TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 195, 1974

GENERALIZED HYPERCOMPLEX FUNCTION THEORY(1)

BY

ROBERT P. GILBERT AND GERALD HILE

ABSTRACT. Lipman Bers and Ilya Vekua extended the concept of an analytic function by considering the distributional solutions of elliptic systems of two equations with two unknowns and two independent variables. These solutions have come to be known as generalized (or pseudo) analytic functions. Subsequently, Avron Douglis introduced an algebra and a class of functions which satisfy (classically) the principal part of an elliptic system of $2r$ equations with $2r$ unknowns and two independent variables. In Douglis' algebra these systems of equations can be represented by a single "hypercomplex" equation. Solutions of such equations are termed hyperanalytic functions. In this work, the class of functions studied by Douglis is extended in a distributional sense much in the same way as Bers and Vekua extended the analytic functions. We refer to this extended class of functions as the class of *generalized hyperanalytic functions*.

1. The equation and the algebra. A. Douglis [1] showed that an elliptic system of the first order in two independent variables, with sufficient smoothness requirements on the coefficients of the first order terms, can be decomposed into the canonical subsystems

$$
\begin{aligned}
&u_{0,x} - v_{0,y} + \cdots &&= g_0,\\
&u_{0,y} + v_{0,x} + \cdots &&= h_0,\\
&u_{k,x} - v_{k,y} + au_{k-1,x} + bu_{k-1,y} + \cdots &&= g_k,\\
&u_{k,y} + v_{k,x} + av_{k-1,x} + bv_{k-1,y} + \cdots &&= h_k,\\
& && k = 1, 2, \cdots, r-1,
\end{aligned}
\tag{1.1}
$$

where the dots represent zero order terms. He called the system a *generalized Beltrami system* if it is homogeneous and contains no terms of zero order. He showed that with a certain commutative algebra such a system can be written in a

Received by the editors December 22, 1972 and, in revised form, March 29, 1973.

AMS (MOS) subject classifications (1970). Primary 30A92, 30A93, 30A96, 30A97, 35J45; Secondary 35C15.

Key words and phrases. Hypercomplex variables, pseudo analytic functions of Bers and Vekua, elliptic systems.

(1) This research was supported in part by the Air Force Office of Scientific Research through AF-AFOSR Grants No. 71-2205A, and No. 74-2592.

brief form. This algebra is generated by the two elements i and e , subject to the multiplication rules

$$i^2 = -1, \qquad ie = ei, \qquad e^r = 0.$$

The elements of this algebra are linear combinations with real coefficients of the $2r$ independent elements

$$e^k, \qquad ie^k, \qquad k = 0, 1, \cdots, r-1,$$

where $e^0 = 1$. A particular element c of the algebra can be written

$$c = \sum_{k=0}^{r-1} c_k e^k, \tag{1.2}$$

where each c_k is a complex number. The conjugate of c, $\bar{c}$, is given by

$$\bar{c} = \sum_{k=0}^{r-1} \bar{c}_k e^k.$$

If $c_0 = 0$, then c does not have a multiplicative inverse and is called *nilpotent.* A *hypercomplex function* is a map from the plane into this algebra and has the form

$$w(x, y) = \sum_{k=0}^{r-1} w_k(x, y) e^k, \tag{1.3}$$

where each w_k is complex valued. Then with the identification

$$w_k = u_k + iv_k \tag{1.4}$$

the generalized Beltrami system can be written

$$\mathfrak{D}w = 0 \tag{1.5}$$

where $\mathfrak{D}$ is the differential operator

$$\mathfrak{D} = \mathfrak{D}_x + i\mathfrak{D}_y + ea\mathfrak{D}_x + eb\mathfrak{D}_y.$$

We will treat somewhat more general systems in this paper. One such system is represented by

$$\mathfrak{D}w + Aw + B\bar{w} = 0, \tag{1.6}$$

where A and B are hypercomplex functions. With

$$\begin{aligned} &A = \sum_{k=0}^{r-1} A_k e^k, && B = \sum_{k=0}^{r-1} B_k e^k, \\ &A_k = \tfrac{1}{2}(p_k + s_k) + \tfrac{1}{2}i(r_k - q_k), && B_k = \tfrac{1}{2}(p_k - s_k) + \tfrac{1}{2}i(r_k + q_k), \end{aligned} \tag{1.7}$$

