

## SOME MAPPING THEOREMS

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**ABSTRACT.** Various mapping theorems are proved, culminating in the following result for mappings  $f$  from a closed  $(2k + 1)$ -manifold  $M$  to another,  $N$ : If "almost all" point-inverses of  $f$  are strongly acyclic in dimensions less than  $k$  and if "almost all" point-inverses of  $f$  have Euler characteristic equal to one, then all but finitely many point-inverses are totally acyclic. (Here "almost all" means "except on a zero-dimensional set in  $N$ ".) More can be said when  $k = 1$ : If  $f$  is a monotone map between closed 3-manifolds and if the Euler characteristic of almost-all point-inverses is one, then all but finitely many point-inverses of  $f$  are cellular in  $M$ ; consequently  $M$  is the connected sum of  $N$  and some other closed 3-manifold and  $f$  is homotopic to a spine map. Other results include an acyclicity criterion using the idea of "nonalternating" mapping and the following result for PL maps  $\phi$  between finite polyhedra  $X$  and  $Y$ : If the Euler characteristic of each point-inverse of  $\phi$  is the integer  $c$  then  $\chi(X) = c\chi(Y)$ .

We begin with a clarification of the term "mapping theorem": this is to mean a theorem in which "local" assumptions are made on a map  $f$  (i.e., assumptions are made on point-inverses of  $f$ ) and "global" conclusions are drawn. The global conclusions may be topological (e.g., deducing that the domain and range of  $f$  are homeomorphic) or algebraic (e.g., concluding  $f$  has degree  $\pm 1$ ). The classical Vietoris mapping theorem [3] is the best known example of the sort of result we have in mind.

Global conclusions can sometimes be obtained using "finiteness" theorems. As an illustration, consider a map  $f: M^n \rightarrow N^n$  between closed topological manifolds. If the set  $C_f$  of points  $y \in N$  for which  $f^{-1}(y)$  is not cellular in  $M$  can be shown to be finite,  $n \neq 4$ , then results of S. Armentrout [2] and L. C. Siebenmann [10] imply that  $M$  is homeomorphic to the connected sum of  $N$  and another manifold. (See [8] for a more complete discussion of finiteness theorems.)

For mappings between closed even-dimensional manifolds there is a finite-

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ness theorem as follows (cf. [7]): If each  $f^{-1}(y)$  has  $(k - 1)$ -connected shape,  $n = 2k$ , then  $C_f$  is finite ( $k \neq 2$ ). In §2, after reviewing the even-dimensional case in more detail in §1, we give an analogous result for odd dimensions: If each  $f^{-1}(y)$  has  $(k - 1)$ -connected shape and Euler characteristic one,  $n = 2k + 1$ , then  $C_f$  is finite (no restriction on  $k$ ). For  $n = 3$ , this generalizes (and uses) a result of A. Wright [17] and thus seems to explain Wright's theorem as a phenomenon about odd-dimensional manifolds.

For functions of a real variable, the concepts of monotone map and nonalternating map coincide. In §3, we show how this equivalence is a special case of a general fact for mappings between odd-dimensional manifolds.

A result of E. G. Skljarenko [11] states that a map between compact ANR's which is locally acyclic almost everywhere (i.e., except on a set of dimension  $\leq 0$ ) is in fact locally acyclic except on a finite set. After slightly modifying Skljarenko's result in §4, we prove in §5 a finiteness theorem for maps  $f$  between closed  $K$ -orientable manifolds ( $K$  a field): If  $f$  is locally  $(k - 1)$ -acyclic almost everywhere, and  $2k \geq n$ , then  $f$  is locally acyclic except on a finite set. A similar result holds with  $2k + 1 = n$  and a local Euler characteristic hypothesis. As applications some of the finiteness theorems of §§1, 2 are improved and a further generalization of Wright's theorem is given.

The paper concludes with §6, where we consider PL maps  $f: X \rightarrow Y$  between finite polyhedra. A proof of the following elementary formula is sketched: Suppose  $Y_0$  is a subpolyhedron of  $Y$  such that  $\chi(f^{-1}(y)) = c_0$  for all  $y \in Y_0$  and  $\chi(f^{-1}(y)) = c$  for all  $y \in Y \setminus Y_0$ ; then  $\chi(X) = c\chi(Y) + (c_0 - c)\chi(Y_0)$ .

