TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 195, 1974

EXTREMAL PROBLEMS IN CLASSES OF ANALYTIC UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSIONS(1)

BY

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ABSTRACT. This work solves many of the classical extremal problems posed in the class of functions $\Sigma_{K(\rho)}$, the class of functions in Σ with $K(\rho)$-quasiconformal extensions into the interior of the unit disk where $K(\rho)$ is a piecewise continuous function of bounded variation on $[r, 1]$, $0 \le r < 1$. The approach taken is a variational technique and results are obtained through a limiting procedure. In particular, sharp estimates are given for the Golusin distortion functional, the Grunsky quadratic form, the first coefficient, and the Schwarzian derivative. Some extremal problems in $S_{K(\rho)}$, the subclass of functions in S with $K(\rho)$-quasiconformal extensions to the exterior of the unit disk, are also solved.

1. Introduction. Recently, interest has grown in the classes of functions which are elements of S and Σ (the usual normalized classes of schlicht functions in the interior or, respectively, exterior of the unit disk) and possess quasiconformal extensions into the complement of their respective domain of analyticity. We will say that a homeomorphism f is a $K(\rho)$-quasiconformal mapping where $K(\rho)$ is a real valued function, $1 \le K(\rho) \le L < \infty$, if f is a generalized (L^2) solution of the Beltrami equation, $f_{\bar{z}} = \mu f_z$ with $|\mu(z)| \le k(\rho)$ a.e., where $\rho = |z|$, $k(\rho) = (K(\rho) - 1)/(K(\rho) + 1)$.

A very natural class to consider is Σ_K, the subclass of functions in Σ with $K(\rho)$-quasiconformal extensions to $|z| \le 1$ where $K(\rho) \equiv K$, a constant. R. Kühnau [5] has applied an extremal metric method and variational technique to study functionals defined on this class and the related class S_K of functions in S with K-quasiconformal extensions to $|z| \ge 1$. He has also indicated implicit bounds for some functionals defined on classes of quasiconformal mappings whose

Received by the editors March 17, 1973 and, in revised form, September 19, 1973.

AMS (MOS) subject classifications (1970). Primary 30A32, 30A34, 30A38, 30A40, 30A60.

Key words and phrases. Quasiconformal mapping, Schiffer's variational method, extremal problems, Golusin's distortion theorem, Grunsky inequalities, Schwarzian derivative.

(1) The results in this paper constitute an extension of the author's Ph. D. dissertation, written under the direction of Professor G. Schober at Indiana University.

dilatations are bounded by a piecewise continuously differentiable function which is identically 1 on some region of its domain (hence conformal) [5].

In this work, we will give explicit bounds for functionals posed on the class $\Sigma_{K(\rho)}$, the subclass of functions in Σ with $K(\rho)$-quasiconformal extensions into the interior of the unit disk where $K(\rho)$ is a piecewise continuous function of bounded variation on $[r, 1]$, $0 \leq r < 1$. The approach taken is first to pose the Golusin distortion problem in the class $\Sigma_{\overline{K}}$, where $K(\rho)$ is some piecewise constant function which we shall denote by $\overline{K}$. Thus, $\Sigma_{\overline{K}}$ is the subclass of functions in Σ with K_j-quasiconformal extensions to $r_j < |z| < r_{j-1}$, K_j constant, $j = 1, \cdots, N$, where $r = r_N < \cdots < r_0 = 1$. (Note: for $N = 1$, we shall denote this class by $\Sigma_{K,r}$.) To solve the problem we apply a variational method originally developed by M. Schiffer [11] to study functionals defined on families of class $C^{(1)}$, K-quasiconformal mappings and later extended by Schiffer and G. Schober [12], [13] to general families of K-quasiconformal mappings. Then through a limiting procedure we apply the results in $\Sigma_{\overline{K}}$ to obtain extreme values in the class $\Sigma_{K(\rho)}$. As corollaries we obtain the Grunsky inequalities and estimates for the first coefficient, the logarithm of the derivative, and Schwarzian derivative in the class $\Sigma_{K(\rho)}$. The results are sharp, and for most of the problems either an explicit extremal function is given or a necessary condition satisfied by it is stated. Furthermore, if we let $K \to \infty$, all solutions can be observed to agree with classical results.

Through an analogous scheme, we also solve extremal problems in $S_{K(\rho)}$, the subclass of functions in S with $K(\rho)$-quasiconformal extenstions to the exterior of the unit disk.

2. Schiffer's theorem. Presently we shall define the variation to be used and state the main conclusions of Schiffer and Schober's work. We refer the reader to [11], [10], [12], [13], [8] for details.

Suppose Ω is a simply connected domain. We assume that $\overline{K}$ is a real valued, piecewise constant function on Ω, which assumes the value K_j, $1 \leq K_j < \infty$, on a fixed open, connected subset Ω_j, $j = 1, \cdots, N$. Let $\mathfrak{F}$ be a normalized family of $\overline{K}$-quasiconformal mappings of Ω. We assume that $\mathfrak{F}$ is normalized in a standard fashion so that $\mathfrak{F}$ is compact in the topology of locally uniform convergence.

Next, suppose λ is a continuous real valued functional defined on $\mathfrak{F}$. Then there exist $f \in \mathfrak{F}$ such that

$$\lambda(f) = \max_{\mathfrak{F}} \lambda. \tag{1}$$

To obtain necessary conditions that f must satisfy, we begin by constructing variations as in [10]. Let χ_ν be the characteristic function of the disk $|w - w_\nu| < \rho_\nu$, and let $|a_\nu| \leq \epsilon < 1$, $\nu = 1, \cdots, \mathfrak{N}$. Assume these disks are disjoint and contained in a fixed bounded set B. Now define

$$\mu = \sum_{\nu=1}^{\mathfrak{N}} a_\nu \chi_\nu. \tag{2}$$

Then the homeomorphic solution of the Beltrami equation

$$\Phi_{\bar{w}} = \mu \Phi_w \tag{3}$$

with $\Phi(w) = w + o(1)$ near ∞ is a $[(1+\epsilon)/(1-\epsilon)]$-quasiconformal mapping of the plane [6]. For an appropriate choice of parameters, the function Φ will generate variations for the family.

