ON A PROBLEM OF GRONWALL FOR BAZILEVIČ FUNCTIONS

BY

JOHN L. LEWIS

ABSTRACT. Let \( B(\alpha, \beta) \), a positive, \( \beta \) real, denote the class of normalized univalent Bazilevič functions in \( K = \{ z : |z| < 1 \} \) of type \( \alpha, \beta \). Let \( B = \bigcup_{\alpha, \beta} B(\alpha, \beta) \). Let \( a, 0 \leq a \leq 2, \) and \( \alpha, 0 < \alpha < \infty \), be fixed and suppose that \( f(z) = z + az^2 + \cdots \) is in \( B(a, 0) \). In this paper for given \( z_0 \in K \), the author finds a sharp upper bound for \( |T_{zn}| \). Also, a sharp asymptotic bound is obtained for \( (1 - r)^2 \max_{|z| = r} |f(z)| \). Finally, a sharp asymptotic bound is found for \( (1 - r)^2 \max_{|z| = r} |f(z)| \) when \( f \) is in \( B \) with second coefficient \( a \).

1. Introduction. Let \( S \) denote the class of univalent functions \( f \) in \( K = \{ z : |z| < 1 \} \) with the normalization \( f(0) = 0, f'(0) = 1 \). Let \( M(r, f) = \max_{|z| = r} |f(z)|, 0 < r < 1 \). Then in [3] Gronwall proposed the following problem.

Problem 1. Given \( a, 0 < a < 2, \) and \( r_0, 0 < r_0 < 1, \) if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( S \) with \( |a_2| = a \), find the maximum of \( M(r_0, f) \).

In [6] Jenkins solved Problem 1 using quadratic differentials. The maximum is attained by a unique function \( G = G(\cdot, a, r_0) \) in \( S \) with second coefficient \( a \) and \( M(r_0, G) = G(r_0, a, r_0) \). \( G \) maps \( K \) onto a region whose complement is a forked slit, symmetric about the real axis. Depending on \( a \) and \( r_0 \), the handle of the fork is either (a) on the negative real axis, (b) at \( \infty \), or (c) along the positive real axis. As \( r_0 \to 1 \) Jenkins showed

\[
(1.1) \quad (1 - r_0)^2 M(r_0, G) \to 4b^{-2} \exp(2 - 4b^{-1}), \quad b = 2 - (2 - a)^{1/2}.
\]

Let \( f \) be as in Problem 1. Then Hayman (see [4, Theorem 1.5]) has shown that \( r^{-1}(1 - r)^2 M(r, f) \) is a decreasing function of \( r, 0 < r < 1 \). It follows from this fact and (1.1) that

\[
(1.2) \quad \lim_{r \to 1} (1 - r)^2 M(r, f) \leq 4b^{-2} \exp(2 - 4b^{-1}), \quad b = 2 - (2 - a)^{1/2},
\]

with equality holding for some \( f \) in \( S \) with \( |a_2| = a \).
In this paper we consider Problem 1 in certain classes of univalent functions. More specifically, for given $\alpha$, $0 < \alpha < \infty$, and $\beta$, $-\infty < \beta < \infty$, let $B(\alpha, \beta)$ denote the normalized Bazilević functions of type $\alpha, \beta$ (see [1]). That is, 

$$f \in B(\alpha, \beta) \text{ if } f(0) = 0, \quad f'(0) = 1, \text{ and}$$

$$f(z) = z \left[ (\alpha + i\beta)z^{-\alpha-i\beta} \int_0^z \left( \frac{\phi(\zeta)}{\zeta} \right)^{\alpha} P(\zeta)\zeta^{\alpha+i\beta-1} d\zeta \right]^{1/(\alpha+i\beta)}$$

where $z \in K - \{0\}$, $\phi$ is starlike univalent with $\phi(0) = 0$, $|\phi'(0)| = 1$, and $P$ has positive real part with $|P(0)| = 1$. The symbol $(\phi(\zeta)/\zeta)\alpha$ denotes an analytic $\alpha$ power of $\phi(\zeta)/\zeta$ in $K$. All other powers inside the brackets are principal values, and the integral is taken along the line segment from 0 to $z$. The power outside the brackets is the analytic $(\alpha + i\beta)^{-1}$ power of the function inside the brackets which approaches 1 as $z \to 0$. We note that $B(1, 0)$ is the class of normalized close to convex functions. We put $B = \bigcup_{\alpha, \beta} B(\alpha, \beta)$ and call $f \in B$ a Bazilević function.