(1.6) becomes the system

$$
\begin{aligned}
&u_{0,x} - v_{0,y} + p_0 u_0 + q_0 v_0 = 0,\\
&u_{0,y} + v_{0,x} + r_0 u_0 + s_0 v_0 = 0,\\
&u_{k,x} - v_{k,y} + a u_{k-1,x} + b u_{k-1,y} + \sum_{l=0}^{k} (p_l u_{k-l} + q_l v_{k-l}) = 0,\\
&u_{k,y} + v_{k,x} + a v_{k-1,x} + b v_{k-1,y} + \sum_{l=0}^{k} (r_l u_{k-l} + s_l v_{k-l}) = 0,\\
&\qquad k = 1, \cdots, r-1.
\end{aligned}
\tag{1.8}
$$

A more general system, of which (1.6) is a special case, is represented by

$$
\mathfrak{D} w + \sum_{k=0}^{r-1} e^k \sum_{l=0}^{k} (A_{kl} w_l + B_{kl} \bar{w}_l) = 0 \tag{1.9}
$$

where here each A_{kl}, B_{kl} is *complex valued.* The distinguishing feature of this system is that the zero order terms in the kth equation involve only the unknown functions $u_0, \cdots, u_k, v_0, \cdots, v_k$.

We discuss briefly norms of hypercomplex numbers in our algebra. For c given by (1.2), Douglis defined the norm

$$
|c| = \sum_{k=0}^{r-1} |c_k|. \tag{1.10}
$$

Then the following hold for any hypercomplex numbers c and d:

$$
|cd| \le |c|\,|d|, \tag{1.11}
$$

$$
|c + d| \le |c| + |d|. \tag{1.12}
$$

Furthermore, writing c as $c = c_0 + E$, where E is the nilpotent part of c, we have the formula for the inverse

$$
\frac{1}{c} = \frac{1}{c_0}\left[1 - \frac{E}{c_0} + \frac{E^2}{c_0^2} - \cdots + (-1)^{r-1} \frac{E^{r-1}}{c_0^{r-1}}\right] \tag{1.13}
$$

provided $c_0 \neq 0$. Thus we also have the inequality

$$
\left|\frac{1}{c}\right| \le \frac{1}{|c_0|} \sum_{k=0}^{r-1} \left(\frac{|E|}{|c_0|}\right)^k. \tag{1.14}
$$

Moreover, we have $1 = |c \cdot 1/c| \le |c|\,|1/c|$ and thus

$$1/|c| \leq |1/c| \,. \tag{1.15}$$

Finally, we define certain spaces of hypercomplex functions. In general, a function w, given by (1.3), will be said to lie in a given function space if each of the complex functions w_k is in that space. For example, we say $w \in L^p(\mathfrak{G})$, where $\mathfrak{G}$ is some domain in the plane, if w is defined in $\mathfrak{G}$ and each w_k is in $L^p(\mathfrak{G})$. This criterion is obviously equivalent to the statement $|w| \in L^p(\mathfrak{G})$. The L^p norm of a hypercomplex function w thus will be defined by

$$|w, \mathfrak{G}|_p = \left(\iint_{\mathfrak{G}} |w|^p \, dx \, dy\right)^{1/p}. \tag{1.16}$$

If w is bounded in $\mathfrak{G}$ we have the norm $|w, \mathfrak{G}|_\infty = \sup_{z \in \mathfrak{G}} |w(z)|$.

The spaces $C^m(\mathfrak{G})$ and $C^m(\overline{\mathfrak{G}})$ consist of those functions whose derivatives up to the mth order are continuous in $\mathfrak{G}$ and the closure of $\mathfrak{G}$, respectively. We write $C^0(\mathfrak{G}) = C(\mathfrak{G})$, $C^0(\overline{\mathfrak{G}}) = C(\overline{\mathfrak{G}})$.

We say $w \in C^m_\alpha(\overline{\mathfrak{G}})$, where $0 < \alpha \leq 1$, if w and its derivatives up to the mth order are Hölder-continuous in $\overline{\mathfrak{G}}$ with exponent α (i.e., there exists a positive constant M such that for $z_1, z_2, \in \overline{\mathfrak{G}}$, $|w(z_1) - w(z_2)| \leq M|z_1 - z_2|^\alpha$, and a similar inequality holds for the derivatives of w to order m).