1. Definitions and a review of the even-dimensional case. For the following definitions suppose that  $f: M \rightarrow N$  is a map with compact point-inverses. Recall three "singular sets"

$$A_i(f; R) = \{y \in N \mid f^{-1}(y) \text{ does not have property } i - uv(R)\},$$

$$A^i(f; R) = \{y \in N \mid \tilde{H}^i(f^{-1}(y); R) \neq 0\},$$

$$C_f = \{y \in N \mid f^{-1}(y) \text{ is not cellular in } M\}.$$

(A compact set  $X \subset M$  has property  $i - uv(R)$  if, for any open set  $U$  containing  $X$ , there is an open set  $V$  with  $X \subset V \subset U$  such that  $\tilde{H}_i(V; R) \rightarrow \tilde{H}_i(U; R)$  is zero. An equivalent condition is that the reduced Skljarenko homology of  $X$  vanish in dimension  $i$ ; see [12].) The map  $f$  is called *strongly acyclic in dimension  $i$  (over  $R$ )* if  $A_i(f; R) = \emptyset$  and *strongly acyclic (over  $R$ )* if  $A(f; R) = \emptyset$ , where

$$A(f; R) = \bigcup_{i \geq 0} A_i(f; R) = \bigcup_{i \geq 0} A^i(f; R).$$

The first equality is a definition. The second follows from (3.3) of [7]. The above terminology is slightly different from that used in [9], where the following is proved:

**Theorem 1.1.** *Suppose  $f: M^n \rightarrow N^n$  is a proper map between  $R$ -orientable  $n$ -manifolds and that  $f$  is strongly acyclic in dimensions less than  $k$ .*

- (1) *If  $2k > n$  then  $f$  is strongly acyclic.*
- (2) *If  $2k = n$  then  $A(f; R)$  is a locally finite subset of  $N^n$ .*

Below we give analogues of (1) for  $2k = n$  and  $2k + 1 = n$  (§3) and (2) for  $2k + 1 = n$  (§2). Another even-dimensional result we shall analogize is the following, proved in [7].

**Theorem 1.2.** *Suppose  $f: M^n \rightarrow N^n$  is a proper map between  $n$ -manifolds and that  $f^{-1}(y)$  has property  $UV^{k-1}$  for each  $y \in N^n$ .*

- (1) *If  $2k > n \neq 4$  then  $f$  is cellular (when  $n = 3$ , we need to assume each  $f^{-1}(y)$  has a neighborhood containing no fake cubes).*
- (2) *If  $2k = n \neq 4$  then  $C_f$  is a locally finite set in  $N^n$ .*

See [1] or [7] for explanations of the terminology used in (1.2).

**Conventions.** A manifold is understood to be a connected, locally Euclidean metric space (without boundary points).  $R$  always means a principal ideal domain. A double arrow  $M \rightarrow N$  indicates a surjective map. Otherwise, our notation is that of [14].

2. **Finiteness theorems in odd dimensions.** We will use local Euler characteristic assumptions in this section. When  $G$  is a module over our PID  $R$ , we define  $\text{rank } G$  to be the minimum number of generators of  $\text{Hom}_R(G, R)$ , i.e., the rank of the free part of  $G$ . If and only if  $X$  is a compactum with  $\text{rank } \check{H}^i(X; R)$  finite for all  $i$  and zero for all but finitely many  $i$ , we write

$$\chi(X; R) = \sum_i (-1)^i \text{rank } \check{H}^i(X; R).$$

The question of dependence on  $R$  will be ignored.

**Theorem 2.1.** *Suppose that  $f: M^{2k+1} \rightarrow N^{2k+1}$  is a proper map between  $R$ -orientable manifolds. If  $f$  is strongly acyclic in dimensions less than  $k$  and if  $\chi(f^{-1}(y); R) = 1$  for each  $y \in N^{2k+1}$ , then  $A(f; R)$  is a locally finite set in  $N^{2k+1}$ .*

**Proof.** The set  $A^{k+1}(f; R)$  is locally finite by Theorem 2.3 of [9] and

$A^q(f; R) = \emptyset$  for  $q \neq k, k + 1$  by (1.3) of [9]. We claim that  $A(f; R) = A^{k+1}(f; R)$ . To see this, let  $y \in N \setminus A^{k+1}(f; R)$ . Then

$$\chi(f^{-1}(y); R) = 1 + (-1)^k \text{rank } \check{H}^k(f^{-1}(y); R)$$

and hence  $\text{rank } \check{H}^k(f^{-1}(y); R) = 0$ . Since  $f^{-1}(y)$  has property  $(k - 1) - uv(R)$ , it follows from the homology/cohomology universal coefficient theorem that  $\check{H}^k(f^{-1}(y); R) = 0$ . Therefore  $y \in N \setminus A(f; R)$ .

**Remarks.** 1. The assumption  $\chi(f^{-1}(y); R) = 1$  in (2.1) can be weakened to the inequality  $\chi(f^{-1}(y); R) \geq 1$  ( $k$  odd) or  $\chi(f^{-1}(y); R) \leq 1$  ( $k$  even) without altering the conclusion. In the context of (2.1), where  $\chi(f^{-1}(y); R) = 1 + (-1)^k \beta_k + (-1)^{k+1} \beta_{k+1}$ , this means assuming  $\beta_k \leq \beta_{k+1}$  instead of  $\beta_k = \beta_{k+1}$ . (Note that  $\beta_{k+1}$  is finite; see the remark following the proof of (2.2) in [9].) Under this weaker assumption, one can use the conclusion of the theorem to show that  $\chi(f^{-1}(y); R) = 1$  for all  $y$  so that the weaker hypothesis never arises in practice.