In particular, a solution for (3) is of the form [6]

$$\Phi(w) = w + \sum_{\nu=1}^{\mathfrak{N}} \frac{a_\nu \rho_\nu^2}{w - w_\nu} - \sum_{\nu=1}^{\mathfrak{N}} a_\nu \chi_\nu(w) \frac{\rho_\nu^2 - |w - w_\nu|^2}{w - w_\nu} + O(\epsilon^2) \tag{4}$$

where $O(\epsilon^2)/\epsilon^2$ is uniformly bounded depending only on the bounded set B.

Now suppose $f \in \mathfrak{F}$. Further, in addition to assuming that the disjoint disks $|w - w_\nu| < \rho_\nu$, $\nu = 1, \cdots, \mathfrak{N}$, are in a fixed bounded set B, we also assume that they are in $f(\Omega)$. We then denote $f^* = l \circ \Phi \circ f$ where l is an appropriate Möbius transformation so that f^* possesses the normalization of $\mathfrak{F}$.

Definition. We will say a real valued functional λ on $\mathfrak{F}$ has a continuous variational derivative on $\mathfrak{F}$ if

(i) whenever $f, f^* \in \mathfrak{F}$, there exists an analytic function $A(\cdot; f) \not\equiv 0$ defined in $f(\Omega)$ such that

$$\lambda(f^*) = \lambda(f) + \sum_{\nu=1}^{\mathfrak{N}} \rho_\nu^2 \operatorname{Re}\{a_\nu A(w_\nu; f)\} + O(\epsilon^2) \tag{5}$$

where $O(\epsilon^2)/\epsilon^2$ is uniformly bounded depending only on the fixed bounded set B, and

(ii) the functions $A(w; \cdot)$ are continuous on $\mathfrak{F}$, i.e., if $f_n \in \mathfrak{F}$ and $f_n \to f$ locally uniformly on Ω, then $A(w; f_n) \to A(w; f)$ locally uniformly in $f(\Omega)$ as $n \to \infty$.

Schiffer's theorem [11], [12], [13], [8], [10]. *Suppose*

(i) $\bar{K}$ *is a real valued, piecewise constant function on a simply connected domain* Ω, *which assumes the value* K_j, $1 \le K_j < \infty$, *on an open connected subset* Ω_j, $j = 1, \cdots, N$.

(ii) $\mathfrak{F}$ *is a normalized, compact family of* $\bar{K}$*-quasiconformal mappings on* Ω.

(iii) λ *is a continuous real valued functional on* $\mathfrak{F}$ *with a continuous variational derivative of* $\mathfrak{F}$.

Then an extremal function f *for the problem* $\operatorname{Max}_{\mathfrak{F}} \lambda$ *exists and satisfies*

(a) $J \circ f - k_j \overline{J \circ f}$ *is locally analytic on* Ω_j, $j = 1, \cdots, N$, *except possibly for zeros of* $A(f; f)$, *where* $J(w; f) = \int \sqrt{A(w; f)}\, dw$ *and* $k_j = (K_j - 1)/(K_j + 1)$.

(b) *$f(\Omega)$ is bounded by analytic curves satisfying the differential equation $dJ = \sqrt{A(w; f)}\, dw = real$.*

3. **Golusin distortion theorem.** We shall now apply the results of the preceding section to the class $\Sigma_{\overline{K}}$ to derive bounds for the extremal problem usually referred to as the Golusin distortion theorem [2]. We begin by observing that for $g \in \Sigma_{\overline{K}}$ (since $\Sigma_{\overline{K}}$ is a subclass of Σ), it makes sense to consider

$$\log \frac{g(z) - g(w)}{z - w} \tag{6}$$

for $|w| > 1$, provided we are referring to the branch (single valued in this domain) of the logarithm function that approaches zero when at least one of the points z or w approaches infinity. For $z = w$ we replace the difference quotient by the derivative. It then follows that (6) is analytic for $|z|$, $|w| > 1$ and has a series expansion of the form

$$\sum_{m,n=1}^{\infty} d_{mn} z^{-m} w^{-n}. \tag{7}$$

As in the classical case, we shall refer to the coefficients, d_{mn} defined in (7) as the Grunsky coefficients [4] of the function g.

Thus we define the functional λ on $\Sigma_{\overline{K}}$ by

$$\lambda(g) = \operatorname{Re}\left\{ \sum_{b,l=1}^{M} \gamma_b \gamma_l \log \frac{g(z_b) - g(z_l)}{z_b - z_l} \right\} \tag{8}$$

where γ_l, z_l, $l = 1, \cdots, M$, are finite sequences of complex numbers, such that $|z_l| > 1$.

We observe that since λ attains its maximum in the compact subclass, $\Sigma_{\overline{K}}^{t}$, of functions in $\Sigma_{\overline{K}}$ with Laurent series expansions of the form

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}, \tag{9}$$

then, by translation invariance, it attains its maximum in $\Sigma_{\overline{K}}$. Next, if we apply the variation Φ as defined by equation (4) to a mapping $g \in \Sigma_{\overline{K}}$, hence deriving the corresponding mapping g^*, it can easily be seen that

$$\lambda(g^*) = \lambda(g) - \sum_{\nu=1}^{\mathfrak{N}} a_\nu \rho_\nu^2 \left[\sum_{l=1}^{M} \gamma_l \frac{1}{(g(z_l) - w_\nu)} \right]^2 + O(\epsilon^2), \tag{10}$$

which implies that λ has a continuous variational derivative.