The space $C^m_\alpha(\mathfrak{G})$ is defined to contain those functions which are in $C^m_\alpha(\overline{\mathfrak{G}}_1)$ for every *bounded* subdomain $\mathfrak{G}_1$ of $\mathfrak{G}$.

We also use the space $B^m(\mathbf{C})$, which consists of the functions which, along with their derivatives to order m, are *continuous* and *bounded* in the whole complex plane $\mathbf{C}$. If the function and its derivatives are also Hölder-continuous in $\mathbf{C}$ with exponent α, then we say the function is in $B^m_\alpha(\mathbf{C})$.

2. Generating solutions. Douglis introduced the notion of a *generating solution* for the operator $\mathfrak{D}$. For our purposes, we assume that a and b are defined in the whole plane $\mathbf{C}$ and make the following definition:

Definition 2.1. A hypercomplex function t is a generating solution for $\mathfrak{D}$ if

(i) t has the form $t(z) = z + \sum_{k=1}^{r-1} t_k(z) e^k = z + T(z)$ ($T(z)$ nilpotent),

(ii) $T \in B^2(\mathbf{C})$,

(iii) $\mathfrak{D}t = 0$ in $\mathbf{C}$.

Notice that if E is a nilpotent constant, then $t + E$ is also a generating solution.

Definition 2.2. Let f be a complex valued function in a domain $\mathfrak{G}$. The integral operator $I_{\mathfrak{G}}$ is given formally by

$$(I_{\mathfrak{G}} f)(z) = -\frac{1}{2\pi} \iint_{\mathfrak{G}} \frac{f(\zeta)}{\zeta - z} \, d\xi \, d\eta \tag{2.1}$$

(with $\zeta = \xi + i\eta$, $z = x + iy$). In the case that $\mathfrak{G}$ is the whole plane $\mathbf{C}$, we write $I_{\mathbf{C}} \equiv I$.

The following result is given by Vekua [3, pp. 56–64].

Theorem 2.3. *Let f be a complex valued function which is in $B^1_\alpha(\mathbf{C})$ for some α, $0 < \alpha < 1$, and in $L^p(\mathbf{C})$ for some p, $1 \le p < 2$. Then $If \in B^2_\alpha(\mathbf{C})$ and moreover*

$$(\mathfrak{D}_x + i\mathfrak{D}_y)(If) = f. \tag{2.2}$$

Using a procedure of Douglis, we can prove now the following existence theorem.

Theorem 2.4. *If a and b are in $B^1_\alpha(\mathbf{C})$ for some α, $0 < \alpha < 1$, and in $L^p(\mathbf{C})$ for some p, $1 \le p < 2$, then there exists a generating solution for $\mathfrak{D}$.*

Proof. Define t by $t_0(z) = z$, and for $k = 1, \cdots, r-1$,

$$t_k = -I[(a\mathfrak{D}_x + b\mathfrak{D}_y)t_{k-1}].$$

We assume that the first and second order derivatives of t_{k-1} are in $B_\alpha(\mathbf{C})$ (true of course for $k = 1$). Then the function $(a\mathfrak{D}_x + b\mathfrak{D}_y)t_{k-1}$ satisfies the conditions in Theorem 2.3. Thus $t_k \in B^2_\alpha(\mathbf{C})$. Furthermore the equation $\mathfrak{D}t = 0$ can be written componentwise as

$$\begin{aligned} &(\mathfrak{D}_x + i\mathfrak{D}_y)t_0 = 0, \\ &(\mathfrak{D}_x + i\mathfrak{D}_y)t_k = -(a\mathfrak{D}_x + b\mathfrak{D}_y)t_{k-1}, \qquad k = 1, \cdots, r-1. \end{aligned} \tag{2.3}$$

Thus t is a generating solution.

Henceforth we will assume that a and b satisfy the conditions of Theorem 2.4 so that the existence of a generating solution is assured.