2. Under the hypothesis of (2.1), take  $R = \mathbb{Z}$  or  $\mathbb{Z}_2$ . Then  $\text{deg } f = \pm 1$ , so  $f_{*i}$  is an isomorphism for  $i \neq k, k + 1$  and an epimorphism for all  $i$ . Moreover,

$$\ker f_{*k} \cong \bigoplus_{y \in N} \check{H}^{k+1}(f^{-1}(y); R), \quad \text{and} \quad \ker f_{*k+1} \cong \bigoplus_{y \in N} \check{H}^k(f^{-1}(y); R).$$

See the analysis in §7 of [9].

3. When  $R = \mathbb{Z}$  or  $\mathbb{Z}_2$ , the orientability hypotheses in (2.1) may be dropped. See §4 of [9].

**Theorem 2.2.** *Suppose  $f: M^{2k+1} \rightarrow N^{2k+1}$  is a proper map between manifolds and that  $f^{-1}(y)$  has property  $UV^{k-1}$  for each  $y \in N^{2k+1}$ . If  $\chi(f^{-1}(y); \mathbb{Z}_2) = 1$  for all  $y \in N^{2k+1}$ , then  $C_f$  is a locally finite set in  $N^{2k+1}$ .*

**Proof.** Suppose first that  $k \geq 2$ . We have  $A(f; \mathbb{Z}_2)$  locally finite by (2.1) and  $A^{k+1}(f; \mathbb{Z})$  locally finite by Theorem 2.3 of [9]. Let  $F = A(f; \mathbb{Z}_2) \cup A^{k+1}(f; \mathbb{Z})$ . We claim that  $F = A(f; \mathbb{Z})$ . To prove this, let  $y \in N \setminus F$ . Then

$$\check{H}^i(f^{-1}(y); \mathbb{Z}) = 0 \text{ for } i \neq k \quad \text{and} \quad \check{H}^i(f^{-1}(y); \mathbb{Z}_2) = 0 \text{ for all } i.$$

It follows from the universal coefficient theorem for Čech cohomology that  $\check{H}^k(f^{-1}(y); \mathbb{Z}) = 0$ . This proves the claim and the local finiteness of  $A(f; \mathbb{Z})$ . Now we are finished since  $f^{-1}(y)$  has  $UV^\infty$  for each  $y \in N \setminus A(f; \mathbb{Z})$ . (See §4 of [7].)

Now assume  $k = 1$ . Then  $A(f; \mathbb{Z}_2)$  is locally finite, hence zero-dimensional, so a result of [17] applies.

**Remarks.** 1. If  $f$  is as in (2.2) with  $M$  and  $N$  closed manifolds, then we can find a closed,  $(k - 1)$ -connected manifold  $K$  such that  $M$  is homeomorphic to the connected sum  $N \# K$ . Conversely, if  $M = N \# K$ , where  $K$  is  $(k - 1)$ -con-

nected, we can construct a map  $f: M \rightarrow N$  which satisfies (2.2). See the discussion in §7 of [7].

2. An interesting aspect of (2.2) is that its statement makes no dimensional restrictions on the manifolds and thus seems to explain the "Wright phenomenon" (cf. [17]) as a statement about manifolds in general. The dichotomy between  $k = 1$  and  $k > 1$  occurs in the proof for two reasons: first, of course, the Poincaré conjecture, and second, the fact that  $UV^{k-1}$  implies  $1 - UV$  in the latter situations while  $UV^0$  is merely the statement that  $f$  is monotone.

3. Both (1.1) and (2.1) can be generalized in the case where  $R$  is a field by requiring only that  $\dim A_i(f; R) \leq 0$  for  $i < k$  (and that  $\dim \{y \in N \mid \chi(f^{-1}(y)) \neq 1\} \leq 0$  in (2.1)). See §5 below.

4. Some kind of hypothesis on  $f^{-1}(y)$  in dimension  $k$  is necessary in both (2.1) and (2.2). See §6 of [7].

3. **Nonalternating mappings.** Suppose that  $f: M \rightarrow N$  is a mapping. We shall say that  $f$  is *nonalternating in dimension  $k$*  provided that, for each pair,  $y, z \in N$  with  $y \neq z$ , there exists a neighborhood  $V$  of  $f^{-1}(y)$  in  $M$  such that  $\tilde{H}_k(V) \rightarrow \tilde{H}_k(M - f^{-1}(z))$  is the zero homomorphism. (Throughout §3,  $R$  is assumed to be a fixed principal ideal domain, and all homology/cohomology is taken with coefficients in  $R$ . Otherwise, our notation follows that of previous sections.) Notice that if  $M$  and  $N$  are locally compact ANR's and each  $f^{-1}(y)$  is compact, then "nonalternating in dimension zero" agrees with the classical notion of nonalternating. See [16].

Similarly, we shall say  $f$  is *weakly acyclic in dimension  $k$*  if each  $f^{-1}(y)$  has a neighborhood  $V$  in  $M$  such that  $\tilde{H}_k(V) \rightarrow \tilde{H}_k(M)$  is zero.

The following result is a corollary to R. Soloway's version of the Vietoris mapping theorem for singular homology. (See [13, Theorem 5].)

