HOMOGENEITY AND EXTENSION PROPERTIES OF EMBEDDINGS
OF $S^1$ IN $E^3$

BY

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ABSTRACT. Two properties of embeddings of simple closed curves in $E^3$ are explored in this paper. Let $S^1$ be a simple closed curve and $f(S^1) = S$ an embedding of $S^1$ in $E^3$. The simple closed curve $S$ is homogeneously embedded or alternatively $f$ is homogeneous if for any points $p$ and $q$ of $S$, there is an automorphism $h$ of $E^3$ such that $h(S) = S$ and $h(p) = q$. The embedding $f$ or the simple closed curve $S$ is extendible if any automorphism of $S$ extends to an automorphism of $E^3$. Two classes of wild simple closed curves are constructed and are shown to be homogeneously embedded. A new example of an extendible simple closed curve is constructed. A theorem of H. G. Bothe about extending orientation-preserving automorphisms of a simple closed curve is generalized.

1. Introduction. A topological space $X$ is homogeneous if for any points $p$ and $q$ of $X$, there is a homeomorphism $b$ of $X$ onto itself such that $b(p) = q$. This property is a property of $X$ and does not depend on how $X$ is embedded in another space. In this paper we will discuss two properties, one very similar in description to homogeneity, but which do depend on an embedding.

Suppose $S^1$ is a simple closed curve and $f(S^1) = S$ is an embedding of $S^1$ in $E^3$. Since $S^1$ is homogeneous as a space, the following two definitions are non-trivial. An automorphism is an onto homeomorphism.

Definition 1.1. The simple closed curve $S$ is homogeneously embedded, or alternatively $f$ is homogeneous if for any points $p$ and $q$ of $S$, there is an automorphism $h$ of $E^3$ such that $h(S) = S$ and $h(p) = q$.

Definition 1.2. The embedding $f$ or the simple closed curve $S$ is extendible if any automorphism of $S$ extends to an automorphism of $E^3$.

An extendible simple closed curve is clearly homogeneously embedded. It follows that tame simple closed curves in $E^3$ are homogeneously embedded. Invertible knots [6] are extendible, and orientation-preserving homeomorphisms of all tame simple closed curves in $E^3$ extend to $E^3$. To have either property,
a wild simple closed curve has to be wild at each of its points. Thus the Fox-
Artin simple closed curve, for example, cannot have either property.

In this paper we will describe examples of simple closed curves and show
that they have one or both of the properties. More specifically, in §2 we will
describe two types of examples. In §3 we show that all of the examples are ho-
genously embedded and prove that homogeneously embedded simple closed curves
satisfy a stronger condition. Bothe has made a study of extendible simple closed
curves in [3]. In §4 we describe Bothe's examples and then construct a new
extendible simple closed curve. We also extend one of Bothe's major theorems.

Notation. If \( A \) is a set in \( E^3 \) then \( \overline{A} \) is the closure of \( A \). The usual metric
on \( E^3 \) is \( \rho \). If \( \epsilon \) is positive, then \( N(A, \epsilon) \) is the set of points in \( E^3 \) whose dis-
tance from \( A \) is less than \( \epsilon \).

Let \( Z \) be the integers and \( Z^+ \) the positive integers. Suppose \( g_i \) are automor-
phisms of \( E^3 \) for \( i \in Z^+ \). Then \( \prod_{i=1}^{n} g_i = g_n \circ g_{n-1} \circ \cdots \circ g_1 \). If the limit as
\( j \to \infty \) of \( f_j = \prod_{i=1}^{j} g_i \) exists, then we call the limit the infinite left product and
denote it \( \prod_{i=1}^\infty g_i \).

Suppose \( K_1 \) and \( K_2 \) are oriented knots. Then \( K_1 + K_2 \) is the oriented knot
such that there is a 2-sphere \( R \) and an arc \( \alpha \) in \( R \) such that:

1. \( R \cap K = \{x, y\} \ (x \neq y) \).
2. \( \alpha \) is an arc from \( x \) to \( y \).
3. \( (\text{Int} R \cap K) \cup \alpha \) is the knot \( K_1 \).
4. \( (\text{Ext} R \cap K) \cup \alpha \) is the knot \( K_2 \).