Following are some inequalities concerning the generating solution which will be used later. We denote generic constants by $M(\cdot)$, where inside the parenthesis are listed whatever entities determine M.

$$|t_x(z)|,\ |t_y(z)| \le M(a, b), \tag{2.4}$$

$$|1/(i + eb)| \le M(b), \tag{2.5}$$

$$\left|\frac{1}{t(\zeta) - t(z)}\right| \le \frac{M(a, b)}{|\zeta - z|} \qquad (z \ne \zeta), \tag{2.6}$$

$$\left|\frac{t_x(z)}{i + eb(z)}\right| \le M(a, b), \tag{2.7}$$

$$\left|\frac{t_x(z)}{i + eb(z)} \cdot \frac{1}{t(\zeta) - t(z)}\right| \le \frac{M(a, b)}{|\zeta - z|} \qquad (z \ne \zeta). \tag{2.8}$$

In each case above, $M(a, b)$ arises from bounds on b and the derivatives of t. For (2.5), the formula (1.14) is used. To obtain (2.6), again use (1.14) to arrive at the inequality

$$\left|\frac{1}{t(\zeta)-t(z)}\right| \le \frac{1}{|\zeta - z|} \sum_{k=0}^{r-1} \left(\frac{|T(\zeta)-T(z)|}{|\zeta - z|}\right)^k .$$

The terms in the summation are bounded because the derivatives of T are bounded. Inequalities (2.7) and (2.8) follow from the preceding ones and applications of (1.11).

3. The hypercomplex function theory of Douglis. In this section are stated some results of Douglis [1], and a few immediate consequences, concerning solutions of the equation $\mathfrak{D}w = 0$. Such solutions, if they are continuously differentiable, we will call *hyperanalytic*, for reasons which will become clear. The next few theorems were proved by Douglis.

Definition 3.1. A domain $\mathfrak{G}$ is *regular* if it is bounded and its boundary Γ consists of a finite number of simple closed curves with piecewise continuous tangent.

Theorem 3.2 (Green's identity). *If $\mathfrak{G}$ is a regular domain, and w and v are hypercomplex functions in $C^1(\overline{\mathfrak{G}})$, then*

$$\iint_{\mathfrak{G}} \frac{t_x}{i+eb}\,[w(\mathfrak{D}v) + (\mathfrak{D}w)v]\,dx\,dy = -\int_\Gamma wv\,dt(z). \tag{3.1}$$

Definition 3.3. Let v be a hypercomplex function in a domain $\mathfrak{G}$. The integral operator $J_{\mathfrak{G}}$ is defined by

$$(J_{\mathfrak{G}}v)(z) = \frac{1}{2\pi i}\iint_{\mathfrak{G}} \frac{t_\xi(\zeta)}{i+eb(\zeta)}\,\frac{v(\zeta)}{t(\zeta)-t(z)}\,d\xi\,d\eta. \tag{3.2}$$

When $\mathfrak{G}$ is the whole plane $\mathbf{C}$, we write $J_{\mathbf{C}} \equiv J$.

Theorem 3.4 (Cauchy integral representation). *Let $\mathfrak{G}$ be a regular domain. If $w \in C^1(\overline{\mathfrak{G}}) \cap C(\overline{\mathfrak{G}})$, then for $z \in \mathfrak{G}$,*

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta) + J_{\mathfrak{G}}(\mathfrak{D}w)(z). \tag{3.3}$$

Note that if $\mathfrak{D}w = 0$ in $\mathfrak{G}$, (3.3) becomes

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta). \tag{3.4}$$

Thus a hyperanalytic function is at least twice continuously differentiable, since for $z \notin \Gamma$ (3.4) may be differentiated under the integral sign as many times as t is continuously differentiable.

Definition 3.5. Let f be an analytic function of z, and let E be nilpotent. We define the hypercomplex function

$$f(z+E) = \sum_{k=0}^{r-1} \frac{1}{k!} E^k f^{(k)}(z). \tag{3.5}$$

Theorem 3.6. *Let f be analytic in a domain $\mathfrak{G}_*$. Let w, given by* (1.3), *be in $C^1(\mathfrak{G})$ with the values of w_0 contained in $\mathfrak{G}_*$. Then in $\mathfrak{G}$,*

$$df(w(z)) = f'(w(z))dw(z). \tag{3.6}$$

Thus if f is analytic in a domain $\mathfrak{G}$, and t is a generating solution, the following formulas hold in $\mathfrak{G}$.

$$(f \circ t)(z) = \sum_{k=0}^{r-1} \frac{1}{k!} (T(z))^k f^{(k)}(z), \tag{3.7}$$

$$\mathfrak{D}(f \circ t)(z) = f'(t(z))\mathfrak{D}t(z) = 0. \tag{3.8}$$

The composition $f \circ t$ therefore is *hyperanalytic*. The relationship between analytic functions is given more completely in the next theorem.