Functions $f \in B$ have been shown to be univalent by Bazilević (see also [7], [9]). They are the largest class of univalent functions given by an explicit representation formula. We remark that the extremal functions $G(\cdot, a, r_0)$ with image domains as in (a) or (c) are not Bazilević. This can be seen by using a geometric characterization of $B(\alpha, \beta)$ due to Sheil-Small (see [9, Theorem 2]). Therefore it is worthwhile to consider Problem 1 in $B$. To do this we first suppose that $\alpha$ is fixed, $0 < \alpha < \infty$, and $\beta = 0$.

For given $\alpha$, $0 \leq \alpha \leq 2$, we introduce the functions $F(\cdot, \sigma)$, $1 \leq \sigma \leq 2\alpha + 1$, defined by

$$F(z, \sigma) = \left\{ \alpha \int_0^z \zeta^{\alpha-1}(1 + 2c\zeta + \zeta^2) \right\}^{1/\alpha} \left( 1 - \zeta \right)^{2\alpha+2-\sigma}(1 + \zeta)^{\sigma}$$

where $z \in K$ and

$$2c = (\alpha + 1)\alpha - 2\alpha - 2 + 2\sigma.$$

Let

$$r_1 = \max \{\alpha - \frac{1}{2}(\alpha + 1)a, 1\}, \quad r_2 = \max \{\alpha + 1 - \frac{1}{2}(\alpha + 1)a, 1\}.$$

For $r_1 \leq \sigma \leq r_2$ we observe that $F(\cdot, \sigma)$ is in $B(\alpha, 0)$ and $F(z, \sigma) = z + az^2 + \cdots$. Then we shall prove

Theorem 1. Let $a$ and $r_0$ be fixed numbers where $0 \leq a \leq 2$ and $0 < r_0 < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $B(\alpha, 0)$ with $|a_2| = a$. Then
Grönewall's Problem for Bazilevič Functions

(1.7) \[ M(r_0, f) \leq \max_{r_1 < \sigma < r_2} F(r_0, \sigma). \]

If \( a \geq 2a/(\alpha + 1) \), then \( r_2 = 1 \) and \( M(r_0, f) \leq F(r_0, 1) \). Equality holds if and only if for some real \( \theta \), \( f(z) = e^{-i\theta}F(e^{i\theta}z, 1) \), \( z \in K \). As \( r \to 1 \),

\[
\lim_{r \to 1} (1 - r)^2 M(r, f) \leq \xi(\alpha) = [\max\{0, \frac{1}{2}(2(1 - \alpha) + (\alpha + 1)\alpha)\}]^{1/\alpha}.
\]

If \( \xi(\alpha) > 0 \), then equality holds if and only if for some real \( \theta \) we have \( f(z) = e^{-i\theta}F(e^{i\theta}z, 1), z \in K \).

From Theorem 1 we see that if \( a \geq 2a/(\alpha + 1) \), then the maximum in (1.7) is attained by \( F(\cdot, 1) \) independent of \( r_0 \). However if \( 0 \leq a < 2a/(\alpha + 1) \), then it follows from our proof (see (2.4)) that for \( r_0 \) near zero the maximum in (1.7) is attained only for \( \sigma \) near \( r_2 \), while for \( r_0 \) near 1 the maximum is attained only for \( \sigma \) near \( r_1 (r_1 < r_2) \). In general for a fixed \( r_0 \) and \( a \), the question of whether the maximum in (1.7) occurs for exactly one value of \( \sigma \) is still open.

Most of the proof of Theorem 1 (§2) is based upon simple properties of subordination and the observation that it suffices to consider functions with real coefficients. Similar properties of subordination have been used by Grönewall [3] and Finkelstein [2] to solve Problem 1 for starlike univalent functions. However, we must use a more difficult argument to prove uniqueness in (1.8) for \( \xi(\alpha) > 0 \) (see §3).