Having satisfied the hypotheses of Schiffer's theorem, we set

$$A(w; f) = -\left[\sum_{l=1}^{M} \gamma_l \frac{1}{w - g(z_l)}\right]^2 \tag{11}$$

and construct

$$J(w; f) = \int \sqrt{A(w; f)}\, dw = i \sum_{l=1}^{M} \gamma_l \log(w - g(z_l)). \tag{12}$$

Hence, the extremal property of g implies

$$q_j(z) = i \sum_{l=1}^{M} \gamma_l \log[g(z) - g(z_l)] + k_j i \sum_{l=1}^{M} \bar{\gamma}_l \overline{\log[g(z) - g(z_l)]} \tag{13}$$

is analytic in $r_j < |z| < r_{j-1}$, which in turn implies each $q_j(z)$ has a Laurent series expansion of the form

$$q_j(z) = \sum_{n=-\infty}^{\infty} c_{j,n} z^n \tag{14}$$

for $r_j < |z| < r_{j-1}$, $j = 1, \cdots, N$.

For $j = 1$, we may by continuity on $|z| = 1$ evaluate the contour integral

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{|z|=1} \frac{q_1(z) + k_1 \overline{q_1(z)}}{z^{n+1}}\, dz \\ &= \frac{1 - k_1^2}{-i} \sum_{l=1}^{M} \gamma_l \frac{1}{2\pi i} \oint_{|z|=1} z^{-(n+1)} \left(\log\left[\frac{g(z) - g(z_l)}{z - z_l}\right] + \log\left[1 - \frac{z}{z_l}\right] + \log(-z_l)\right) dz. \end{aligned} \tag{15}$$

This simplifies for $n > 0$ to

$$c_{1,n} + k_1 \bar{c}_{1,-n} = \frac{1 - k_1^2}{i} \sum_{l=1}^{M} \gamma_l \frac{1}{n} z_l^{-n} \tag{16}$$

and for $n < 0$ to

$$c_{1,-n} + k_1 \bar{c}_{1,n} = \frac{1 - k_1^2}{-i} \sum_{l=1}^{M} \gamma_l \left[\sum_{m=1}^{\infty} d_{mn} z_l^{-m}\right]. \tag{17}$$

Clearly, we now need a relationship between $c_{1,n}$ and $c_{1,-n}$. Motivated by the relationship

$$q_{j+1}(z) = \frac{1 - k_{j+1} k_j}{1 - k_j^2} \cdot \left[q_j(z) + \left[\frac{k_j - k_{j+1}}{1 - k_{j+1} k_j}\right] \overline{q_j(r_j^2/\bar{z})}\right] \tag{18}$$

on $|z| = r_j$, we may continue q_j analytically across $|z| = r_j$. Therefore

$$c_{j+1,n} = \frac{1-k_{j+1}k_j}{1-k_j^2}\left[c_{j,n} + \frac{k_j - k_{j+1}}{1-k_{j+1}k_j}\, r_j^{-2n}\,\overline{c_{j,-n}}\right] \tag{19}$$

for $j = 1, \cdots, N-1$ and all n. We now introduce the following notation. Let

$$S^p(\overline{K}, \overline{r}, \nu, n) = \sum_{j_p=p}^{\nu} \cdots \sum_{j_1=1}^{j_2-1}\left[\prod_{t=1}^{[(p+1)/2]} [r_j^{(2t-1)}]^n\right]\left[\prod_{t=1}^{[p/2]} [r_j^{(2t)}]^{-n}\right]\left[\prod_{t=1}^{p} h_j^{(t)}\Delta k_j^{(t)}\right] \tag{20}$$

where, to avoid double subscripting, we have let

$$\Delta k_j^{(t)} = k_{j_t} - k_{j_t+1}, \tag{21}$$

$$h_j^{(t)} = 1/(1-k_{j_t+1}k_{j_t}) \tag{22}$$

and

$$r_j^{(t)} = r_{j_t}, \qquad t = 1, \cdots, p. \tag{23}$$

Then it follows easily by finite induction from (19) that

$$\begin{aligned} c_{j+1,n} = &\left\{\prod_{p=1}^{j} \frac{1-k_{p+1}k_p}{1-k_p^2}\right\} \\ &\cdot\left\{\left[1+\sum_{p=1}^{[j/2]} S^{2p}(\overline{K}, \overline{r}, j, 2n)\right]c_{1,n} + \sum_{p=1}^{[(j+1)/2]} S^{2p-1}(\overline{K}, \overline{r}, j, -2n)\overline{c}_{1,-n}\right\} \end{aligned} \tag{24}$$

for all n.

If $r_N = r > 0$, we may apply part (b) of Schiffer's theorem, §2. Then the image domain of g is bounded by finitely many analytic arcs which satisfy the differential equation

$$dJ = \text{real}. \tag{25}$$

In particular, since

$$J \circ g = q_N - k_N\overline{q}_N \tag{26}$$

we also have

$$dq_N = \text{real} \tag{27}$$

on $|z| = r$ except possibly at finitely many singularities. But then $q_N(z) - q_N(r)$ is real and continuous everywhere on $|z| = r$, so that by the Schwarz reflection principle there are no singularities and (27) implies that the coefficients (14) satisfy

$$\bar{c}_{N,-n} = r^{2n} c_{N,n} \quad \text{for } n \neq 0. \tag{28}$$

Finally, if $r_N = r = 0$, then (28) is still true for $n > 0$ since q_N is analytic at the origin.

Thus we are able to combine (24) and (28) to relate $\bar{c}_{1,-n}$ and $c_{1,n}$. For simplicity in the computation, we define

$$I_n = \sum_{p=1}^{[(N+1)/2]} S^{2p-1}(\bar{K}, \bar{r}, N, n) \Bigg/ \left(1 + \sum_{p=1}^{[N/2]} S^{2p}(\bar{K}, \bar{r}, N, -n)\right) \tag{29}$$

where we set $k_{N+1} = 1$. Hence we have

$$\bar{c}_{1,-n} = -c_{1,n} I_{2n} \tag{30}$$

for $n > 0$. Combining (30) with (16) and (17) we have

$$\sum_{l=1}^{M} \gamma_l \log \frac{g(z) - g(z_l)}{z - z_l} = \sum_{l=1}^{M} \bar{\gamma}_l \sum_{n=1}^{\infty} \frac{k_1 - I_{2n}}{1 - k_1 I_{2n}} \frac{1}{n} (\bar{z}_l z)^{-n}. \tag{31}$$