Theorem 3.7. *If w is hyperanalytic in $\mathfrak{G}$ and in $C(\overline{\mathfrak{G}})$, where $\mathfrak{G}$ is a regular domain, then w can be represented in $\mathfrak{G}$ by*

$$w(z) = \sum_{p=0}^{r-1} \sum_{k=0}^{r-1} \frac{1}{k!} (T(z))^k f_p^{(k)}(z) e^p = \sum_{p=0}^{r-1} f_p(t(z)) e^p. \tag{3.9}$$

where each f_p is analytic in $\mathfrak{G}$.

Some consequences of the preceding theorems now follow.

Theorem 3.8. *The zeros of a hyperanalytic function are isolated, unless the function is identically zero.*

Proof. Let w be hyperanalytic in a neighborhood of a point z_0, with $w(z_0) = 0$. Since $t(z) - T(z_0)$ is also a generating solution, (3.9) yields, in a neighborhood of z_0,

$$w(z) = \sum_{p=0}^{r-1} \sum_{k=0}^{r-1} \frac{1}{k!} (Tz - Tz_0)^k f_p^{(k)}(z) e^p \tag{3.10}$$

and

$$0 = w(z_0) = \sum_{p=0}^{r-1} f_p(z_0)e^p.$$

If we assume w is not identically zero, then some f_p is not identically zero, and since each f_p is analytic, some f_p is nonzero in a deleted neighborhood of z_0. Let $\tilde{p}$ be the smallest p such that f_p has this property. Then in (3.10) the coefficient of $e^{\tilde{p}}$ is $f_{\tilde{p}}(z)$, and therefore $w(z) \neq 0$ in some deleted neighborhood of z_0.

Theorem 3.9. *A hyperanalytic function which is entire and bounded is a constant.*

(To prove this theorem, one proceeds exactly as in proving the analogous theorem for analytic functions, using the Cauchy integral representation and bounds on the function to show that the partial derivatives vanish. Thus, this proof will not be repeated here.)

We conclude this section with brief remarks about the special hypercomplex function $\exp(z + E)$ for which (3.5) gives

$$\exp(z + E) = (\exp z) \sum_{k=0}^{r-1} \frac{E^k}{k!}. \tag{3.11}$$

Because the coefficient of e^0 $(=1)$ in this expression is $\exp z$, $\exp(z + E)$ is never nilpotent. It is straightforward to check that the identity

$$\exp(z_1 + E_1) \exp(z_2 + E_2) = \exp(z_1 + z_2 + E_1 + E_2) \tag{3.12}$$

is true. We use the expansion (3.11) and the fact that $(E_1)^j (E_2)^k = 0$ if $j + k \geq r$.

4. **The operator $\mathfrak{D}$.** The operator $\mathfrak{D} = \mathfrak{D}_x + i\mathfrak{D}_y + ea\mathfrak{D}_x + eb\mathfrak{D}_y$ will now be defined in a Sobolev sense. In accordance with formula (3.1) we make the following definition.

Definition 4.1. Let $\mathfrak{G}$ be a domain in the plane, and let w and v be hypercomplex functions in $L^1_{\mathrm{loc}}(\mathfrak{G})$. Then $v = \mathfrak{D}w$ in $\mathfrak{G}$ if for all hypercomplex functions ϕ in $C^1_c(\mathfrak{G})$

$$\iint_{\mathfrak{G}} \frac{t_x}{i + eb} [w(\mathfrak{D}\phi) + v\phi]\, dx\, dy = 0 \tag{4.1}$$

($C^1_c(\mathfrak{G})$ is that class of functions in $C^1(\mathfrak{G})$ with compact support in $\mathfrak{G}$).

Because of linearity, it is clear that it is equivalent to ask that (4.1) hold for all *complex* functions ϕ in $C_c^1(\mathfrak{G})$. Moreover, the following theorem is true.