In §4 we first show if \( f \in B(\alpha, \beta) \) and \( \beta \neq 0 \), then \( M(r, f) = o((1 - r)^{-2}) \) as \( r \to 1 \). We then obtain easily from Theorem 1:

**Theorem 2.** Let \( a \) be fixed, \( 0 \leq a \leq 2 \). Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in \( B \) with \( |a_2| = a \). Then

\[
\lim_{r \to 1} (1 - r)^2 M(r, f) \leq \max_{0 < \alpha < \infty} \xi(\alpha) = \xi(\alpha_0),
\]

where \( \alpha_0 \) is the root of the equation

\[
\alpha(a - 2) - [2(1 - \alpha) + a(1 + \alpha)] \log[\frac{1}{2}(2(1 - \alpha) + a(1 + \alpha))] = 0.
\]

Equality holds only if for some real \( \theta \), \( f(z) = e^{-i\theta}F(e^{i\theta}z, 1), z \in K \), where \( F(\cdot, 1) \) is defined as in (1.4) with \( \alpha = \alpha_0 \).

We remark that (1.10) implies \( a_0 \), for \( 0 \leq a < 2 \), is an increasing function of \( a \) and \( a_0 \to +\infty \) as \( a \to 2 \). Also, we remark that the extremal function \( F(\cdot, 1) \) in Theorem 2 for \( 0 \leq a < 2 \) is in \( B(\alpha, \beta) \) only when \( \alpha = \alpha_0 \) and \( \beta = 0 \), since otherwise it could not be extremal. Finally we remark for \( f \) as in Theorem 2 that \( \lim_{n \to \infty} a_n / n \leq \xi(\alpha_0) \). This inequality follows directly from Theorem 2 and a theorem of Hayman (see [4, Theorem 5.7]).
In §5 we compare the bound given by (1.9) with the bound given by (1.2) for different values of \( a \). We also discuss mapping properties of the function \( F(\cdot, 1) \) for \( a = \frac{1}{2}, 1, 2, \) and \( 0 < a < 2 \) in this section.

2. Proof of Theorem 1. The proof of Theorem 1 is based upon the following lemmas (see Finkelstein [2] for similar lemmas).

**Lemma 1.** Let \( \omega \) be analytic in \( K \) with \( |\omega(z)| \leq 1, z \in K, \omega(0) = 0 \) and \( \omega'(0) = b, -1 \leq b \leq 1 \). If \( \omega(r) \) is real, \( 0 < r < 1 \), then \( \omega(r) \leq r(r + b)/(1 + rb) \), \( 0 < r < 1 \), with equality holding only for \( \omega(z) = z(z + b)/(1 + bz) \), \( z \in K \).

**Proof.** Apply Schwarz’s lemma to the function \( \left[ \omega(z) - bz \right]/\left[ z - \omega(z) \right] \), \( z \in K \).

**Lemma 2.** Let \( Q \) be an analytic function in \( K \) with positive real part and \( Q(0) = 1, Q'(0) = 2b, -1 \leq b \leq 1 \). If \( Q(r) \) is real for \( 0 < r < 1 \), then
\[
Q(r) \leq \frac{1 + 2br + r^2}{1 - r^2}, \quad 0 < r < 1.
\]
Equality holds only for \( Q(z) = (1 + 2bz + z^2)/(1 - z^2) \), \( z \in K \).

**Proof.** From the hypotheses on \( Q \) we have \( Q = (1 + \omega)/(1 - \omega) \), where \( \omega \) is as in Lemma 1. Applying Lemma 1 and using the fact that the function \( x \rightarrow (1 + x)/(1 - x) \) is increasing for \( -1 \leq x < 1 \), we get Lemma 2.

**Lemma 3.** Let \( \psi \) be starlike univalent in \( K \) with \( \psi(0) = 0, \psi'(0) = 1, \) and \( \psi''(0) = 4b, -1 \leq b \leq 1 \). If \( \psi(r) \) is real, \( 0 < r < 1 \), then
\[
\psi(r) \leq r(1 - r)^{-1} - b(1 + r)^{b-1}, \quad 0 < r < 1.
\]
Equality holds only for \( \psi(z) = z(1 - z)^{-1} - b(1 + z)^{b-1} \).

**Proof.** From the hypotheses on \( \psi \) we have \( z\psi'(z)/\psi(z) = Q(z) \), where \( Q \) is as in Lemma 2. Applying Lemma 2 and integrating we get Lemma 3.

