_m$ induce a basis of $k[[X, Y]]/(f, f_X, f_Y) = D(k[\epsilon])$.

Proposition 8.8. *Under the above conditions, a formal versal equisingular deformation is given over $k[[t_1, \cdots, t_s]]$ by a lifting of the form $f + \sum_{i=1}^m \alpha_i(t) g_i$, where $\alpha_i(t) = \alpha_i(t_1, \cdots, t_m)$ has linear term t_i if $i \leq s$, and has no linear term if $i > s$.*

Proof. The result follows easily from the following lemma, applied to $\overline{ES} \subset D$.

Lemma 8.9. *Let $F_1 \subset F_2$ be an inclusion of versal functors, where $h_S \longrightarrow F_1$ and $h_T \longrightarrow F_2$ are the smooth morphisms yielding versality. Then there exists a (nonunique) surjection $T \longrightarrow S$ such that the induced inclusion $h_S \longrightarrow h_T$ yields a commutative diagram*

$$\begin{array}{ccc} h_S & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ h_T & \longrightarrow & F_2 \end{array}$$

Proof. Let $S_n = S/m_S^n$. The canonical elements of $h_S(S_n)$ induce a compatible sequence of elements in $F_2(S_n)$. Since $h_T(S_{n+1}) \to h_T(S_n) \times_{F_2(S_n)} F_2(S_{n+1})$ is surjective, we may lift these elements compatibly to elements in $h_T(S_n)$; but T is complete, so this process yields a homomorphism $T \to S$. If now $\alpha \in h_S(A)$, some $A \in \underline{C}$, then α is the image of the canonical element in a map $h_S(S_n) \to h_S(A)$. Commutativity of the above diagram follows easily. Since $h_S(k[\epsilon]) = F_1(k[\epsilon]) \subset F_2(k[\epsilon]) = h_T(k[\epsilon])$, we have $T \to S$ is surjective, whence $h_S \subset h_T$.

Remark 8.10. One cannot in general eliminate terms g_i with $i > s$, as seen by the formal equisingular deformation given by

$$[(X^4 - Y^4) + t(X^4 + X^3Y + X^2Y^2 + XY^3)]^2 - X^{10} \quad \text{(Example 6.8).}$$

9. Equisingularity and topological equivalence.

(9.1) In this section we prove that π_1 of the complement of a plane algebroid curve remains constant during formal equisingular lifting. This result is the algebraic analogue of the fact that two equisingular curves over the complex numbers yield knots of the same knot type in $\mathbf{R}^3$. Zariski has proved an analogous result for convergent power series rings, relying heavily on topological arguments [12, Theorem 6.1].

We need to use a theorem on deformations of branched covers. Recall that if $(d) \subset R$, R any domain, a lifting $(\bar{d}) \subset R \otimes A$ is called *trivializable* if there is a $\sigma \in G(A)$ such that $(\sigma(\bar{d})) = (d)$. If $R \to S$ is finite and free, then a deformation $R_A \to \bar{S}$ is called *trivializable* if there is a $\sigma \in G(A)$ such that $\sigma[R_A \to \bar{S}]$ is the class of the trivial deformation $R_A \to S_A$. For details, see [8, §§1.3 and 1.4].

Theorem A [8, Theorem 2.4.1]. *Let R be a regular excellent k-domain, S normal, and $R \to S$ a finite free homomorphism with discriminant $(d) \subset R$. Then to every trivializable lifting $(\bar{d})$ of (d) to R_A, there exists a trivializable deformation $R_A \to \bar{S}$ with discriminant $(\bar{d})$, and this deformation is unique up to unique R_A-isomorphism.*

The whole point of Theorem A is the uniqueness of the trivializable deformation class. Recall that if R is a finitely generated k-domain, then the trivializable liftings of (d) are precisely those $(\bar{d})$ that do not change the formal nature of the singularities of $R/(d)$.