A simple closed curve \( S \) in \( E^3 \) is tame if there is an automorphism of \( E^3 \)
taking \( S \) onto a polyhedral simple closed curve. A simple closed curve \( S \) is wild
if it is not tame.

A monotone map is a map whose point inverses are compact and connected.

2. The examples. The examples we will give will all be toroidal, that is, the
intersection of a decreasing sequence of solid tori. We describe two types of
simple closed curves. Simple closed curves of Type 1 are actually a subset of
those of Type 2 but are sufficiently interesting to be described separately. We
show that Bing's example is a simple closed curve, and an analogous method
works for the other examples.

Examples of Type 1. Let \( K_i \) be a knot and \( n_i \) an integer for \( i \) a nonnegative
integer. Suppose \( n_i \geq 1 \) for \( i \) positive and \( n_0 > 1 \). We construct a simple closed
curve \( (K_i, n_i) \) as follows.

Let \( T_0 \) be a tame solid torus tied in a \( K_0 \) knot. We divide \( T_0 \) up into \( n_0 \)
solid cylinders \( \{C^0_j\} \). In each \( C^0_j \) we put \( n_1 \) solid cylinders \( \{C^1_j\} \) each with the
knot \( K_1 \) tied in it. This is done so that the union of the \( \{C^1_j\} \) is another solid
torus \( T_1 \), each \( C^1_j \) lies in some \( C^0_r \), and \( T_1 \cap C^0_j \cap C^0_{j+1} \) is a disk for \( j = 1, 2, \ldots \).
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Here and throughout the paper we take our subscript mod the appropriate integer. The torus $T_i$ is constructed in a similar fashion with the knot $K_i$ and integer $n_i$. If the diameters of $C_j$ go to zero as $i \to \infty$ then $\bigcap_{i=1}^{\infty} T_i$ is a simple closed curve and we call it $(K_i, n_i)$.

Examples of Type 2. For these examples we remove the restriction in the previous examples that each $T_i \cap C_j \cap C_{j+1}^{-1}$ is one disk. We assume for each $i$ that $T_i \cap C_j \cap C_{j+1}^{-1}$ is a finite number of disks. We still assume that $T_{i+1} \cap C_j^i$ are congruent for all $j$ and, in fact, the congruences between all adjacent $T_{i+1} \cap C_j^i$ are realized by a simple slide of $T_i$. To insure that we get a simple closed curve we also assume the diameters of the $\{C_j^i\}$ go to zero as $i \to \infty$ and that $T_{i+1} \cap C_j^i$ has only one spanning cell. That is, only one of the cells in $T_{i+1} \cap C_j^i$ hits both the left and right bases of $C_j^i$.

To better understand examples of Type 2 we refer to a well-known example of Bing. This example, $J$, was constructed by Bing in [2] as an example of a simple closed curve that pierces no disk. In this example $T_0$ is unknotted, and each stage is constructed in the same manner. The construction is illustrated in Figure 1.

![Figure 1](image)

To show $J$ is a simple closed curve we will exhibit a specific homeomorphism $f$ of $S^1$ onto $J$. This construction generalizes to simple closed curves of Type 2, and we use this fact in §3.

(2) This figure originally appeared in [2]. It is reprinted here with the permission of the author and the journal.
We construct our homeomorphism by defining it on a dense set of points. We think of $S^1$ as the unit circle in the plane, and we divide it into six equal parts with points $a_1^0, a_2^0, \ldots, a_6^0$. We put an orientation on each $T_i$ so that as $i$ increases the $C_i^0$ go around $T_0$ clockwise. For a given $C_j^i$ we can define a point on $C_j^{i-1} \cap C_j^i$ as follows. In $C_j^i$ there is a unique $C_j^{i+1}$ which hits $C_j^{i-1} \cap C_j^i$ and lies on the spanning cell. Similarly, there is a unique such cylinder in $C_r^{i+1}$. The intersection of the sequence of such cylinders is a point of $C_j^{i+1} \cap C_j^i$. Thus, for each of the $C_j^i$ we get a point $b_j^0$ in this manner. We define $f(a_j^0) = b_j^0$. If there are $m$ cylinders at the next stage we get points $b_1^1, b_2^1, \ldots, b_m^1$. Six of them will be the $b_j^0$'s. We break $S^1$ into $m$ pieces with points $a_1^1, a_2^1, \ldots, a_m^1$ with $a_0^1 = a_1^1$ and $a_r^1 = a_1^{1+r/m/6}$ for $r = 1, 2, \ldots, 5$. Equivalently, the $a_j^0$'s are evenly spaced among the $a_j^1$'s. The images under $f$ of the $a_j^1$'s will be the $b_j^1$'s in such a way that going around $S^1$ clockwise corresponds to going around $T_1$ clockwise. Renumber the $b_j^1$'s, if necessary, so that $f(a_j^0) = b_j^1$.