We now prove (1.7) of Theorem 1. Let \( a \) and \( r_0 \) be fixed numbers where \( 0 < a < 2 \) and \( 0 < r_0 < 1 \). Let \( f \in B(a, 0) \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) with \( |a_2| = a \). By making a rotation, if necessary, we may assume that \( |f(r_0)| = M(r_0, f) \). Next suppose that \( f \) has the form (1.3) with \( \beta = 0 \), where \( \phi \) and \( P \) are as defined there. We put
\[
\psi'(z) = z \left[ \frac{\phi(z)}{z} \cdot \overline{\phi(z)} \right]^{\frac{1}{2}}, \quad Q(z) = [P(z) \cdot \overline{P(z)}]^{\frac{1}{2}},
\]
\[
g(z) = z \left[ \alpha z^{-a} \int_0^z \left( \frac{\psi'(\zeta)}{\zeta} \right)^{a} Q(\zeta)^{a-1} d\zeta \right]^{1/\alpha},
\]
when \( z \in K \). Here the analytic \( \frac{1}{2} \) powers are \( 1 \) at \( z = 0 \). It is easily checked that \( \psi \) is starlike univalent with \( \psi(0) = 0, \psi'(0) = 1 \), and that \( Q \) has positive real part with \( Q(0) = 1 \). From this fact we see that \( g \in B(\alpha, 0) \) provided the powers in the expression for \( g \) are given the same interpretation as in (1.3). Since 
\[
\psi(\zeta) = |\phi(r)| \quad \text{and} \quad Q(\zeta) = |P(r)|, \quad 0 \leq r < 1,
\]
we have the inequality
\[
M(r_0, f) \leq g(r_0).
\]
(2.1)

Also, \( g(z) = z + \text{Re} a_2 z^2 + \cdots \) and \( Q, \psi \), have real coefficients.

From the above remarks we see that \( Q \) and \( \psi \) satisfy the hypotheses of Lemmas 2 and 3. Applying these lemmas we obtain for some \( \sigma \) that
\[
g(r_0) \leq \alpha \int_0^{r_0} \frac{(1 + s^2 + 2c_0 s)s^{\alpha-1}}{(1 - s)^{2a+2-\sigma}(1 + s)^{\sigma}} ds.
\]
(2.2)

where \( 2c_0 = (a + 1) \text{Re} a_2 + 2\sigma - 2a - 2, \ 1 \leq \sigma \leq 2a + 1, \) and \( -1 \leq c_0 \leq 1 \). Let \( F(\cdot, \sigma) \) and \( c \) be as in (1.4) and (1.5). Then since \( c_0 \leq c \), it follows from (2.1) and (2.2) that
\[
M(r_0, f) \leq g(r_0) \leq F(r_0, \sigma).
\]
(2.3)

Let \( r_1 \) and \( r_2 \) be as in (1.6). Then if \( \sigma > r_2 \), we consider \( F(r_0, \cdot) \) as a function of \( \sigma \) for \( r_2 < \sigma \leq 2a + 1 \). We obtain since \( c > 0 \) that
\[
\frac{\partial F^\alpha}{\partial \sigma} (r_0, \sigma) = \alpha \int_0^{r_0} \frac{(1 + s^2 + 2cs)s^{\alpha-1}}{(1 - s)^{2a+2-\sigma}(1 + s)^{\sigma}} \left[ \frac{2s}{(1 + s^2 + 2cs)} + \log \left( \frac{1 - s}{1 + s} \right) \right] ds < 0,
\]
(2.4)

for \( r_2 < \sigma \leq 2a + 1 \). Hence \( F(r_0, \sigma) < F(r_0, r_2) \), \( r_2 < \sigma \leq 2a + 1 \).

If \( 1 \leq \sigma < r_1 \), then \( c_0 \leq c < -1 \), which is a contradiction to the fact that \(-1 \leq c_0 \leq 1\). We conclude that (1.7) is true.

If \( \alpha > 2a/(a + 1) \), then \( r_2 = 1 \), and from (1.7) we have \( M(r_0, f) \leq F(r_0, 1) \).

From Lemmas 1–3 and (2.4), we deduce that equality can hold only for rotations of \( F(\cdot, 1) \).