(9.2) If $f \in k[[X, Y]] = R$ and $F \in k[[X, Y, t_1, \cdots, t_s]]$ are reduced such that $F(X, Y, 0, \cdots, 0) = f(X, Y)$, then we say $V(F)$ defines a *formal equisingular lifting* of $V(f)$ if the image F_n of F in $k[[X, Y, t_1, \cdots, t_s]]/(t_1, \cdots, t_s)^n$ defines an equisingular lifting in the usual sense, for all n. Let $S = k[[t_1, \cdots, t_s]]$ and $S_n = S/m_S^n$, and denote by $\sigma_n: S_n[[X, Y]] \to S_n$ the equisingular section for $V(F_n) \subset \operatorname{Spec} S_n[[X, Y]]$. Since equisingular sections are unique, the σ_n are induced by a unique section $\sigma: S[[X, Y]] \to S$. By Proposition 1.6, F is equimultiple along σ in the sense that $F \in (\operatorname{Ker} \sigma)^r$, r = multiplicity of f. Let $\bar{p}_1: \bar{B}_1 \to \operatorname{Spec} S[[X, Y]]$ be the blowing-up along σ. Then, as for artinian S, the reduced total transform $R(F)$ of $V(F)$ is obtained by taking $r - 1$ copies of the exceptional subscheme of $\bar{B}_1$ out of $\bar{p}_1^{-1}(V(F))$. Further, $R(F) \subset \bar{B}_1$ yields formal equisingular liftings of the singularities of $R(f) \subset B_1$. Thus we may repeat the process and obtain a sequence of blowings-up $\bar{p}: \bar{B} \to \operatorname{Spec} S[[X, Y]]$ such that the reduced total transform of $V(F)$ has only ordinary double points as singularities; this will of course induce the same situation on the truncations $p_n: B_n \to \operatorname{Spec} S_n[[X, Y]]$. Therefore, this definition of formal equisingular lifting agrees with Zariski's via [11, Theorem 7.4].

(9.3) Let $R(F_n) \subset B_n$ be the reduced total transform of F_n via p_n. Recall (4.5) that p_n is induced via flat base change from a morphism $r_n: C_n \to \operatorname{Spec} S_n[X, Y]$, where C_n is an algebraic k-scheme, whence $B_n \to \operatorname{Spec} S_n$ is a locally trivial deformation of $B \to \operatorname{Spec} k$ in the Zariski topology. In order to apply Theorem A, we will have to know $R(F_n) \subset B_n$ is locally a trivializable lifting of $R(f)$; since our definition of equisingularity guarantees only formal trivializability, an algebraicity result is needed. But an old result (e.g., [2, Theorem B]) implies that after change of coordinates in $k[[X, Y]]$, the ideal (f) is generated by a polynomial (which we call f). Further, if $g_1, \cdots, g_m$ are polynomials inducing a basis of $D(k[\epsilon])$, then $(f + \Sigma s_i g_i) \subset k[[X, Y, s_1, \cdots, s_m]]$ defines a formal versal deformation of $k[[X, Y]]/(f)$. Thus, after an S-automorphism of $S[[X, Y]]$ inducing the identity $\bmod m_S$, (F) is generated by an element in $k[X, Y][[T_1, \cdots, T_s]]$. Therefore, $R(F_n) \subset B_n$ is induced by the reduced total transform in C_n of F_n, considered as an element of $S_n[X, Y]$. Since formally trivializable implies locally trivializable for liftings on an algebraic k-scheme [8, 2.1.5], it follows that the liftings $R(F_n) \subset B_n$ of $R(f)$ are trivializable locally in the Zariski topology.

Theorem 9.4. *Let $F \in k[[X, Y, t_1, \cdots, t_s]]$ define a formal equisingular lifting of $f(X, Y) = F(X, Y, 0, \cdots, 0)$. Then the natural map of algebraic fundamental groups*

$$\pi_1(\operatorname{Spec} k[[X, Y]] - V(f)) \to \pi_1(\operatorname{Spec} k[[X, Y, t_1, \cdots, t_s]] - V(F))$$

is an isomorphism.