**Theorem S.** *Suppose that  $M$  and  $N$  are locally compact ANR's and that  $f: M \rightarrow N$  is a proper map which is strongly acyclic in dimensions less than  $k$ . Then  $f_{*i}: H_i(M) \rightarrow H_i(N)$  is an isomorphism for  $i \leq k - 1$  and an epimorphism for  $i = k$ . If, in addition,  $f$  is weakly acyclic in dimension  $k$ , then  $f_{*k}$  is an isomorphism.*

We will be applying Theorem S to certain types of maps between manifolds.

**Theorem 3.1.** *Suppose  $f: M^n \rightarrow N^n$  is a proper map between  $R$ -orientable  $n$ -manifolds,  $k < n$ . If  $f$  is strongly acyclic in dimensions less than  $k$  and weakly acyclic in dimension  $k$  then  $\check{H}^i(f^{-1}(y)) = 0$  for  $i \geq n - k$  and all  $y \in N^n$ .*

**Proof.** Let  $y \in N$ , and consider

$$\begin{array}{ccc}
 H_i(M - f^{-1}(y)) & \rightarrow & H_i(M) \\
 f|_* \downarrow & & \downarrow f_* \\
 H_i(N - \{y\}) & \longrightarrow & H_i(N),
 \end{array}$$

a commutative diagram. By Theorem S,  $f_*$  is an isomorphism when  $i \leq k$  and  $f|_*$  is an isomorphism when  $i < k$  and an epimorphism when  $i = k$ . Since the lower horizontal map is an isomorphism for  $i < n$ , we can conclude that the upper horizontal map is an isomorphism when  $i < k$  and an epimorphism when  $i = k$ . It follows from the homology sequence of  $(M, M - f^{-1}(y))$  that  $H_i(M, M - f^{-1}(y)) = 0$  for  $i \leq k$ ; the result follows from duality.

**Corollary 3.2.** *Suppose  $f: M^{2k} \rightarrow N^{2k}$  is a proper map between R-orientable manifolds which is strongly acyclic in dimensions less than  $k$ . If  $f$  is weakly acyclic in dimension  $k$ , then  $f$  is strongly acyclic.*

Corollary 3.2 is not surprising in view of the local finiteness of  $A(f)$  noted above. We should point out, however, that one cannot conclude, in Theorem 3.1, that  $f$  is strongly acyclic in dimension  $k$ : there is a map  $f: S^{2k+1} \rightarrow S^{2k+1}$  which is strongly acyclic in dimensions less than  $k$  (and, obviously, weakly acyclic in dimension  $k$ ) which is not strongly acyclic in dimension  $k$ . (See §6 of [7].)

Changing from weakly acyclic to nonalternating, we can obtain an acyclicity criterion in odd dimensions as follows.

**Theorem 3.3.** *Suppose  $f: M^{2k+1} \rightarrow N^{2k+1}$  is a proper map between R-orientable manifolds. If  $f$  is strongly acyclic in dimensions less than  $k$  and nonalternating in dimension  $k$ , then  $f$  is strongly acyclic.*

**Proof.** Let  $y \in N$ . By (3.1), we have  $\check{H}^i(f^{-1}(y)) = 0$  for  $i > k$ ; and by §3 of [7]  $\check{H}^i(f^{-1}(y)) = 0$  for  $i < k$ . It suffices, therefore, to show that  $\check{H}^k(f^{-1}(y)) = 0$ . We claim first that

$$H_c^k(M) \rightarrow \check{H}^k(f^{-1}(y))$$

is zero. For the proof, let  $U$  be a neighborhood of  $f^{-1}(y)$  such that

$$\alpha: H_k(U) \rightarrow H_k(M)$$

is zero, and let  $V$  be a neighborhood of  $f^{-1}(y)$  in  $U$  such that

$$\beta: H_{k-1}(V) \rightarrow H_{k-1}(U)$$

is zero. Consider the commutative diagram below (from the universal coefficient theorem):

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Ext } H_{k-1}(V) & \rightarrow & H^k(V) & \rightarrow & \text{Hom } H_k(V) \rightarrow 0 \\
 & & \beta^\# \uparrow & & \uparrow & & \uparrow \\
 0 & \rightarrow & \text{Ext } H_{k-1}(U) & \rightarrow & H^k(U) & \rightarrow & \text{Hom } H_k(U) \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \alpha^* \\
 0 & \rightarrow & \text{Ext } H_{k-1}(M) & \rightarrow & H^k(M) & \rightarrow & \text{Hom } H_k(M) \rightarrow 0.
 \end{array}$$