We repeat this process for all stages and we get a one-to-one function of a dense subset of $S^1$ onto a dense subset of $J$. This function is continuous on the union over all $i$ and $j$ of $a_j^i$ since, if two points are close enough, they will be forced into adjacent cylinders at a stage far out in the construction. Since the diameters of the cylinders go to zero, we get continuity. Thus we get a continuous function on all of $S^1$ by extending $f$ to the closure of the union of the $a_j^i$. This extension is one-to-one, since distinct points of $S^1$ are separated by an $a_j^i$ so they are mapped into different $C_j^i$'s. Thus $f$ is a one-to-one and continuous function of $S^1$ onto $J$ which implies it is a homeomorphism.

For examples of Type 1 and Type 2 to be of interest in this paper we need to know they are wild. The fact that many examples of these types are wild follows from work done by Edwards in [5]. We state his Corollary 5.

Corollary 2.1 (Edwards). Suppose that $\{B_n^1\}$ is a sequence of tame solid tori in $S^3$ with $B_{n+1} \subset \text{Int} B_n$ for $n \geq 1$, such that no two of the tori $\{B_n^1\}$ are concentric. If $\bigcap_{n=1}^{\infty} B_n$ is a simple closed curve $K$, then $K$ is wildly embedded in $S^3$.

3. Homogeneously embedded simple closed curves. In this section we show that simple closed curves of Type 2 are homogeneously embedded. Following this we show that if a simple closed curve $S$ is homogeneously embedded, then the automorphism $b$ taking $p$ to $q$ can be assumed orientation preserving on $S$.

Theorem 3.1. Simple closed curves of Type 2 are homogeneously embedded.

Proof. Suppose $S$ is a simple closed curve of Type 2. An immediate consequence of the construction of $S$ is that for any integer $m$, positive integer $i$, and
positive number \( \epsilon \), there is an automorphism \( h^i \) of \( E^3 \) such that:

1. \( h^i|E^3 - N(T_i, \epsilon) = \text{identity} \),
2. \( h^i(C_i^j) = C_i^{j+m} \) for all \( j \),
3. \( h^i(T_r) = T_r \) for \( r \geq i \).

This can be done with a slide along \( T_i \) fixed outside \( N(T_i, \epsilon) \). From (3) we know \( h^i(S) = S \).

As mentioned in §2 there is a homeomorphism \( f \) of \( S^1 \) onto \( S \) which takes the \( a^j_i \)'s to the \( b^j_i \)'s. We note that \( f^{-1}h^i \) takes \( a^j_i \) to \( a^j_{i+m} \) for all \( j \). In fact \( f^{-1}h^i \) is just a rotation of \( S^1 \) taking \( a^j_i \) to \( a^j_{i+m} \). This follows, since it takes all \( a^j_i \)'s to other \( a^j_i \)'s for \( r \geq i \).

Suppose we have distinct points \( p \) and \( q \) of \( S \). We must find an automorphism \( b \) of \( E^3 \) with \( b(S) = S \) and \( b(p) = q \). In \( S^1 \), \( f^{-1}(p) \) and \( f^{-1}(q) \) are limits of \( a^j_i \)'s for increasing \( i \), and we call them \( p^i \)'s and \( q^i \)'s for convenience. There is a rotation \( g_1 \) of \( S^1 \) which takes \( p^1 \) to \( q^1 \). There is a rotation \( g_2 \) of \( S^1 \) which takes \( g_1(p^2) \) to \( q^2 \). Similarly there is a \( g_i \) taking \( \Pi_{i=1}^{j-1} g_i(p^j) \) to \( q^j \). The infinite left product \( g = \Pi_{i=1}^{\infty} g_i \) exists and is an automorphism of \( S^1 \) taking \( f^{-1}(p) \) to \( f^{-1}(q) \).