Next we prove (1.8). First suppose that \( \alpha < (a + 1)\alpha/2 < 1 \). Then from (1.5) we see that \( c > (a + 1)\alpha/2 - \alpha > -1 \) for \( r_1 = 1 \leq \sigma \leq r_2 \). It follows from this inequality that the integrand in (2.4) is negative on \([s_0, 1]\) provided \( s_0 \) is near enough 1. Also the choice of \( s_0 \) does not depend on \( \sigma, 1 \leq \sigma \leq r_2 \). Hence there
exists a positive constant, $A$, which does not depend on $\sigma$, such that
\[
\max_{1 \leq r \leq 2} F(r, \sigma) \leq A + F(r, 1), \quad 0 < r < 1.
\]
Using this inequality, (1.7), and l'Hôpital's rule we get
\[
\lim_{r \to 1} (1 - r)^2 M(r, f) \leq \lim_{r \to 1} (1 - r)^2 F(r, 1) = \left[ \lim_{r \to 1} (1 - r)^2 a F^2(r, 1) \right]^{1/\alpha}.
\]

(2.5)

If $\alpha - (\alpha + 1)\alpha/2 \geq 1$, we note for fixed $\sigma$ and $r_0$ that the value of $F(r_0, \sigma)$ increases with $\alpha$, as follows from (1.5). Next, given $a_1$, $2 \geq a_1 > 2(\alpha - 1)/(\alpha + 1)$ we replace $a$ by $a_1$ in (1.4). We obtain for fixed $\sigma$, $1 \leq \sigma \leq 2\alpha + 1$, a new function $G = G(\sigma, \sigma)$ in $K$. Furthermore, from the above remark and (2.3) it is clear that $M(r_\sigma, f) \leq F(r_\sigma, \sigma) \leq G(r_\sigma, \sigma)$ where $1 \leq \sigma \leq 2\alpha + 1$. Using this inequality and arguing as in the proof of (2.5) we find that
\[
\lim_{r \to 1} (1 - r)^2 M(r, f) \leq \left[ \max_{1 \leq \sigma \leq 2\alpha + 1} (1 - r)^2 a F^2(r, 1) \right]^{1/\alpha}.
\]

Letting $a_1 \to 2(\alpha - 1)/(\alpha + 1)$, we get $\lim_{r \to 1} (1 - r)^2 M(r, f) = 0$. Hence (1.8) is true.

3. A uniqueness proof. It remains to show for $\xi(\alpha) > 0$ that equality holds in (1.8) only for $f(z) = e^{-i\theta} F(e^{i\theta} z, 1)$, $z \in K$, $\theta$ real. This can be shown by the following argument. Suppose that $\lim_{r \to 1} (1 - r)^2 M(r, f) = \xi(\alpha)$. Then by Hayman (see [4, Theorem 5.6]) there exists $\psi$, $0 \leq \psi < 2\pi$, such that
\[
(3.1) \quad \lim_{r \to 1} (1 - r)^2 |f(re^{i\psi})| = \xi(\alpha).
\]
By making a rotation if necessary, we may assume that $\psi = 0$. We recall that $f$ has the form (1.3) with $\beta = 0$, where $\phi$ and $P$ are as defined there. We claim that
\[
(3.2) \quad \phi(z) = e^{ir} z/(1 - z)^2, \quad z \in K,
\]
for some real $r$. Indeed, by a theorem of Pommerenke [8] either $\phi(z) = e^{ir} z/(1 - e^{ir}) z^2$ for some real $r$ and $y$ or $M(r, \phi) = o(1 - r)^{-2}$. The second possibility cannot occur, since it would imply $\xi(\alpha) = 0$. From (3.1) the first possibility can occur only if $y = 0$. Hence our claim is true.

Next we note that $f(z) = z + az^2 + \cdots$. Otherwise defining $g$ as previously and arguing as in §2, we could obtain a contradiction to (3.1). It follows from this fact and (3.2) that
(3.3) \[ P(z) = e^{-ia \alpha}(1 + 2cz + \cdots) \]

where \( c \) is as in (1.5) with \( \alpha = 1 \). To complete the proof of uniqueness we show that 
\( e^{ia \alpha} = 1 \) and

(3.4) \[ P(z) = (1 + 2cz + z^2)/(1 - z^2), \quad z \in \mathbb{D}. \]

First from the Herglotz representation formula we see that \( P \) may be written in the form