Proof. We must show that every finite étale cover of $\operatorname{Spec} k[[X, Y]] - V(f)$ lifts uniquely to an étale cover of $\operatorname{Spec} S[[X, Y]] - V(f)$. We shall do this via the natural isomorphisms

$$\bar{B} - R(F) \simeq \operatorname{Spec} S[[X, Y]] - V(F) \quad \text{and} \quad B - R(f) \simeq \operatorname{Spec} k[[X, Y]] - V(f).$$

Let X be a finite étale cover of $B - R(f)$, and let Y be the normalization of B in the quotient field of X. Then $Y \longrightarrow B$ is finite and flat (since two-dimensional normal local rings are Cohen-Macaulay), and has its discriminant D supported on a subset of $R(f)$. There is a unique locally trivializable lifting of D to a Cartier divisor $D_n \subset B_n$ such that $D_n \subset R(F_n)$, since D consists of components of $R(f)$ counted with various multiplicities. Since $k[[X, Y]]$ is excellent, so is B; the B_n's are locally trivial deformations of B, so Theorem A implies there is a (unique) locally trivializable deformation $Y_n \longrightarrow B_n$ with discriminant D_n, for all n.

View $\mathcal{O}_{Y_n}$ as a (finite flat) $\mathcal{O}_{B_n}$-algebra, and note that $\bar{p}: \bar{B} \longrightarrow \operatorname{Spec} S[[X, Y]]$ is proper. The Grothendieck existence theorem [EGA, III. 5.1.4] implies there is a unique finite $\mathcal{O}_{\bar{B}}$-algebra $\mathcal{O}_{\bar{Y}}$ inducing the $\mathcal{O}_{Y_n}$. For, one first lifts the $\mathcal{O}_{Y_n}$ to $\mathcal{O}_{\bar{Y}}$ as an $\mathcal{O}_{\bar{B}}$-module, and then applies the Grothendieck existence theorem again to the algebraic structure maps $\mathcal{O}_{Y_n} \otimes \mathcal{O}_{Y_n} \longrightarrow \mathcal{O}_{Y_n}$ to make $\mathcal{O}_{\bar{Y}}$ an $\mathcal{O}_{\bar{B}}$-algebra. An easy local argument shows the corresponding finite map $\bar{Y} \longrightarrow \bar{B}$ is flat, whence $\bar{Y} \longrightarrow \bar{B}$ is étale off the discriminant divisor $\bar{D}$. But since $\bar{Y} \longrightarrow \bar{B}$ induces $Y_n \longrightarrow B_n$, it follows that $\bar{D}$ induces D_n for all n, so $\bar{D} \subset R(\bar{F})$. Consequently, $\bar{Y}$ induces a finite étale cover of $\bar{B} - R(\bar{F})$ lifting $X \longrightarrow B - R(f)$.

For uniqueness, suppose $\bar{X}_i \longrightarrow \bar{B} - R(\bar{F})$, $i = 1, 2$, are finite étale covers lifting X. Let $\bar{Y}_i$ be the normalization of $\bar{B}$ in the function field of $\bar{X}_i$; then $\bar{Y}_i \longrightarrow \bar{B}$ is finite, and Lemma 9.5 below shows it is flat as well. It suffices to prove $\bar{Y}_1$ and $\bar{Y}_2$ are isomorphic over $\bar{B}$, or (by the Grothendieck existence theorem again) that the induced $\bar{Y}_{i,n}$ are (compatibly) isomorphic over the B_n. But $\bar{Y}_{i,n} \longrightarrow B_n$ is finite, flat, and étale off $R(F_n)$; one also checks that these maps induce $Y \longrightarrow B$ over $\operatorname{Spec} k$. Since the discriminants involved are liftings of $D_1 = D \subset R(f)$, they are locally trivializable liftings. Therefore, everything will follow from Theorem A once we show the deformations $\bar{Y}_{i,n} \longrightarrow B_n$ are locally trivializable, or even formally trivializable [8, 2.1.6]; in fact, by [8, 2.4.5], it suffices to check this over nonsingular points x of D. However, $\bar{Y}_i \longrightarrow \bar{B}$ induces a finite free extension of $\hat{\mathcal{O}}_{\bar{B},x}$ branched along a regular parameter; by Abhyankar's lemma, this extension is obtained by extracting an mth root of a parameter. Therefore, the induced branched covers of $\hat{\mathcal{O}}_{B_n,x}$ are trivializable.