Corresponding to each \( g_i \) there are \( h^i_m \)'s so that \( f g_i = h^i_m |S \). For each \( i \) we choose one such \( h^i_m \), call it \( h^i_m \), with the following properties. First by choosing \( \epsilon \) small enough we insure that \( h^i_m|E^3 - T_i \) is the identity. Secondly we choose the \( h^i_m \)'s so that the distance that cylinders slide under \( h^i \) goes to zero as \( i \) increases. Then \( b = \Pi_{i=1}^{\infty} h^i_m \) is an automorphism of \( E^3 \) such that \( b(S) = S \) and \( b(p) = q \). Clearly \( b \) is an automorphism on \( E^3 - S \), since each point of \( E^3 - S \) is moved finitely many times. We get that \( b \) is an automorphism of \( S \), since \( b|S = f g f^{-1} \). If \( b \) is continuous, we are done. This follows easily from the construction noting that points in the same cylinder at the \( i \)th stage stay in the same cylinder at the \( i \)th stage after applying \( b_1, b_2, \cdots, b_r \). Thus \( S \) is homogeneously embed-

We now prove a theorem that implies that homogeneously embedded simple closed curves satisfy a stronger condition.

**Theorem 3.2.** Suppose \( S \) is a homogeneously embedded simple closed curve in \( E^3 \). Then for any points \( p \) and \( q \) of \( S \) there is an automorphism \( b \) on \( E^3 \) such that \( b(S) = S \), \( b(p) = q \), and \( b \) is orientation preserving on \( S \).

**Proof.** We let \( A_p \) be the set of all points \( x \) of \( S \) such that there is an automorphism \( b \) of \( E^3 \) with \( b(p) = x \), \( b(S) = S \), and \( b \) orientation preserving on \( S \). Similarly, let \( B_p \) be the set such that \( b \) is orientation reversing on \( S \). Since \( S \) is homogeneous, \( A_p \cup B_p = S \). If \( B_p \) is empty we are done. Suppose \( f \) is an automorphism of \( E^3 \) with \( f(S) = S \) and such that \( f \) reverses the orientation on \( S \). Then \( f|S \) has a fixed point \( x \).
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We show \( x \in A_p \cap B_p \). Since \( x \in S \), \( x \) is either in \( A_p \) or \( B_p \). If \( x \in A_p \) then there is an orientation-preserving homeomorphism \( h \) such that \( h(p) = x \). Then \( fh \) is orientation reversing and \( fb(p) = x \) so \( x \in B_p \). If \( x \in B_p \) then there is an orientation-reversing homeomorphism \( b \) such that \( b(p) = x \). Then \( fb \) is orientation preserving and \( fb(p) = x \) so \( x \in A_p \). Thus \( x \in A_p \cap B_p \). Therefore there is an orientation-preserving \( h_1 \) and an orientation-reversing \( h_2 \) such that \( h_1(p) = b_1(p) = x \). So \( b_2^{-1} b_1(p) = p \) and \( b_2^{-1} b_1 \) is orientation reversing.

We can now show \( B_p \subseteq A_p \) and thus \( A_p = S \). Suppose \( g \) is any automorphism which reverses the orientation on \( S \). Then \( g(p) \) is in \( A_p \), since \( g(p) = g(b_2^{-1} b_1(p)) \) and \( g b_2^{-1} b_1 \) is orientation preserving.

Conjecture. Suppose \( S \) is a homogeneously embedded simple closed curve in \( E^3 \). Then for any points \( p \) and \( q \) of \( S \) there is an automorphism of \( E^3 \) such that \( h(S) = S \), \( h(p) = q \), and \( h \) is orientation preserving on \( E^3 \). One might also require \( b \) to be orientation preserving on \( S \) as in Theorem 3.2.