(3.5) \[ P(z) = \cos \alpha \int_{-\pi}^{\pi} \left( \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} \right) d\gamma(\theta) - i \sin \alpha, \quad z \in \mathbb{D}, \]

where \( \gamma \) is a nondecreasing function on \([-\pi, \pi]\) which is continuous on the right and \( \gamma(\pi) - \gamma(-\pi) = 1 \). Second from Theorem 2 of Hayman [5] and its subsequent proof we see that

(3.6) \[ \lim_{\rho \to 1} \frac{1 - \rho}{1 + \rho} P(\rho) = \cos \alpha [\gamma(0^+) - \gamma(0^-)] = \lambda. \]

Using (3.1) and (3.2) it follows that

\[
(1 - \rho)^{2\alpha} |f(\rho)|^2 = \lambda \alpha (1 - \rho)^{2\alpha} \int_0^{\pi} \frac{\rho^{\alpha - 1}(1 + \rho)}{(1 - \rho)^{2\alpha + 1}} d\rho + o(1)
\]

\[
= \lambda \alpha + o(1) \to e^{\alpha}(\alpha) = \frac{\lambda}{2}(1 + c),
\]

as \( \rho \to 1 \). Hence,

(3.7) \[ \lambda = (1 + c)/2. \]

Let \( \gamma_1(\theta) = 0 \) for \(-\pi \leq \theta < 0\) and \( \gamma_1(\theta) = \gamma(0^+) - \gamma(0^-) \) for \( 0 \leq \theta \leq \pi \). Then since \( \gamma \) is continuous on the right, the function \( \gamma_2 = \gamma - \gamma_1 \) is increasing on \([-\pi, \pi]\). Using this fact and (3.5)-(3.7) it follows that

\[
H(z) = P(z) - \frac{(1 + c)}{2} \left( \frac{1 + z}{1 - z} \right) + i \sin \alpha = \cos \alpha \int_{-\pi}^{\pi} \left( \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} \right) d\gamma_2(\theta)
\]

has positive real part and \( \cos \alpha [\gamma_2(\pi) - \gamma_2(-\pi)] = \cos \alpha - (1 + c)/2 \). Using (3.3) and comparing coefficients in the above formula for \( H \), we get

\[
H(z) = \cos \alpha - \frac{1}{2}(1 + c) + [2ce^{-i\alpha} - (1 + c)] z + \cdots
\]

\[
= \cos \alpha r - \frac{1}{2}(1 + c) + 2 \cos \alpha \left( \int_{-\pi}^{\pi} e^{-i\theta} d\gamma_2(\theta) \right) z + \cdots.
\]

Hence
\[\begin{align*}
[2c \cos \alpha r - (1+c)] &= 2 \cos \alpha r \int_{-\pi}^{\pi} \cos \theta d\gamma_2(\theta) \\
&\geq -2 \cos \alpha r [\gamma_2(\pi) - \gamma_2(-\pi)] = -2 \cos \alpha r + (1+c),
\end{align*}\]

where equality holds only if \(\gamma_2\) is constant on \([-\pi, \pi]\). Since \(-1 < c \leq 1\), the above inequality implies that \(\cos \alpha r \geq 1\). Hence \(\gamma_2\) is constant on \([-\pi, \pi]\) and \(\cos \alpha r = 1\). We conclude that

\[H(z) = \frac{1}{2}(1-c)(1-z)/(1+z), \quad z \in K,\]

and thereupon that (3.4) is true. The proof of uniqueness is now complete.

4. Proof of Theorem 2. We begin the proof of Theorem 2 by showing that if \(f \in B(\alpha, \beta)\) and

\[\lim_{r \to 1} (1-r)^2 M(r, f) > 0,\]

then \(\beta = 0\). (4.1) implies, as in §3, that

\[\lim_{r \to 1} (1-r)^2 |f'(r)| > 0\]

for some real \(\psi\). Again we may assume \(\psi = 0\). We observe \((1+r)^{-1}(1-r)^3 |f'(r)|\) is a decreasing function of \(r\) for \(0 < r < 1\), as follows easily from the well-known inequality,

\[(\partial / \partial \rho) \log |f'(\rho)| \leq (2\rho + 4)/(1 - \rho^2), \quad 0 < \rho < 1.\]

(For a proof see Hayman [4, p. 5].) Also, \(|f(r)| \leq \int_0^r |f'(\rho)| d\rho\). Using these observations we deduce that

\[\lim_{r \to 1} (1-r)^3 |f'(r)| > 0,\]

since otherwise we could contradict (4.2).