Lemma 9.5. *Let* $k[[X_1, \cdots, X_n]] \longrightarrow S$ $(n \geq 2)$ *be a finite injective map, with* S *normal, such that the map on the spectra is étale over* $\operatorname{Spec} k[[X_1, \cdots, X_n]] - V(X_1X_2)$. *Then* $k[[X_1, \cdots, X_n]] \longrightarrow S$ *is flat.*

Proof. By one form of Abhyankar's lemma (e.g., [8, 2.3.3]),

$$\pi_1(\operatorname{Spec} k[[X_1, \cdots, X_n]] - V(X_1X_2)) \simeq \hat{Z} \oplus \hat{Z}.$$

Thus, every such cover is obtained by taking a certain finite free map $k[[X_1, X_2]] \to S_1$, étale off $V(X_1X_2)$, and considering the étale cover induced off $V(X_1X_2)$ by $k[[X_1, \cdots, X_n]] \to \bar{S} = S_1[[X_3, \cdots, X_n]]$. Since S_1 is normal, so is $\bar{S}$; since $\bar{S}$ agrees with S off $V(X_1X_2)$, then $\bar{S} = S$, whence it is flat over $k[[X_1, \cdots, X_n]]$.

10. **A counterexample.**

(10.1) We had originally hoped to prove Theorem 9.4 by showing that if $k[[X, Y]] \to T$ is finite and flat with discriminant (d), and if $(\bar{d}) \subset A[[X, Y]]$ is an equisingular lifting (so (d) should be reduced), then there is a finite flat cover $A[[X, Y]] \to \bar{T}$ with discriminant $(\bar{d})$ lifting the original "branched cover". The problem is that, in the notation of the proof of Theorem 9.4, $\Gamma(\bar{Y})$ will be a finite normal $\Gamma(\bar{B}) = S[[X, Y]]$-algebra, but it will not be flat in general; equivalently, $\Gamma(\bar{Y}) \to \Gamma(Y)$ need *not* be surjective (see [9, p. 69]). We outline a counterexample suggested by Mumford, in which (d) is an ordinary multiple point; for details, see [9, p. 152].

(10.2) We start with a family of nonsingular space curves $\mathcal{C} \subset \mathbf{P}^n \times X$ over X, a nonsingular affine variety, with the following "jump phenomenon" at a closed point $x \in X$; the natural map $\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1)) \otimes_{\mathcal{O}_X} k(x) \to \Gamma(C_x, \mathcal{O}_{C_x}(1))$ is not surjective. To see that such families exist, let C be a nonhyperelliptic curve of genus $g \geq 5$. Then the canonical bundle K is very ample via $C \to \operatorname{Proj}(H^0(X, K)^\vee) = \mathbf{P}^{g-1}$; further, for generic hyperplane $L \subset H^0(X, K)$, any choice of $g-1$ independent sections induces a closed immersion $C \to \mathbf{P}^{g-2}$. Let S be the $(2g-2)$th symmetric power of C, the variety parametrizing effective divisors of degree $2g - 2$; let J be the component of the Picard group of C parametrizing invertible sheaves of degree $2g-2$, $\Phi: S \to J$ the natural map, and $x \in J$ the point corresponding to K. A divisor $D \in |K|$ corresponds to a point $s \in \Phi^{-1}(x)$. One can identify the tangent space map $\Phi_*: T_{S,s} \to T_{J,x}$ with the map on cohomology $H^0(N_D) \to H^1(\mathcal{O}_X)$ [5, p. 165], whence it has rank $g-1$. Therefore, one can find near s a nonsingular $(g-1)$-dimensional variety V_s such that $\Phi|V_s$ is a closed immersion; let W_s be the image. We may view V_s as a global section of the invertible sheaf on $C \times W_s$ induced from the universal invertible sheaf M of degree $2g-2$ on $C \times J$; this section extends s. Thus, if $s_1, \cdots, s_{g-1} \in H^0(X, K)$ induce a projective embedding $C \to \mathbf{P}^{g-2}$, we may intersect the W_{s_i} near x, and find a nonsingular curve X through x on J over which the sections s_i extend. Shrinking X perhaps, we get a family of closed immersions $r: C \times X \to \mathbf{P}^{g-2} \times X$ such that $r^*(\mathcal{O}(1))$ is the invertible sheaf on $C \times X$ induced from M on $C \times J$. Using Riemann-Roch, one checks that this family of embeddings has the desired jump phenomenon at $x \in X$.