(The notation is as in §2 of [7].) We have  $\alpha^* = 0$  and  $\beta^\# = 0$ . Since the rows of the diagram are exact, it follows that  $H^k(M) \rightarrow H^k(V)$  is zero, and hence that  $H^k(M) \rightarrow \check{H}^k(f^{-1}(y))$  is zero. The claim now follows easily from the fact that  $H^k(\hat{M}) \rightarrow H^k_c(M)$  is a functorial isomorphism ( $k \neq 0$ ), where  $\hat{M}$  is the one-point compactification of  $M$ . (See [14, pp. 331, 332].) Now consider the diagram

$$\begin{array}{ccccccc}
 H^k_c(M) & \longrightarrow & \check{H}^k(f^{-1}(y)) & & & & \\
 \downarrow D & & \downarrow D & & & & \\
 H_{k+1}(M) & \xrightarrow{j_*} & H_{k+1}(M, M - f^{-1}(y)) & \xrightarrow{\partial} & H_k(M - f^{-1}(y)) & \xrightarrow{i_*} & H_k(M) \\
 & & & & \downarrow f|_* & & \downarrow f_* \\
 & & & & H_k(N - \{y\}) & \longrightarrow & H_k(N)
 \end{array}$$

in which  $D =$  duality isomorphism and both  $f|_*$  and  $f_*$  are isomorphisms by Theorem S. The diagram commutes up to sign, and the long row is exact. It follows that  $i_*$  is an isomorphism and hence that  $\partial = 0$ . The above paragraph implies that  $j_* = 0$ , so we have

$$0 = H_{k+1}(M, M - f^{-1}(y)) \simeq \check{H}^k(f^{-1}(y)).$$

Therefore,  $\check{H}^*(f^{-1}(y)) = 0$ .

We conclude by remarking that there exist maps  $f: S^{2k} \rightarrow S^{2k}$  which are strongly acyclic in dimensions  $\leq k - 2$ , nonalternating in dimension  $k - 1$ , but not strongly acyclic in dimension  $k - 1$ : suspend the "join" example of §6 of [7].

**Remark.** A technique of Soloway [13] can be used to conclude the properness of the map  $f$ , by merely assuming each point-inverse of  $f$  is compact, in each of the situations (1.1), (1.2), (2.1), (2.2) and (3.2) (but definitely not in (3.1)).

**Question.** Suppose  $f: M^{2k+1} \rightarrow N^{2k+1}$  is a map with compact point-inverses, strongly acyclic in dimensions less than  $k$ , and nonalternating in dimension  $k$ . Is  $f$  proper?

**4. Almost acyclic mappings.** The following theorem differs from a result of Skljarenko's [11] only in its dependence on the integer  $k$ . For the statement, we take  $G$  to be a finitely generated  $R$ -module and  $A^q(f; G)$  to be the set of values  $y$  for which  $\check{H}^q(f^{-1}(y); G) \neq 0$ . Following Skljarenko, if  $A$  is a subset of the space

$Y$ , we define  $\text{rd } A$  to be

$$\text{rd } A = \max\{\dim C \mid C \text{ is closed in } Y \text{ and } C \subset A\}$$

where  $\dim C$  means the covering dimension of  $C$ .

**Theorem 4.1.** *Let  $f: X \rightarrow Y$  be a closed map between paracompact Hausdorff spaces. Suppose further that, for some integer  $k \geq 0$ , the following hold:*

- (i)  $\check{H}^q(X; G)$  is finitely generated for  $q \leq k$ ;
- (ii)  $\check{H}^q(Y; G)$  is finitely generated for  $q \leq k + 1$ ; and
- (iii)  $\text{rd } A^q(f; G) \leq 0$  for  $q \leq k$ .

*Then  $A^q(f; G)$  is finite for  $q \leq k$  and  $\check{H}^q(f^{-1}(y); G)$  is finitely generated for  $q \leq k$  and  $y \in Y$ .*

The proof is based on that of Skljarenko and uses the Leray spectral sequence of  $f$ . We outline the major steps.

**Lemma 4.2.** *Let  $\{E_r^{**}\}$  be a convergent first quadrant spectral sequence. If  $E_2^{p,q} = 0$  whenever  $p > 0$  and  $0 < q \leq k$ , then there exists an exact sequence*

$$E_2^{1,0} \rightarrow \dots \rightarrow E_2^{p,0} \rightarrow H^p \rightarrow E_2^{0,p} \rightarrow E_2^{p+1,0} \rightarrow \dots \rightarrow E_2^{k+2,0},$$

*the maps  $E_2^{p,0} \rightarrow H^p \rightarrow E_2^{0,p}$  being edge homomorphisms and  $E_2^{0,p} \rightarrow E_2^{p+1,0}$  being the map  $d_{2,p+1}^{0,p}$  of [6].*

**Proof.** Simply apply three propositions from Chapter XV of [6]: (5.7), (5.9), and (5.9a).

Now suppose that  $f: X \rightarrow Y$  is a closed, surjective map between paracompact Hausdorff spaces and assume that  $\text{rd } A^q(f; G) \leq 0$ ,  $q \leq k$ . Following Skljarenko [11], we define

$$\mathcal{Q}^q = R^q/G, \quad 1 \leq q \leq k, \quad \text{and} \quad \mathcal{Q}^0 = R^0/G/G.$$

(Here,  $R^q f$  is the  $q$ th right derived functor of the direct image functor. See [5].) Finally, let  $\Gamma^q = \Gamma(Y, \mathcal{Q}^q)$ ,  $0 \leq q \leq k$ , i.e.,  $\Gamma^q$  is the module of sections of the sheaf  $\mathcal{Q}^q$  over  $Y$ .