Before proceeding further we adopt the following notation. If \(b\) is analytic in \(K\), \(b(z) \neq 0\), \(z \in K\), and \(b(0) = 1\), we let \(\log b\) denote the analytic logarithm of \(b\) in \(K\) for which \(\log b(0) = 0\). Furthermore we let \(\arg b(z) = \text{Im} \log b(z)\). In terms of this notation we have

\[\begin{align*}
|\log r^{-1} f(r)| &\leq -2(1 + o(1)) \log (1-r), \quad r \to 1, \\
|\log f'(r)| &\leq -3(1 + o(1)) \log (1-r), \quad r \to 1.
\end{align*}\]
(4.4) is a simple consequence of the well-known inequality

\[
\left| \frac{f'(z)}{f(z)} \right| \leq \frac{1 + \rho}{\rho(1 - \rho)}, \quad |z| = \rho, \ 0 < \rho < 1,
\]

and (4.5) is a consequence of the inequality

\[
\left| \frac{zf''(z)}{f'(z)} - \frac{2\rho^2}{(1 - \rho^2)} \right| \leq \frac{4\rho}{(1 - \rho^2)}, \quad |z| = \rho, \ 0 < \rho < 1
\]

(see Hayman [4, p. 51 for inequalities (4.4a), (4.5a)).

For completeness we give the proof of (4.5). From (4.5a) we see that

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{4 + 2\rho}{(1 - \rho^2)}, \quad |z| = \rho, \ 0 < \rho < 1.
\]

Using this inequality we obtain

\[
|\log f'(r)| = \left| \int_0^r \frac{f''(\rho)}{f'(\rho)} \, d\rho \right| \leq \int_0^r \left| \frac{f''(\rho)}{f'(\rho)} \right| \, d\rho \leq \int_0^r \frac{4 + 2\rho}{(1 - \rho^2)} \, d\rho
\]

\[= \log (1 + r) - 3 \log (1 - r).
\]

From (4.2) and (4.4) it is easily seen that

\[
\arg \left( \frac{f(r)}{r} \right) = -o(1) \log (1 - r), \quad r \to 1,
\]

and from (4.3) and (4.5) that

\[
\arg f'(r) = -o(1) \log (1 - r), \quad r \to 1.
\]

Next we note that

\[
(4.8) \quad (\alpha + i\beta - 1) \log z^{-1} f(z) + \log f'(z) = \log [(z^{-1} \phi(z))^\alpha P(z)].
\]

(4.8) is a consequence of the representation formula (1.3) for \( f \). Using (4.6), (4.7), (4.2), and taking imaginary parts in (4.8) for \( z = r \), we get

\[
(\beta + o(1)) \log |r^{-1}/r| = -2(\beta + o(1)) \log (1 - r) = \arg [(r^{-1} \phi(r))^\alpha P(r)].
\]

The right-hand side of this equality remains bounded as \( r \to 1 \), since \( \phi \) is star-like univalent and \( P \) has positive real part. Hence we must have \( \beta = 0 \) and \( f \in B(\alpha, 0) \).

We now prove Theorem 2. From the above argument we may assume \( f \in B(\alpha, 0) \).
and \( f(z) = z + \sum_{n=0}^{\infty} a_n z^n \), with \(|a_2| = a\). Then from Theorem 1 it follows that

\[
\lim_{r \to 1} (1 - r)^2 M(r, f) \leq \max_{0 < \alpha < a_0} \xi(\alpha) = \xi(a_0).
\]

Using the differential calculus we deduce for \(0 < a < 2\) that \(a_0\) is the unique root of equation (1.10).

5. Some remarks on Theorem 2. The following table gives a comparison between the bound in (1.2) for \(T(\alpha) = 4b^{\alpha - 2} \exp(2 - 4b^{-1})\), \(b = 2 - (2 - a)^{1/2}\), and the bound \(\xi(a_0)\) in (1.9) for different values of \(a\). From the table it can be seen that the maximum difference is about \(0.032\).