(10.3) Denote by C the embedding of C_x in $\mathbf{P}^n$, and let $f_0: C \to \mathbf{P}^1$ be a "generic projection" onto a line; thus, if C has degree d in $\mathbf{P}^n$ and genus g, there are $b = 2(g + d - 1)$ distinct branch points on $\mathbf{P}^1$. Shrinking X perhaps, we may extend to a family of generic projections $f\colon \mathcal{C} \to \mathbf{P}^1 \times X$, where the discriminant of f is a finite étale cover of X of degree b. Writing $X = \operatorname{Spec} S$, and supposing $m \subset S$ corresponds to C, consider the induced map of *affine cones* (see [EGA, II.8.8]):

$$\begin{array}{ccc} \bigoplus_{i=0}^{\infty} \Gamma(\mathbf{P}^1 \times X, \mathcal{O}_{\mathbf{P}^1 \times X}(i)) & \to & \bigoplus_{i=0}^{\infty} \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(i)). \\ \| & & \| \\ S[X, Y] & & \overline{T} \end{array}$$

(10.4) One checks that $S[X, Y] \to \overline{T}$ is a finite map of normal domains, and it is étale off a subscheme $Z \subset \operatorname{Spec} S[X, Y]$ which, over $\operatorname{Spec} S$, yields a family of ordinary b-tuple points. But $\overline{T}/m\overline{T} \to \bigoplus_{i=0}^{\infty} \Gamma(C, \mathcal{O}_C(i)) = T$ is a strict inclusion (by the jump phenomenon) and a finite birational map (since $\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(i)) \to \Gamma(C, \mathcal{O}_C(i))$ is surjective for all large i); since T is normal, $\overline{T}/m\overline{T}$ is not normal. Then $g\colon \operatorname{Spec} \hat{T} \to \operatorname{Spec} k[[X, Y]]$ (taking completions over the vertex of the cone) is a finite flat morphism whose discriminant (d) is an ordinary multiple point of $\leq b$ branches; let $(d_n) \subset \operatorname{Spec} S/m^n[[X, Y]]$ be the equisingular lifting induced by Z. Then g cannot lift for all n to a cover with discriminant (d_n). If it did, there would be a finite flat map $\operatorname{Spec} T_1 \to \operatorname{Spec} \hat{S}[[X, Y]]$, isomorphic off the pull-back of Z to $\operatorname{Spec} \hat{\overline{T}} \to \operatorname{Spec} \hat{S}[[X, Y]]$, and inducing g. But since $\hat{\overline{T}}$ is normal, there would have to be a factorization $\hat{S}[[X, Y]] \to T_1 \to \hat{\overline{T}}$; but reduction mod the maximal ideal of $\hat{S}$ yields a contradiction.

Added in proof. The first part of the proof of Proposition 6.6 is incomplete as it stands. One must check that if $i + j - p \geq q - p - 1$, but $pi + qj < pq$, then the reduced total transform is not equisingular (if $a_{ij} \neq 0$), even though one has a trivial deformation of the proper transform. These inequalities occur only if $i + j = q - 1$ and $j(q - p) < p$; of course $j \geq 1$, so $p > q - p$. Letting $r = q - p$, we claim $X(X^r + T^p + \epsilon \sum a_j X^{r-1} T^j)$ is equisingular only if $a_j = 0$ for $j < p/r$. Since $ES(k[\epsilon])$ is an ideal, it suffices to show $X(X^r + T^p + \epsilon X^{r-1} T^s)$ is not equisingular, where s is the largest integer $< p/r$. This is proved by induction on pr, by blowing up and treating separately the cases $r < p - r$, $r = p - r$, and $r > p - r$. One uses that g singular, $(f + \epsilon h) \cdot g$ equisingular imply (by 3.2 and 3.4) that $f + \epsilon h$ is equisingular along the trivial section.

Concerning §10, we have proved that the cover $k[[X, Y]] \to T$ does lift (with prescribed equisingular deformation of the discriminant), provided T has a singularity for which the "geometric genus" equals the "arithmetic genus of the fundamental cycle" (e.g., if T has a rational singularity). For, it will follow automatically in this case that $\Gamma(\overline{Y}) \to \Gamma(Y)$ is surjective.

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