4. Extendible simple closed curves. In [3], Bothe constructs examples of extendible simple closed curves. However, he uses the term homogeneous for this property. To prove his examples are extendible, he needs the following theorem.

Theorem 4.1 (Bothe). Suppose \( S \) is a simple closed curve in \( E^3 \) with the following property: For any two distinct points \( p \) and \( q \) of \( S \) and an arc \( B \) between them on \( S \), and positive number \( \epsilon \), there is an automorphism \( h \) of \( E^3 \) such that:

1. \( h(S) = S \),
2. \( \rho(b(p), b(q)) < \epsilon \),
3. \( h(x) = x \) for all \( x \in E^3 - N(B, \epsilon) \).

Then all orientation-preserving automorphisms of \( S \) extend to automorphisms of \( E^3 \).

In this section we describe a class of examples which satisfy the hypothesis of Theorem 4.1. We then describe Bothe’s examples and give a new example of an extendible simple closed curve. Finally, we prove a theorem analogous to Theorem 4.1 but for monotone maps of \( S \).

Theorem 4.2. Suppose \( K \) is a knot and \( K_i = K \) for \( i > m \). Suppose \( n_i \) are integers greater than one for \( i \geq 0 \). Then all orientation-preserving automorphisms of \( (K_i, n_i) \) extend to automorphisms of \( E^3 \).

Proof. Let \( S \) be some \( (K_i, n_i) \) as above. We will show that \( S \) satisfies the hypothesis of Theorem 4.1. Let \( p, q, \epsilon, B \) be as in Theorem 4.1. Let \( B' \) be an arc of \( S \) with endpoints \( p' \) and \( q' \) such that \( B \subseteq B' \subseteq S \cap N(B, \epsilon) \). We begin by
building a chain of solid cylinders from \( p' \) to \( q' \) along \( B' \). Let \( r > m \) be so large that the diameter of each \( C_j^r \) is less than \( \epsilon/2 \). We begin our chain with all cylinders at the \( r \)th stage which contain no points of \( S - B \). To these we add cylinders of \( \{C_j^{r+1}\} \) which contain no points of \( S - B \) and are not contained in cylinders already in the chain. If we continue this process for later stages, we get the required chain.

Naming one of the largest cylinders \( C_0 \), we number the cylinders going to \( p' \) \( C_{-1}, C_{-2}, C_{-3}, \ldots \) and those going to \( q' \) \( C_1, C_2, C_3, \ldots \). If \( F = \bigcup_{i \in \mathbb{Z}} C_i \cup \{p', q'\} \) then \( F \) is a three cell in \( N(B', \epsilon/2) \). There is an automorphism \( g \) of \( E^3 \) with the following properties:

1. \( g(S) = S \)
2. \( g(C_i) = C_{i+1} \)
3. \( g = \text{identity on } (E^3 - N(F, \epsilon/2)) \cup \overline{S - B'} \).

The automorphism \( g \) slides the boundary of \( F \) along leaving \( p' \) and \( q' \) fixed. The interior of each \( C_i \) is moved to that of \( C_{i+1} \) with the possible necessity of shrinking or expanding. Note this process does rely on the fact that we have the same knot from some stage on. By repeated use of \( g \) we can take the \( C_i \) containing \( p \) to the one containing \( q \) and thus bring the image of \( p \) within \( \epsilon \) of \( q \). This composition is the automorphism necessary to show \( S \) satisfies the hypothesis of Theorem 4.1.

Bothe's examples. Let \( N \) be a knot and \( N' \) the reflection of \( N \) through a plane. Let \( R_i = N + N' \) for all \( i > 0 \). That is, \( R_i \) is the knot \( N \) followed by its reflection. Suppose \( R_0 \) is the trivial knot. Bothe shows that \( S = (R_i, n_i) \) is extendible. By Theorem 4.2 any orientation-preserving automorphism of \( S \) extends to an automorphism of \( E^3 \). The simple closed curve \( S \) is constructed in such a way that there is a canonical automorphism \( R \) which reflects \( E^3 \) about a plane and takes \( S \) to \( S \). The automorphism \( R \) also reverses the orientation on \( S \). Suppose we have any orientation-reversing homeomorphism of \( S, g \). Then \( Rg \) is orientation preserving on \( S \) so it extends to an automorphism \( G \) of \( E^3 \). Thus \( R^{-1}G \) extends \( g \), since \( R^{-1}G|S = R^{-1}Rg = g \). This shows all automorphisms of \( S \) extend and \( S \) is extendible.