<table>
<thead>
<tr>
<th>(a)</th>
<th>(a_0)</th>
<th>(\xi(a_0))</th>
<th>(T(\alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.63</td>
<td>0.069</td>
<td>0.093</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.127</td>
<td>0.157</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.205</td>
<td>0.237</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.300</td>
<td>0.331</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.407</td>
<td>0.434</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.520</td>
<td>0.541</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.634</td>
<td>0.649</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>0.743</td>
<td>0.752</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>0.843</td>
<td>0.848</td>
<td></td>
</tr>
<tr>
<td>1.8</td>
<td>0.931</td>
<td>0.933</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>0.968</td>
<td>0.969</td>
<td></td>
</tr>
<tr>
<td>1.99</td>
<td>26.9</td>
<td>0.997</td>
<td></td>
</tr>
</tbody>
</table>

We now discuss the mapping properties of \(F(\cdot, l)\). If \(a = 2\) and \(0 < a < \infty\), then \(F(z, 1) = z/(1 - z)^2\), \(z \in K\). In general for a fixed \(a\), \(0 < a < \infty\), and \(a\), \(0 \leq a < 2\), the image domain corresponding to \(K\) under the mapping \(F(\cdot, 1)\) does not have a simple geometric description. Therefore we consider only the cases \(a = \frac{1}{2}, 1, 2\), and \(0 \leq a < 2\), where such a description is possible. If \(0 \leq a < 2\) and \(c\) is as in (1.5) with \(\sigma = 1\), we let

\[
\gamma = \alpha \int_0^r e^{i\theta} \frac{z^{-1}}{(1 - z)^{a+1}} \frac{(1 + 2cz + z^2)}{(1 + z)} \, dz, \quad z \in K,
\]

where \(\cos \theta = -c\) and \(0 < \theta < \pi\).

First, if \(a = \frac{1}{2}\), and \(0 \leq a < 2\), then the function \(F^{1/2}(z, 1)\) is univalent in \(D = K \cap \{z: \text{Im } z > 0\}\). Here the square root is intended as a principal value. Let
Ω = F^{1/2}(D, 1). Then the boundary of Ω consists of the line segments:

l_1 = \{w: \text{Re } w \geq 0, \text{Im } w = 0\}, \quad l_2 = \{w: \text{Im } w \geq 0, \text{Re } w = 0\},

l_3 = \{w: \text{Re } w = \text{Re } \gamma, \text{Im } w \geq \text{Im } \gamma\},

as we find from examining the tangent to the boundary. It follows that F(\cdot, 1) maps D onto the region in the w = u + iv plane whose complement is the closure of the lower half plane and the parabolic arc

\[ u = (\text{Re } \gamma)^2 - (v/2 \text{ Re } \gamma)^2, \quad v \geq 2 \text{ Im } \gamma \text{ Re } \gamma. \]

Using the reflection principle we deduce that F(\cdot, 1) maps K onto the region whose complement is the parabolic arcs,

\[ u = (\text{Re } \gamma)^2 - (v/2 \text{ Re } \gamma)^2, \quad v \geq 2 \text{ Im } \gamma \text{ Re } \gamma \]

\[ u = (\text{Re } \gamma)^2 - (v/2 \text{ Re } \gamma)^2, \quad v \leq -2 \text{ Im } \gamma \text{ Re } \gamma. \]

As \( a \to 2 \) the parabolic arcs approach the line segment \((- \infty, -\frac{1}{2} \gamma)\).

Second if \( a = 1, \) and \( 0 \leq a < 2 \), then F maps K onto the region whose complement is the line segments:

\[ \text{Re } u \leq \text{Re } \gamma, \quad \text{Im } u = \text{Im } \gamma, \quad \text{Re } u \leq \text{Re } \gamma, \quad \text{Im } u = -\text{Im } \gamma. \]

Third, if \( a = 2 \), then from an analysis similar to the case \( a = \frac{1}{2} \), we find that F(\cdot, 1) maps K onto a region whose complement is the hyperbolic arcs:

\[ 2uv = \text{Im } \gamma, \quad u^2 - v^2 \geq \text{Re } \gamma, \quad v \geq 0, \]

\[ 2uv = -\text{Im } \gamma, \quad u^2 - v^2 \geq \text{Re } \gamma, \quad v \leq 0. \]

Again as \( a \to 2 \) the hyperbolic arcs approach the line segment \((- \infty, -\frac{1}{2} \gamma)\).

The author wishes to thank Professor Roger W. Barnard for many helpful conversations during the writing of this paper.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506