Certainly, by letting $z = z_b$, multiplying by γ_b and summing over b, (31) yields maximum of the functional (8) over $\Sigma_{\bar{K}}$. Replacing each γ_j by $e^{i\theta}\gamma_j$ where θ is arbitrary yields the following:

Lemma 1. *For $g \in \Sigma_{\bar{K}}$ where $\gamma_1, \cdots, \gamma_M$ are arbitrary complex numbers and $z_1, \cdots, z_M$ distinct complex numbers such that $|z_l| > 1$, $l = 1, \cdots, M$, then*

$$\left| \sum_{b,l=1}^{M} \gamma_b \gamma_l \log \frac{g(z_b) - g(z_l)}{z_b - z_l} \right| \leq \sum_{b,l=1}^{M} \gamma_b \bar{\gamma}_l \sum_{n=1}^{\infty} \frac{k_1 - I_{2n}}{1 - k_1 I_{2n}} \frac{1}{n} (z_b \bar{z}_l)^{-n}. \tag{32}$$

We note that this result is sharp. Clearly for $|z| > 1$, any g satisfying (31) is extremal. To extend g to $|z| < 1$ so that $g \in \Sigma_{\bar{K}}$ we use (18) and the following relationship, which is equivalent to (13),

$$\sum_{l=1}^{M} \gamma_l \log[g(z) - g(z_l)] = \frac{-i}{1 - k_j} [q_j(z) + k_j \overline{q_j(z)}] \tag{33}$$

for $r_j < |z| < r_{j-1}$. $q_1(z)$ is obtained from (13) for $j = 1$ and the continuity of (31) on $|z| = 1$.

4. Description of limiting procedure. Suppose

$$\lambda: \Sigma_{K(\rho)} \to R \tag{34}$$

is a functional which we want to extremize over $\Sigma_{K(\rho)}$. We begin by letting P_N

be a partition of $[r, 1]$ into N intervals $[r_j, r_{j-1}]$, $j = 1, \cdots, N$, where $r \le r_N < \cdots < r_0 \le 1$, such that points of discontinuity of $K(\rho)$ are not included in the partition points and such that P_{N+1} is a refinement of P_N and $\|P_N\| \to 0$ as $N \to \infty$. For each partition P_N, let $\Sigma_{\overline{K}^m}$ and $\Sigma_{\overline{K}^M}$ be the classes of functions whose dilatation quotients in $r < |z| < 1$ are bounded, respectively, by piecewise constant functions $\overline{K}^m$ and $\overline{K}^M$ defined on (r_j, r_{j-1}) by

$$(35) \qquad K_j^m = \inf_{\rho \in (r_j, r_{j-1})} K(\rho), \qquad K_j^M = \sup_{\rho \in (r_j, r_{j-1})} K(\rho), \qquad j = 1, \cdots, N.$$

We assume that λ can be extended to $\Sigma_{\overline{K}^M}$. Then $\sup_{\Sigma_{\overline{K}^M}} \lambda$ is nonincreasing and $\sup_{\Sigma_{\overline{K}^m}} \lambda$ is nondecreasing as a function of N. Hence if

$$(36) \qquad \lim_{N \to \infty} \sup_{\Sigma_{\overline{K}^m}} \lambda = \lim_{N \to \infty} \sup_{\Sigma_{\overline{K}^M}} \lambda,$$

then these limits must be $\sup_{\Sigma_{\overline{K}(\rho)}} \lambda$ since

$$(37) \qquad \Sigma_{\overline{K}^m} \subset \Sigma_{K(\rho)} \subset \Sigma_{\overline{K}^M}.$$

Furthermore, if there exists $g_N \in \Sigma_{\overline{K}^m}$ such that $\lambda(g_N) = \max_{\Sigma_{\overline{K}^m}} \lambda$ and $g_N \to g$ locally uniformly, then by Theorem 5.2 of Lehto and Vertanen [6], we have $g \in \Sigma_{K(\rho)}$. Consequently, if λ is continuous on $\Sigma_{K(\rho)}$, then

$$(38) \qquad \lambda(g) = \max_{\Sigma_{K(\rho)}} \lambda.$$

5. Golusin distortion theorem for the class $\Sigma_{K(\rho)}$. In this section, we shall extend the Golusin distortion theorem to the class $\Sigma_{K(\rho)}$. To begin, consider the following bounded set of points in R^p denoted by $D_p = \{\rho \in R^p : \rho = (\rho^{(1)}, \cdots, \rho^{(p)})$ and $r \le \rho^{(t)} \le \rho^{(t-1)} \le 1$ for $t = 2, \cdots, p\}$, where we use the bracketed superscript to denote the tth component for $t = 1, \cdots, p$.

Now define the following functions on D_p.

$$(39) \qquad \mathcal{K}(\rho) = (k(\rho^{(1)}), \cdots, k(\rho^{(p)}))$$

where, for each coordinate $\rho^{(t)}$, $t = 1, \cdots, p$, k is defined by $k(\rho) = (K(\rho) - 1)/(K(\rho) + 1)$.

$$(40) \qquad \mathcal{H}(\rho) = \prod_{t=1}^{p} h(\rho^{(t)})$$

where

$$(41) \qquad h(\rho) = \frac{1}{1 - k(\rho^+)k(\rho^-)},$$

$$\mathcal{R}(\rho) = \left[\prod_{t=1}^{[(p+1)/2]} \rho^{(2t-1)}\right]\left[\prod_{t=1}^{[p/2]} (\rho^{(2t)})^{-1}\right]. \tag{42}$$