We now describe a new example of an extendible simple closed curve. The first two stages are illustrated in Figure 2.

**Theorem 4.3.** Suppose \( L_0 \) is the trivial knot and \( L_i \) is the trefoil knot for all \( i > 0 \) and \( n_0 \) is even. Then if \( S \) is of the form \( (L_i, n_i) \) then \( S \) is extendible.

**Proof.** From Theorem 4.2 we know all orientation-preserving automorphisms of \( S \) extend to \( E^3 \). As in Bothe's examples we must find one automorphism of \( E^3 \) which takes \( S \) onto itself and reverses the orientation on \( S \). To find such an automorphism we note a particular property of the trefoil knot. If one takes a trefoil
knot plus a trefoil knot and an axis through two points that separate the knots, then an appropriate rotation about the axis takes the simple closed curve to itself and reverses its orientation. Since $n_0$ is even, we can do the same thing for $S$ using a simple rotation. This is the necessary automorphism of $E^3$ which reverses the orientation on $S$, so $S$ is extendible.

The discussions in §§3 and 4 lead naturally to the question of which homogeneously embedded simple closed curves are not extendible.

We mention the following conjectures.

**Conjecture.** Let $K_1$ be the square knot or another appropriate knot. Then no orientation-reversing automorphism of $(E^3, n_1)$ extends to an automorphism of $E^3$.

**Conjecture.** Bing's simple closed curve is not extendible.

**Conjecture.** Let $f$ be an orientation-preserving automorphism of Bing's example which extends to an automorphism of $E^3$. Then $f$ is determined by where any one point goes under $f$. That is to say, only "rigid" automorphisms as described in §3 extend.

We now generalize Theorem 4.1.

**Theorem 4.4.** Suppose $S$ is a simple closed curve in $E^3$ with the following property: For any two distinct points $p$ and $q$ of $S$ and any arc $B$ between them on $S$, and any positive number $\epsilon$, there is an automorphism $b$ of $E^3$ such that:

1. $b(S) = S$,
2. $\rho(b(p), q) < \epsilon$,
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(3) $b(x) = x$ for all $x \in E^3 - N(B, \epsilon)$.

Then any orientation-preserving monotone map of $S$ onto $S$ extends to $F$ taking $E^3$ to $E^3$ and such that $F/E^3 - S$ is a homeomorphism.

Before proving Theorem 4.4 we prove two necessary lemmas. We use $p$, $q$, $\epsilon$, $B$ as stated in Theorem 4.2. If $a$ and $b$ are points of $S$ then $ab$ is the appropriate arc between $a$ and $b$ on $S$ as determined by the context.

Lemma 4.5. Suppose $S$ is a simple closed curve in $E^3$ satisfying the hypothesis of Theorem 4.4. Then there is an automorphism $b$ of $E^3$ such that:

(1) $b(S) = S$,
(2) $b(p) = q$,
(3) $b(x) = x$ for $x \in E^3 - N(B, \epsilon)$.

Proof. We construct $b$ satisfying (1) and (2). This is done so that it is clear that a slight modification insures $b$ satisfies (3) also.

We put an orientation on $S$ so that, with respect to $B$, $p$ is to the "left" of $q$. Let $M_i = N(p, 1/z^i)$ and $N_i = N(q, 1/z^i)$. We pick sequences $\{a^i\}$ and $\{b^i\}$ so that the following hold:

(1) $\{a^i\}$ are to the left and $\{b^i\}$ to the right of $p$,
(2) $a^i p$ and $p b^i$ are in $M_i$,
(3) $a^i$ separates $a^i_{i-1}$ and $b^i$ separates $p b^i_{i-1}$.