**Lemma 4.3.** *Under the above assumptions, there exists an exact sequence*

$$\begin{aligned} 0 \rightarrow \check{H}^0(Y; G) \rightarrow \dots \rightarrow \check{H}^q(Y; G) \rightarrow \check{H}^q(X; G) \rightarrow \Gamma^q \rightarrow \check{H}^{q+1}(Y; G) \\ \rightarrow \dots \rightarrow \check{H}^{k+2}(X; G). \end{aligned}$$

**Proof.** The proof is the same as Skljarenko's. His arguments show that  $\{E_r^{**}\}$  satisfies the hypothesis of (4.2), where  $\{E_r^{**}\}$  is the Leray spectral sequence of  $f$ . This fact yields most of the required sequence, since  $E_2^{p,0} = \check{H}^p(Y; G)$ ,  $H^p = \check{H}^p(X; G)$ , and  $E_2^{0,p} = \check{H}^0(Y; R^p f/G) = \Gamma^p$ . The first few terms

are constructed in the present situation just as they are in [11].

**Remark.** For  $q \leq k$  the module  $\Gamma^q$  is finitely generated if and only if  $A^q(f; G)$  is finite and each  $\check{H}^q(f^{-1}(y); G)$  is finitely generated,  $y \in Y$ . (See [11].) Hence (4.1) follows from (4.3).

The introduction of "relative dimension" is an empty generalization in the case of mappings between manifolds, as we now show that  $rd$  and  $dim$  agree on  $A^i(f; G)$ . (This was suggested by D. R. McMillan, Jr., who pointed out that the same is true for  $A_i(f; G)$ .)

**Theorem 4.4.** *Suppose  $f: X \rightarrow Y$  is a proper map between metric spaces, with  $X$  a locally compact ANR. Then  $A^i(f; G)$  is a countable union of closed subsets of  $Y$ , and hence  $rd A^i(f; G) = \dim A^i(f; G)$ .*

**Proof.** If  $\mathcal{U}$  is an open cover of  $Y$ , define  $B(\mathcal{U})$  to be the set  $\{x \in X \mid \text{if } f(x) \in U \in \mathcal{U} \text{ then } \check{H}^i(f^{-1}(U); G) \rightarrow \check{H}^i(f^{-1}(x); G) \text{ is not zero}\}$ . We claim that  $B(\mathcal{U})$  is a closed subset of  $X$ . To see this, suppose  $x$  is a limit point  $B(\mathcal{U})$ , and suppose  $f(x) \in U \in \mathcal{U}$ . Let  $V_1, V_2, \dots$  be open sets in  $Y$  such that  $\bar{V}_{n+1} \subset V_n \subset U$  for all  $n$  and  $f(x) \in \bigcap_n \bar{V}_n$ . Choose points  $y_n \in V_n \cap f(B(\mathcal{U}))$  for each  $n$ . Considering the diagram

$$\begin{array}{ccc} \check{H}^i(f^{-1}(V_n)) & \leftarrow & \check{H}^i(f^{-1}(U)) \\ & \searrow & \downarrow \\ & & \check{H}^i(f^{-1}(y_n)), \end{array}$$

one sees easily that  $\check{H}^i(f^{-1}(U)) \rightarrow \check{H}^i(f^{-1}(V_n))$  is not zero for each  $n$ , since  $f^{-1}(y_n) \subset B(\mathcal{U})$ . If we choose  $\{V_n\}$  with the additional property that  $\text{image}[\check{H}^i(f^{-1}(V_n)) \rightarrow \check{H}^i(f^{-1}(V_{n+1}))]$  is finitely generated for each  $n$ , then it follows that the map

$$\check{H}^i(f^{-1}(U)) \rightarrow \lim_n \check{H}^i(f^{-1}(V_n)) \simeq \check{H}^i(f^{-1}(x))$$

is nonzero. That  $\{V_n\}$  may be so chosen follows from an argument similar to the one for (2.1) of [9]. Therefore  $x \in B(\mathcal{U})$ .

Taking an appropriate sequence of open covers of  $Y$  shows that  $f^{-1}(A^i(f; G))$  (and, hence,  $A^i(f; G)$ ) is a countable union of closed sets.

The second part of the conclusion follows from the "Sum Theorem" for dimension.

5. Almost acyclic mappings between manifolds. In this section we let  $K$  be a field. We conjecture that  $K$  could be replaced by  $R$  in (5.1).

**Theorem 5.1.** *Let  $f: M^m \rightarrow N^n$  be a map between closed,  $K$ -orientable*

manifolds,  $k < n$ . Suppose  $\dim A^q(f; K) \leq 0$  for  $q < k$ . Then  $A^q(f; K)$  is finite whenever  $q < k$  or  $q \geq m - k$ . Therefore, if  $2k \geq m$ ,  $A(f; K)$  is finite.