We can easliy observe that by the piecewise continuity of $\mathcal{H}(\rho)$, the continuity of $\mathcal{R}(\rho)$ and the bounded variation of $\mathcal{K}(\rho)$, the Lebesgue-Stieltjes integral of $\mathcal{H}(\rho)\ (\mathcal{R}(\rho))^n$ exists with respect to $\mathcal{K}(\rho)$ on D_p and, furthermore, agrees with the Riemann-Stieltjes integral over the regions in which the latter exists. Hence the integral is

$$\int_{D_p} \mathcal{H}(\rho)\ (\mathcal{R}(\rho))^n d\mathcal{K}(\rho)$$

$$= \int_r^1 \int_{\rho^{(p)}}^1 \cdots \int_{\rho^{(2)}}^1 \left[\prod_{t=1}^{[(p+1)/2]} [\rho^{(2t-1)}]^n\right]\left[\prod_{t=1}^{[p/2]} [\rho^{(2t)}]^{-n}\right] \prod_{t=1}^{p} h(\rho^{(t)}) \prod_{t=1}^{p} dk(\rho^{(t)}). \tag{43}$$

By construction and the bounded variation of K, k is a function of bounded variation and thus can be written as the sum of continuous function, say f, and a jump function j. Hence substituting for $dk(\rho^{(t)})$, $t = 1, \cdots, p$, and applying the noncommutative binomial expansion, (43) becomes

$$\int_r^1 \cdots \int_{\rho^{(2)}}^1 \left[\prod_{t=1}^{[(p+1)/2]} [\rho^{(2t-1)}]^n\right]\left[\prod_{t=1}^{[p/2]} [\rho^{(2t)}]^{-n}\right] \prod_{t=1}^{p} h(\rho^{(t)}) \prod_{t=1}^{p} df(\rho^{(t)}) +$$

$$\cdots + \int_r^1 \cdots \int_{\rho^{(2)}}^1 \left[\prod_{t=1}^{[(p+1)/2]} [\rho^{(2t-1)}]^n\right]\left[\prod_{t=1}^{[p/2]} [\rho^{(2t)}]^{-n}\right] \prod_{t=1}^{p} h(\rho^{(t)}) \prod_{t=1}^{p} dj(\rho^{(t)}). \tag{44}$$

Remark 1. (i) If K is continuous, then (43) is simply a Riemann-Stieltjes integral and is equivalent to the first iterated integral of (44).

(ii) If K is a piecewise constant function $\overline{K}$, then k is a jump function so that (44) and equivalently (43) reduce to $S^p(\overline{K}, \overline{r}, N, n)$.

Now, on each $\rho^{(t)}$ coordinate, partition the corresponding intervals $[r^{(t)}, 1^{(t)}]$, $t = 1, \cdots, p$, as we have done in §4, i.e., let $P_N^{(t)}$ be a partition of $[r^{(t)}, 1^{(t)}]$ into N intervals $[r_j^{(t)}, r_{j-1}^{(t)}]$ such that the points of discontinuity of k are not included in the set of partition points and $\|P_N^{(t)}\| \to 0$ as $N \to \infty$. Then

$$\sum_{j_p=1}^{N} \cdots \sum_{j_1=1}^{j_2-1} \left[\prod_{t=1}^{[(p+1)/2]} [r_j^{(2t-1)}]^n\right]\left[\prod_{t=1}^{[p/2]} [r_j^{(2t)}]^{-n}\right]\left[\prod_{t=1}^{p} h_j^{(t)} \Delta k_j^{(t)}\right] \tag{45}$$

is a Lebesgue-Stieltjes sum corresponding to a partition of D_p formed by the cross product of the partitions $P_N^{(t)}$ on each $\rho^{(t)}$ coordinate, $t = 1, \cdots, p$, where

$$\Delta k_j^{(t)} = k(r_j^{(t)}) - k(r_{j+1}^{(t)}), \tag{46}$$

$$b_j^{(t)} = 1/(1 - k(r_j^{(t)})k(r_{j+1}^{(t)})). \tag{47}$$

We note that, for $\xi_j^{(t)} \in (r_{j+1}^{(t)}, r_j^{(t)})$, by the piecewise continuity of k, we have

$$\lim_{r_{j+1}^{(t)} \to \xi_j^{(t)};\, r_j^{(t)} \to \xi_j^{(t)}} \frac{1}{1 - k(r_j^{(t)})k(r_{j+1}^{(t)})} = b(\xi_j^{(t)}) \tag{48}$$

where b is defined by (41). Hence (45) converges to (43) as $N \to \infty$. Furthermore, we observe that (45) is equivalent to our earlier definition (20) of $S^p(\bar{K}, \bar{r}, N, n)$.

Now, fix N in the partitions $P_N^{(t)}$.

Claim.

$$|S^p(\bar{K}, \bar{r}, N, (-1)^{p+1}n)| \le M^p/p! \tag{49}$$

for fixed $M < \infty$, independent of N.

To see this, let us examine (45). We first observe that $b_j^{(t)}$, by definition, is bounded, say by $H < \infty$. Moreover, we note that

$$0 < \left[\prod_{t=1}^{[(p+1)/2]} [r^{(2t-1)}]^{(-1)^{p+1}n}\right]\left[\prod_{t=1}^{[p/2]} [r^{(2t)}]^{(-1)^p n}\right] \le 1. \tag{50}$$

Thus we have

$$\begin{aligned} |S^p(\bar{K}, \bar{r}, N, (-1)^{p+1}n)| &\le H^p \sum_{j_p=1}^{N} \cdots \sum_{j_1=1}^{j_2-1} \left|\prod_{t=1}^{p} \Delta k_j^{(t)}\right| . \\ &\le H^p \int_r^1 \int_{\rho^{(p)}}^1 \cdots \int_{\rho^{(2)}}^1 \prod_{t=1}^{p} |dk(\rho^{(t)})| = \frac{H^p}{p!}\left(\int_r^1 |dk(\rho)|\right)^p. \end{aligned} \tag{51}$$

Consequently our claim is valid.

Therefore, by (49) we may conclude that

$$\sum_{p=1}^{\infty} S^{2p-1}(\bar{K}, \bar{r}, N, n) \quad \text{and} \quad \sum_{p=1}^{\infty} S^{2p}(\bar{K}, \bar{r}, N, -n)$$

converge uniformly independent of N. Furthermore, we have already seen that by the piecewise continuity of k, (45) converges to (43) as $N \to \infty$ and is independent of our choice of approximating step functions. Hence, we may now conclude that

$$\lim_{N\to\infty} I_n = \mathfrak{J}_n = \frac{\sum_{p=1}^{\infty} \int_{D_{2p-1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^n \, d\mathcal{K}(\rho)}{1 + \sum_{p=1}^{\infty} \int_{D_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-n} \, d\mathcal{K}(\rho)} \tag{52}$$

where I_n was defined by (29).