Since $S$ is a simple closed curve satisfying the hypothesis of Theorem 4.4, there is an automorphism $g_i$ of $E^3$ with $b(S) = S$ which fixes $q$ but takes $a^i$ so close to $q$ that $g_i(a^i b^i) \subseteq N_i$. We use the hypothesis again leaving $g_i(a^i)$ fixed, we drag $g_i(b^i)$ to the opposite side of $q$ keeping the condition that $g_i(a^i b^i) \subseteq N_i$. We call the composition of the two automorphisms $b^i$.

We repeat the process to get $b_2$ which takes $b_1(a_2) b_1(b_2)$ into $N_2$ with $b_1(a_2)$ and $b_1(b_2)$ on opposite sides of $q$. Since $b_1(a_2) b_1(b_2)$ is in $N_1 \cap b_1(M_1)$ we can require that $b_2$ be fixed off of $N_1 \cap b_1(M_1)$. Similarly at the $i$th stage we get an automorphism $b_i$ such that if $f_i = \prod_{j=1}^{i} h_j$ we have

(1) $f_i(a^i b^i) \subseteq N_i$,
(2) $f_i(a^i)$ is to the left of $q$,
(3) $f_i(b^i)$ is to the right of $q$,
(4) $f_i$ is identity on $E^3 - (f_{i-1}(M_{i-1}) \cap N_{i-1})$.

We must take the images of $a^i$ and $b^i$ to opposite sides of $q$ to insure that at the following stage $q \in b_i(a^i) b_i(b^i)$ and thus $q \in b_i(M_i) \cap N_i$.

We now show that $b = \prod_{i=1}^{\infty} b_i$ is an automorphism of $E^3$ satisfying (1) and (2) of Lemma 4.5. Since each point of $E^3 - p$ is not in some $M_i$, it can only be moved by finitely many $b_i$ by condition (4) above. Thus $b$ is a homeomorphism.
of \( E^3 - p \) with image in \( E^3 - q \). Clearly \( S \) goes to \( S \) and \( p \) goes to \( q \). We must show that \( b \) is continuous at \( p \). This follows, since \( b^{-1}(N_r(\gamma)) = N_r(\gamma) \) for all \( i > r \) and the diameters of the \( N_r \) go to zero. Thus (1) and (2) are satisfied. We note that by starting our construction with a sufficiently high \( i \) we could get condition (3).

**Lemma 4.6.** Suppose \( S \) is a simple closed curve in \( E^3 \) satisfying the hypothesis of Theorem 4.4. Suppose \( ab \) is an arc of \( S \) and \( \{x_i\}, \{y_i\}, i = 1, 2, \ldots, n \), are points of \( \overline{ab} \) with \( x_i \neq y_j \) for \( i \neq j \), and so that the orientations of the points are \( a, x_1, x_2, x_3, \ldots, x_n, b \) and \( a, y_1, y_2, \ldots, y_n, b \). If \( A \) is an arc in the interior of \( \overline{ab} \) containing \( \{x_i\} \) and \( \{y_i\} \) and \( \epsilon \) is a positive number, then there is an automorphism \( b \) of \( E^3 \) such that:

1. \( b(S) = S \),
2. \( b(x_i) = y_i \),
3. \( b = \) identity on \( E^3 - N(A, \epsilon) \).

**Proof.** This lemma follows with repeated application of Lemma 4.5, never moving anything in \( E^3 - N(A, \epsilon) \). We first take \( x_1 \) to \( y_1 \) on a small neighborhood of \( x_1y_1 \). By our orientation condition, \( y_1 \) cannot separate the image \( x_2 \) from \( y_2 \) in \( \overline{ab} \), so we can take the image of \( x_2 \) to \( y_2 \) keeping \( y_1 \) fixed. We repeat \( n \) times to prove the lemma.

**Corollary 4.7.** Suppose \( S \) is as in Lemma 4.6 and \( S \) is broken up into \( n \) disjoint arcs by \( n \) points \( c_1, c_2, \ldots, c_n \). Suppose there are \( \{x_i\} \) and \( \{y_i\} \) as in Lemma 4.6 on each arc. Then there is an automorphism \( b \) such that:

1. \( b(S) = S \),
2. \( b(x_i) = y_i \) for all arcs,
3. \( b = \) identity on \( (E^3 - N(S, \epsilon)) \cup \{c_n\} \).