**Proof.** By (4.1),  $A^q(f)$  is finite for  $q < k$ . (We suppress  $K$  from notation in the proof.) Let

$$V = N - \bigcup_{q < k} A^q(f), \quad U = f^{-1}(V).$$

Applying Theorem 1.3 of [9], we see that  $A^q(f|U) = \emptyset$  for  $q < k$  and  $q > m - k$  (and, hence,  $A^q(f)$  is finite in these ranges). Also, Theorem 2.3 of [9] implies that  $A^{m-k}(f|U)$  is a locally finite subset of  $U$ .

We wish to show that  $A^{m-k}(f|U)$  is actually finite. Suppose  $Y$  is any finite subset of  $A^{m-k}(f|U)$ , and let  $X = f^{-1}(Y)$ . By Theorem 1.1 of [9], the inclusion-induced map  $H_c^{m-k}(U) \rightarrow \check{H}^{m-k}(X)$  is epic. Let  $\hat{U}$  be the one-point compactification of  $U$ . We have the following diagram, each map being induced by inclusion.

$$\begin{array}{ccc} H_c^{m-k}(U) & \xrightarrow{\alpha} & \check{H}^{m-k}(X) \\ \beta \uparrow & & \uparrow \gamma \\ H^{m-k}(\hat{U}) & \longrightarrow & H^{m-k}(U) \end{array}$$

The map  $\alpha$  is epic, as noted above, and  $\beta$  is an isomorphism (assuming  $m - k \neq 0$ ). Therefore,  $\alpha\beta$  is epic, and hence  $\gamma$  is epic. Let  $d$  be the dimension of  $H^{m-k}(U)$ . (We show in the next paragraph that  $d$  is finite.) Since

$$\check{H}^{m-k}(X) = \bigoplus_{y \in Y} \check{H}^{m-k}(f^{-1}(y)),$$

the cardinality of  $Y$  is no greater than  $d$ ; therefore,  $A^{m-k}(f|U)$  is finite. It follows that

$$A^{m-k}(f) \subset \left[ A^{m-k}(f|U) \cup \bigcup_{q < k} A^q(f) \right],$$

a finite set.

It remains to show that  $H^{m-k}(U)$  is finitely generated. For this, it suffices to show that  $H_{m-k}(U)$  is finitely generated. A portion of the homology sequence of  $(M, U)$  looks like

$$H_{m-k+1}(M, U) \rightarrow H_{m-k}(U) \rightarrow H_{m-k}(M)$$

where

$$H_{m-k+1}(M, U) \simeq \check{H}^{k-1} \left( f^{-1} \left( \bigcup_{q < k} A^q(f) \right) \right) = \bigoplus_{y \in A^{k-1}(f)} \{ \check{H}^{k-1}(f^{-1}(y)) \}.$$

Since  $A^{k-1}(f)$  is finite and  $\check{H}^{k-1}(f^{-1}(y))$  is finitely generated (by (4.1)) for each  $y \in N$ , the result follows.

The following is an odd-dimensional analogue of the last statement in (5.1).

**Theorem 5.2.** *Suppose that  $f: M^{2k+1} \rightarrow N^{2k+1}$  is a map between closed,  $K$ -orientable manifolds. Suppose further that  $\dim A^q(f; K) \leq 0$  for  $q < k$  and that  $\dim \{y | \chi(f^{-1}(y); K) \neq 1\} \leq 0$ . Then  $A(f; K)$  is finite.*

**Proof.** By (5.1) the sets  $A^q(f)$  are finite whenever  $q \neq k$ . We have the inclusion

$$A^k(f) \subset \left[ \{y | \chi(f^{-1}(y)) \neq 1\} \cup \bigcup_{q \neq k} A^q(f) \right],$$

so  $\dim A^k(f) \leq 0$ . The result follows from (4.1).

Applying Wright's results (Theorem 1 of [17]) again we obtain what may be the ultimate generalization of his theorem, at least for maps between closed 3-manifolds.

**Corollary 5.3.** *Let  $f: M^3 \rightarrow N^3$  be a map between closed 3-manifolds. If there exists a zero-dimensional set  $Z \subset N^3$  such that  $f^{-1}(y)$  is connected and  $\chi(f^{-1}(y); \mathbb{Z}_2) \geq 1$  for each  $y \in N^3 \setminus Z$ , then  $C_f$  is a finite set. Consequently  $M^3$  is the connected sum of  $N^3$  and some closed 3-manifold.*

The proof of (5.3) uses Remark 1 of §2 as applied to the proof of (5.2). Note, incidentally, that there are monotone maps  $b: S^3 \rightarrow S^3$  with  $\chi(b^{-1}(y)) \geq -1$  for each  $y$  and  $C_b$  an arc in  $S^3$ .

**Question.** If  $f: M^3 \rightarrow N^3$  is a monotone map with  $\chi(f^{-1}(y)) \geq 0$  for each  $y$ , must  $C_f$  be finite?

**6. PL mappings and the Euler characteristic.** In the preceding sections local Euler characteristic assumptions were used several times, and a natural question seems to appear: If  $f: X \rightarrow Y$  is a map between compact ANR's and  $\chi(f^{-1}(y)) = c$  for all  $y \in Y$ , where  $c$  is constant, what can be said about  $\chi(X)$  as related to  $\chi(Y)$ ? There is an easy answer when everything is PL, and we sketch this answer here.