Remark 2. (i) If $K(\rho)$ is a piecewise constant function with N points of discontinuity then, by Remark 1(ii), it is easily seen that

$$\mathfrak{g}_n = I_n. \tag{53}$$

(ii) If $K(\rho) \equiv K$ on $0 \le r \le \rho \le 1$ then, again by Remark 1(ii), we have

$$\mathfrak{g}_n = -r^n. \tag{54}$$

An immediate consequence of the convergence of (52) independent of approximating step functions is that condition (36), §4, holds for the functional defined by (8). Hence we have

Theorem 2. *Let $g \in \Sigma_{K(\rho)}$. Then for $\gamma_1, \cdots, \gamma_M$, arbitrary complex numbers and $z_1, \cdots z_M$ distinct complex numbers such that $|z_l| > 1$, $l = 1, \cdots, M$, we have*

$$\left| \sum_{b,l=1}^{M} \gamma_b \gamma_l \log \frac{g(z_b) - g(z_l)}{z_b - z_l} \right| \le \sum_{b,l=1}^{M} \gamma_b \bar{\gamma}_l \sum_{n=1}^{\infty} \frac{k(1^-) - \mathfrak{g}_{2n}}{1 - k(1^-)\mathfrak{g}_{2n}} \frac{1}{n} (z_b \bar{z}_l)^{-n} \tag{55}$$

where $\mathfrak{g}_{2n}$ is defined by (52).

Remark 3. (i) By Remark 2, (i) and (ii), we can derive the bounds for the Golusin distortion functional defined on $\Sigma_{K,r}$ and Σ_K [5], respectively, by replacing $\mathfrak{g}_{2n}$ in (55) by the corresponding value (54).

(ii) We note that (55) is sharp since (32) is sharp, hence by equation (38) we could derive a relationship satisfied by an extremal function.

(iii) In particular, extremal functions for the classes $\Sigma_{K,r}$ and Σ_K satisfy the following (note for class Σ_K, $r \equiv 0$ [5]).

$$e^{i\theta} \sum_{l=1}^{M} \gamma_l \log \frac{g(z) - g(z_l)}{z - z_l}$$

$$= \sum_{l=1}^{M} \bar{\gamma}_l \sum_{n=1}^{\infty} \frac{k + r^{2n}}{1 + kr^{2n}} \frac{1}{n} (z\bar{z}_l)^{-n} \quad \text{for } |z| \ge 1 \tag{56}$$

$$= \sum_{l=1}^{M} \left[\sum_{n=1}^{\infty} \frac{r^{2n}}{1 + kr^{2n}} \frac{1}{n} [\bar{\gamma}_l (\bar{z}_l z)^{-n} - k\gamma_l (z_l \bar{z})^{-n} + k\gamma_l (z/z_l)^n - k^2 \bar{\gamma}_l (\bar{z}/\bar{z}_l)^n] - k\bar{\gamma}_l \log(1 - \bar{z}/\bar{z}_l) \right] \quad \text{for } r < |z| \le 1 \tag{57}$$

for some $0 \le \theta < 2\pi$.

6. Corollaries to the Golusin distortion theorem. In a scheme analogous to the preceding work, we can find a bound for the Golusin distortion functional defined on the class $S_{K(\rho)}$ where $K(\rho)$ is now a piecewise continuous function of bounded variation defined on $\{\rho: 1 \le \rho \le R \le \infty\}$.

Theorem 3. *Let $f \in S_{K(\rho)}$. Then for any set of fixed complex constants $\gamma_1, \cdots, \gamma_M$ such that $\gamma_1 + \cdots + \gamma_M = 0$ and $z_1, \cdots, z_M$ distinct complex numbers in $|z| < 1$ we have*

$$\left| \sum_{b,l=1}^{M} \gamma_b \gamma_l \log \frac{f(z_b) - f(z_l)}{z_b - z_l} \right| \leq \sum_{b,l=1}^{M} \gamma_b \bar{\gamma}_l \sum_{n=1}^{\infty} \frac{k(1^+) - \hat{\mathcal{J}}_{2n}}{1 - k(1^+)\hat{\mathcal{J}}_{2n}} \frac{1}{n} (z_b \bar{z}_l)^n, \tag{58}$$

where

$$k(1^+) = (K(1^+) - 1)/(K(1^+) + 1), \tag{59}$$

$$\hat{\mathcal{J}}_n = \sum_{p=1}^{\infty} \int_{\hat{D}_{2p-1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-n} \, d\mathcal{K}(\rho) \Bigg/ \left(1 + \sum_{p=1}^{\infty} \int_{\hat{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{n} \, d\mathcal{K}(\rho)\right) \tag{60}$$

and

$$\hat{D}_p = \{\rho \in R^p : 1 \leq \rho^{(t-1)} < \rho^{(t)} \leq R \leq \infty,\ t = 2, \cdots, p\}. \tag{61}$$

The functions $\mathcal{H}$, $\mathcal{R}$, and $\mathcal{K}$ defined on $\hat{D}_p$, (61), are as formulated by (39), (40), (41), and (42) and K is defined on the domain $\{\rho : 1 \leq \rho \leq R \leq \infty\}$.