This follows by using Lemma 4.6 on the \( n \) arcs taking care not to let moves on one interfere with those on another.

**Proof of Theorem 4.4.** Suppose \( f \) is a monotone map of \( S \) onto \( S \). We must extend \( f \) to a map of \( E^3 \) onto \( E^3 \) such that \( F/E^3 - S \) is a homeomorphism. Since \( f \) is monotone, point inverses of \( f \) are compact and connected and thus must be points or closed arcs. Since \( S \) is separable, at most a countable number are arcs. We call them \( \{a_i\} \) and their images under \( f, \{a_i\} \). Since \( \{a_i\} \) is countable, \( \bigcup_{i=1}^{\infty} a_i \) contains no open subset of \( S \). Thus for each integer \( j \) we can pick a finite number of points \( \{b^j_n\}, n = 1, 2, \ldots, n(j) \), such that:

1. \( b^j_n \in S - \{a_i\} \forall j \) and \( n = 1, 2, \ldots, n(j) \),
2. \( \{b^j_n\} \) is oriented \( b^j_1, b^j_2, \ldots, b^j_{n(j)} \).
(3) \( b^j_n \in \{ b^{j+1}_n \} \) \( \forall j \) and \( n = 1, 2, \ldots, n(j) \),

(4) The diameters of the components of \( K - \{ b^j \} \) are less than \( 1/2^j \).

To simplify notation we let \( x^j_n = f^{-1}(b^j_n) \) and \( N_i = N(S, 1/i) \). For each \( j \) we define an automorphism \( f_j \) of \( E^3 \) such that if \( F_j = \Pi_{r=1}^j f_r \) and \( F_0 = \) identity, we have

1. \( f_j(S) = S \).
2. \( f_j(F_{j-1}(x^j_n)) = b^j_n \) for \( n = 1, 2, \ldots, n(j) \).
3. \( f_j \) moves no point more than \( 1/2^{j-1} \).
4. \( f_j \) is the identity on \( E^3 - F_{j-1}(N_j) \).

The existence of the \( f_j \)'s follows from Corollary 4.7. The arcs we use at the \( j \)th stage are

\[
\frac{b_1^{j-1}b_2^{j-1}}{b_1^{j-1}b_2^{j-1}}, \ldots, \frac{b_n^{j-1}b_1^{j-1}}{b_n^{j-1}b_1^{j-1}}.
\]

We take \( F_{j-1}(x^j_n) \) to \( b^j_n \) in these arcs noting that the endpoints must be and are kept fixed. To get (3) we use the fact that the \( b^j \) break \( S \) into components of diameter less than \( 1/2^j \).

We now show \( F = \Pi_{i=1}^\infty f_i \) is a monotone map extending \( f \) and is a homeomorphism on \( E^3 - S \). Condition (3) insures that \( F \) is well defined and continuous. Condition (4) insures that points of \( E^3 - S \) are moved by finitely many of the \( f_i \), so \( F \) is a homeomorphism on \( E^3 - S \). In fact, \( F \) takes \( E^3 - S \) to itself since

\[
F(E^3 - S) = F \left( \bigcup_{j=1}^{\infty} (E^3 - N_j) \right) = E^3 - \bigcup_{j=1}^{\infty} F(N_j) = E^3 - S.
\]

We need only show that \( F \) extends \( f \). By the construction we have \( F(x^j_n) = F_j(x^j_n) = b^j_n \). The \( x^j_n \) are dense in \( S - \bigcup_{i=1}^{\infty} \alpha_i \) so \( F/S - \bigcup_{i=1}^{\infty} \alpha_i = f/S - \bigcup_{i=1}^{\infty} \alpha_i \). Furthermore, the \( \alpha_i \)'s are mapped to points, since the diameters of their images go to zero. By the continuity of \( F \) and \( f_i \), \( F/\alpha_i = f/\alpha_i \). Thus \( F/S = f \).

Remark. By using a sufficiently small \( \varepsilon \) when applying Corollary 4.7 we can add the condition that \( F \) may be chosen so that it does not move points more than a given distance away from \( S \).

REFERENCES


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