**Theorem 6.1.** *Suppose that  $f: X \rightarrow Y$  is a PL map between (finite) polyhedra and that  $Y_0$  is a subpolyhedron of  $Y$ . If there are integers  $c, c_0$  such that  $\chi(f^{-1}(y)) = c$  for  $y \in Y \setminus Y_0$  and  $\chi(f^{-1}(y)) = c_0$  for  $y \in Y_0$ , then*

$$\chi(X) = c\chi(Y) + (c_0 - c)\chi(Y_0).$$

Here,  $\chi(X)$  denotes the usual Euler characteristic  $\chi(X; \text{rational numbers})$ .

The proof of (6.1) requires the following calculation, in which #  $S$  denotes the cardinality of the set  $S$  and  $\hat{\Delta}$  denotes the barycenter of  $\Delta$ .

**Lemma 6.2.** *If  $f: K \rightarrow \Delta^n$  is a simplicial map of the finite complex  $K$  onto an  $n$ -simplex then*

$$\sum_j (-1)^j \#\{\sigma^j \in K \mid f(\sigma^j) = \Delta^n\} = (-1)^n \chi(f^{-1}(\hat{\Delta}^n)).$$

**Proof.** Let  $H^j = \{\sigma^{n+j} \in K \mid f(\sigma^{n+j}) = \Delta^n\}$  and  $H = \bigcup_j H_j$ . Now  $f^{-1}(\hat{\Delta}^n)$  has a natural triangulation as a subcomplex of a first derived subdivision of  $K$ , but we want instead a cell structure determined by  $K$  as follows. Associate with each  $\sigma \in H^j$  the set  $\Gamma(\sigma) = f^{-1}(\hat{\Delta}^n) \cap \sigma$ . Notice that  $\Gamma(\sigma^{n+j})$  is a  $j$ -dimensional cell and in fact  $\Gamma = \{\Gamma(\sigma) \mid \sigma \in H\}$  is a cell complex whose underlying space is  $f^{-1}(\hat{\Delta}^n)$ . The Euler characteristic of  $f^{-1}(\hat{\Delta}^n)$  can be computed using this cell structure, and we obtain

$$\chi(f^{-1}(\hat{\Delta}^n)) = \sum_j (-1)^j \#\{\gamma^j \in \Gamma\} = \sum_j (-1)^j \#H^j = (-1)^n \sum_j (-1)^{n+j} \#H^j,$$

which completes the proof.

**Proof of (6.1).** First assume that  $(K, K_0)$  and  $(L, L_0)$  are triangulations of  $(X, f^{-1}(Y_0))$  and  $(Y, Y_0)$ , respectively, such that  $f$  and  $f|f^{-1}(Y_0)$  are simplicial.

Assume as a special case that  $Y_0 = \emptyset$ . Let  $\sigma^n$  be a top-dimensional simplex of  $L$ ,  $L_1 = L \setminus \{\sigma^n\}$ , and  $K_1 = f^{-1}(L_1)$ . We have  $\chi(L) = \chi(L_1) + (-1)^n$  and, by (6.2)  $\chi(K) = \chi(K_1) + (-1)^n c$ . By induction, we may assume that  $\chi(K_1) = c\chi(L_1)$ , so  $\chi(K) = c\chi(L_1) + (-1)^n c = c\chi(L)$ .

Now we prove the theorem assuming  $Y_0 \neq \emptyset$  by induction on the number of simplexes of  $L \setminus L_0$ . The case  $L \setminus L_0 = \emptyset$  follows from the special case above, so we proceed to the inductive step. Let  $r^n$  be a top-dimensional simplex of  $L \setminus L_0$ ,  $L_1 = L \setminus \{r^n\}$ , and  $K_1 = f^{-1}(L_1)$ . Using (6.2) and the inductive hypothesis we have

$$\begin{aligned} \chi(K) &= \chi(K_1) + (-1)^n c = c\chi(L_1) + (c_0 - c)\chi(L_0) + (-1)^n c \\ &= c(\chi(L) - (-1)^n) + (c_0 - c)\chi(L_0) + (-1)^n c \\ &= c\chi(L) + (c_0 - c)\chi(L_0). \end{aligned}$$

**Remarks.** Other similar results follow from the same kind of argument. For example, one can replace  $\chi(-; \text{rationals})$  by  $[\chi(-; \text{rationals})]_q$  throughout, where  $[ ]_q$  denotes equivalence class modulo  $q$  and the formula is interpreted in  $\mathbb{Z}_q$ . As another example, one can show the following: If  $f: M^n \rightarrow N^n$  is a PL map between closed, orientable PL manifolds such that  $\chi(f^{-1}(y)) = c$  for all  $y \in N^n$ ,

then  $\deg f = \pm c$ . For a more sophisticated mod 2 version of this last statement, see [15].

Added in proof (March 10, 1974). The question at the end of §5 has been answered affirmatively by T. Knoblauch.

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