We will now state a number of results which follow easily from Theorems 2 and 3. In particular, this next result follows as in Golusin's work [3] by letting $M = 1$ in Theorem 2. Hence (55) yields

Corollary 1. *Let $g \in \Sigma_{K(\rho)}$, $|z_1| > 1$. Then*

$$|\log g'(z_1)| \leq \sum_{n=1}^{\infty} \frac{k(1^-) - \mathcal{J}_{2n}}{1 - k(1^-)\mathcal{J}_{2n}} \frac{1}{n} |z_1|^{-2n}. \tag{62}$$

Note, (62) is sharp since (55) is sharp. In particular, for $K(\rho) \equiv K$ for $0 \leq r \leq \rho \leq 1$ equality is achieved by

$$g(z) = \begin{cases} (z - z_1) \exp \displaystyle\sum_{n=1}^{\infty} \frac{k + r^{2n}}{1 + kr^{2n}} e^{i\theta} \frac{1}{n} (z\bar{z}_1)^{-n} + c & \text{for } |z| \geq 1, \\ (z - z_1)(1 - \bar{z}/\bar{z}_1)^{-ke^{i\theta}} & \\ \cdot \exp \displaystyle\sum_{n=1}^{\infty} \frac{r^{2n}}{1 + kr^{2n}} e^{i\theta} \frac{1}{n} [k(z/z_1)^n - k^2(\bar{z}/\bar{z}_1)^n + (\bar{z}_1 z)^{-n} - k(z_1\bar{z})^{-n}] + c & \\ \qquad\qquad \text{for } r < |z| \leq 1,\ 0 \leq \theta < 2\pi. \end{cases} \tag{63}$$

Corollary 2. *Let $g \in \Sigma_{K(\rho)}$ and $|z| > 1$. Then*

$$\exp\left[-\sum_{n=1}^{\infty} \frac{k(1^-) - \mathcal{J}_{2n}}{1 - k(1^-)\mathcal{J}_{2n}} \frac{1}{n} |z|^{-2n}\right] \leq |g'(z)| \leq \exp\left[\sum_{n=1}^{\infty} \frac{k(1^-) - \mathcal{J}_{2n}}{1 - k(1^-)\mathcal{J}_{2n}} \frac{1}{n} |z|^{-2n}\right]. \tag{64}$$

If in Corollaries 1 and 2 we let $K(\rho) \equiv K$ and let $r \to 0$ then by Remark 2 we have agreement with Kühnau's results [5]. Also, if we let $k \to 1$, they yield the classical results of C. Loewner [7].

7. Grunsky inequalities and coefficient estimates. We now turn to coefficient problems. We begin by modifying the well-known Grunsky inequalities [4] for the classes $\Sigma_{K(\rho)}$ and $S_{K(\rho)}$. Recall definition (7) in which we introduced the term Grunsky coefficients.

We now maximize

$$\left| \sum_{m,n=1}^{M} d_{mn} x_m x_n \right| \tag{65}$$

for all $g \in \Sigma_{K(\rho)}$.

Theorem 4. *Let $g \in \Sigma_{K(\rho)}$. Let $x_1, \cdots, x_M$ be arbitrary complex numbers. Then*

$$\left| \sum_{m,n=1}^{M} d_{mn} x_m x_n \right| \le \sum_{n=1}^{M} \frac{k(1^-) - \mathfrak{g}_{2n}}{1 - k(1^-)\mathfrak{g}_{2n}} \frac{|x_n|^2}{n}. \tag{66}$$

Any extremal function satisfies

$$\begin{aligned}
&e^{i\theta} \sum_{m=1}^{M} \frac{1}{m} \pi_m[g(z)] x_m \\
&= \sum_{m=1}^{M} \frac{1}{m} z^m x_m - \left[\frac{k(1^-) - \mathfrak{g}_{2m}}{1 - k(1^-)\mathfrak{g}_{2m}} \right] z^{-m} \bar{x}_m \qquad \text{for } |z| \ge 1 \\
&= \sum_{m=1}^{M} \frac{1}{m} \frac{1}{1 - k(1^-)\mathfrak{g}_{2m}} \\
&\quad \cdot \left[\left[1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{2m} d\mathcal{K}(\rho) - \mathfrak{g}_{2m} \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p-1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-2m} d\mathcal{K}(\rho) \right] \right. \\
&\quad \cdot [z^m x_m - k(|z|) z^{-m} \bar{x}_m] \\
&\quad + \left[\mathfrak{g}_{2m} \left[1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-2m} d\mathcal{K}(\rho) \right] \right. \\
&\qquad \left. \left. - \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p-1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{2m} d\mathcal{K}(\rho) \right] [z^{-m} \bar{x}_m - k(|z|) \bar{z}^{-m} x_m] \right] \\
&\qquad\qquad \text{for } r < |z| \le 1
\end{aligned} \tag{67}$$

for some $0 \le \theta < 2\pi$ where $\pi_m(g(z))$ is the Faber polynomial of g [14] and

$$\mathfrak{D}_p = \{\rho \in R^p : r < |z| \le \rho^{(t)} < \rho^{(t-1)} \le 1 \text{ for } t = 2, \cdots, p\}. \tag{68}$$

This result follows in a standard way from Theorem 2 (see [1] or [5]) or directly through an application of the variational method [8].

Remark 4. For $g \in \Sigma_{K,r}$, by Remark 1(ii), one substitutes $\mathcal{J}_{2m} = -r^{2m}$ in (66) to obtain bounds and in (67) to obtain extremal functions. In particular, to obtain Kühnau's result [5] for Σ_K, let $r = 0$. Finally to obtain the classical result [4], let $k \to 1$.

Corollary 3. *Suppose $g \in \Sigma_{K(\rho)}$ and $x_1, \cdots, x_M$ are arbitrary complex numbers. Then*

$$\sum_{m=1}^{M} m \left(\frac{1 - k(1^-)\mathcal{J}_{2m}}{k(1^-) - \mathcal{J}_{2m}}\right) \left|\sum_{n=1}^{M} d_{mn} x_n\right|^2 \leq \sum_{m=1}^{M} \left[\frac{k(1^-) - \mathcal{J}_{2m}}{1 - k(1^-)\mathcal{J}_{2m}}\right] \frac{|x_m|^2}{m}. \tag{69}$$

Corollary 3 is obtained through a classical approach (see Pederson [9]) which applies a lemma of Schur [15]. Next, we observe that for the choice $x_1 = 1$, $M = 1$, in Corollary 3, we have the following "area theorem" for the class $\Sigma_{K(\rho)}$:

Corollary 4. *Let $g \in \Sigma_{K(\rho)}$ and have expansion*

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \quad \text{for } |z| > 1. \tag{70}$$

Then

$$\sum_{m=1}^{\infty} m \left(\frac{1 - k(1^-)\mathcal{J}_{2m}}{k(1^-) - \mathcal{J}_{2m}}\right) |b_m|^2 \leq \frac{k(1^-) - \mathcal{J}_2}{1 - k(1^-)\mathcal{J}_2}. \tag{71}$$

This result implies the next result, which is sharp.

Corollary 5. *Suppose $g \in \Sigma_{K(\rho)}$ and has expansion* (70), *then*

$$|b_1| \leq (k(1^-) - \mathcal{J}_2)/(1 - k(1^-)\mathcal{J}_2). \tag{72}$$

Equality is attained by

$$z + e^{i\theta}\left[\frac{k(1^-) - \mathcal{J}_2}{1 - k(1^-)\mathcal{J}_2}\right] z^{-1} \quad \text{for } |z| \geq 1,$$

$$\begin{aligned} &\frac{1}{1 - k(1^-)\mathcal{J}_2}\Bigg[\Bigg[1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^2 \, d\mathcal{K}(\rho) \\ &\qquad - \mathcal{J}_2 \sum_{p=0}^{\infty} \int_{\mathfrak{D}_{2p+1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-2} \, d\mathcal{K}(\rho)\Bigg] [z + e^{i\theta} k(|z|)\bar{z}] \\ &\qquad + \Bigg[\sum_{p=0}^{\infty} \int_{\mathfrak{D}_{2p+1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^2 d\mathcal{K}(\rho) - \mathcal{J}_2 \Bigg[1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-2} d\mathcal{K}(\rho)\Bigg]\Bigg] \\ &\qquad \cdot [e^{i\theta} z^{-1} + k(|z|)\bar{z}^{-1}]\Bigg] \end{aligned} \tag{73}$$

for $r < |z| \leq 1$ where $0 \leq \theta < 2\pi$.

By a standard argument Corollary 5 now yields

Corollary 6. *If $f \in S_{K(\rho)}$ with expansion*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{in } |z| < 1 \tag{74}$$

and $f(z) \neq \infty$ in $|z| < R$, then

$$|a_2| \leq 2(k(1^+) - \hat{\mathfrak{g}}_1)/(1 - k(1^+)\hat{\mathfrak{g}}_1). \tag{75}$$

Equality is attained by

$$f(z) = \begin{cases} z\left[1 + \dfrac{k(1^+) - \hat{\mathfrak{g}}_1}{1 - k(1^+)\hat{\mathfrak{g}}_1} e^{i\theta} z\right]^{-2}, & |z| \leq 1, \\ \dfrac{[1 - k(1^+)\hat{\mathfrak{g}}_1]\,|z|^2}{[\mathfrak{L} + k(|z|)\mathfrak{g}|z|]^2 \bar{z} + 2e^{i\theta}[(\mathfrak{L} + k(|z|)\mathfrak{g})(k(|z|)\mathfrak{L} + \mathfrak{g}|z|)]|z| + [k(|z|)\mathfrak{L} + \mathfrak{g}|z|]^2 z}, & R > |z| \geq 1, \end{cases} \tag{76}$$

where $0 \leq \theta < 2\pi$ and

$$\mathfrak{L} = 1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)\mathcal{R}(\rho)\, d\mathcal{K}(\rho) - \mathfrak{g}_1\left[\sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p-1}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-1}\, d\mathcal{K}(\rho)\right], \tag{77}$$

$$\mathfrak{g} = \hat{\mathfrak{g}}_1\left[1 + \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p}} \mathcal{H}(\rho)(\mathcal{R}(\rho))^{-1}\, d\mathcal{K}(\rho)\right] - \sum_{p=1}^{\infty} \int_{\mathfrak{D}_{2p-1}} \mathcal{H}(\rho)\mathcal{R}(\rho)\, d\mathcal{K}(\rho). \tag{78}$$

Our next result yields the Grunsky inequalities for the class $S_{K(\rho)}$. This bound can be obtained in a standard way from Theorem 3 (e.g., [15] or [5]) or by using the variational technique of §2 or directly [8]. The Grunsky coefficients for the class $S_{K(\rho)}$ are defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} d_{mn} z^{mn} \tag{79}$$

for $f \in S_{K(\rho)}$. Hence we have

Theorem 5. *If $f \in S_{K(\rho)}$, then for arbitrary complex numbers $x_1, \cdots, x_M$*

$$\left|\sum_{m,n=1}^{M} d_{mn} x_m x_n\right| \leq \sum_{n=1}^{M} \frac{k(1^+) - \hat{\mathfrak{g}}_{2n}}{1 - k(1^+)\hat{\mathfrak{g}}_{2n}} \frac{|x_n|^2}{n}. \tag{80}$$

By setting $x_n = nz^{-n-1}$ in Theorem 4 and $x_n = nz^{n-1}$ in Theorem 5 we obtain the following bounds on the Schwarzian derivative $\{F, z\} = (F'''/F') - 3(F''/F')^2/2$.

Corollary 7. *If $g \in \Sigma_{K(\rho)}$, then*

$$(81) \qquad |\{g, z\}| \le 6 \sum_{m=1}^{\infty} \frac{k(1^-) - \mathfrak{g}_{2m}}{1 - k(1^-)\mathfrak{g}_{2m}} m|z^{m+1}|^{-2}, \qquad |z| > 1.$$

If $f \in S_{K(\rho)}$, *then*

$$(82) \qquad |\{f, z\}| \le 6 \sum_{m=1}^{\infty} \frac{k(1^+) - \hat{\mathfrak{g}}_{2m}}{1 - k(1^+)\hat{\mathfrak{g}}_{2m}} m|z^{m-1}|^{2}, \qquad |z| < 1.$$

Remark 5. In (29), (52), and (60) of §§3, 5, and 6, respectively, we tacitly assumed that the quotients were not indeterminate. Should they be indeterminate in some case, we define them by continuity relative to the parameters. Then all solutions to extremal problems considered here remain valid by continuity of the respective functionals.

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