Theorem 4.2. *If w and v are in $L^1_{\mathrm{loc}}(\mathfrak{G})$, then $v = \mathfrak{D}w$ if and only if for all ψ in $C_c^1(\mathfrak{G})$,*

$$\iint_{\mathfrak{G}} [w(\mathfrak{D}^*\psi) + v\psi]\, dx\, dy = 0 \tag{4.2}$$

where the differential operator $\mathfrak{D}^*$ is given by

$$\mathfrak{D}^*\psi = (\mathfrak{D}_x + i\mathfrak{D}_y)\psi + e\mathfrak{D}_x(a\psi) + e\mathfrak{D}_y(b\psi). \tag{4.3}$$

Proof. If ψ_1 and ψ_2 are continuously differentiable hypercomplex functions, then

$$\mathfrak{D}^*(\psi_1\psi_2) = \psi_1(\mathfrak{D}\psi_2) + (\mathfrak{D}^*\psi_1)\psi_2.$$

From Douglis [1, p. 270], we have $\mathfrak{D}^*(t_x/(i + eb)) = 0$. Thus if we let $\psi = t_x\phi/(i + eb)$, we obtain

$$\mathfrak{D}^*\psi = t_x\mathfrak{D}\phi/(i + eb)$$

and therefore (4.1) and (4.2) are equivalent.

We now need some results from Vekua [3] concerning the operators

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right).$$

Vekua defined these operators in a Sobolev sense in the following manner.

Definition 4.4. Let f and g be complex functions in $L^1_{\mathrm{loc}}(\mathfrak{G})$. Then (i) $g = \partial f/\partial \bar z$ in $\mathfrak{G}$, or (ii) $g = \partial f/\partial z$ in $\mathfrak{G}$, if for all complex functions ϕ in $C_c^1(\mathfrak{G})$,

$$\text{(i)} \qquad \iint_{\mathfrak{G}} \left(f\,\frac{\partial \phi}{\partial \bar z} + g\phi\right) dx\, dy = 0,$$

$$\text{(ii)} \qquad \iint_{\mathfrak{G}} \left(f\,\frac{\partial \phi}{\partial z} + g\phi\right) dx\, dy = 0.$$

The following theorem is then obtained [3, p. 72].

Theorem 4.5. *If $f \in L^1_{\mathrm{loc}}(\mathfrak{G})$, $\partial f/\partial \bar z \in L^p_{\mathrm{loc}}(\mathfrak{G})$ for some p, $p > 1$, then $\partial f/\partial z$ exists and is in $L^p_{\mathrm{loc}}(\mathfrak{G})$. Thus f_x and f_y exist and are in $L^p_{\mathrm{loc}}(\mathfrak{G})$.*

Theorem 4.6. *Let $w \in L^1_{\mathrm{loc}}(\mathfrak{G})$, $\mathfrak{D}w = v$, where $v \in L^p_{\mathrm{loc}}(\mathfrak{G})$ for some p, $p > 1$. Then for each k, $w_{k,x}$ and $w_{k,y}$ exist in the Sobolev sense and are in $L^p_{\mathrm{loc}}(\mathfrak{G})$. Moreover, the following formulas hold:*

$$2\partial w_0/\partial \bar z = v_0, \qquad 2\partial w_k/\partial \bar z + aw_{k-1,x} + bw_{k-1,y} = v_k, \qquad k = 1, \cdots, r - 1.$$

Proof. Let ψ be a complex function in $C_c^1(\mathfrak{G})$. By (4.2),

$$\iint_{\mathfrak{G}} [w(\psi_x + i\psi_y + e(a\psi)_x + e(b\psi)_y) + v\psi]\,dx\,dy = 0.$$

Equating powers of e, we obtain the equations

$$\iint_{\mathfrak{G}} [w_0(\psi_x + i\psi_y) + v_0\psi]\,dx\,dy = 0,$$

$$\iint_{\mathfrak{G}} [w_k(\psi_x + i\psi_y) + w_{k-1}((a\psi)_x + (b\psi)_y + v_k\psi]\,dx\,dy = 0,$$

$$k = 1, \cdots, r-1.$$

The first equation states $2\partial w_0/\partial\bar{z} = v_0$, and by Vekua's theorem $w_{0,x}$ and $w_{0,y}$ exist and are in $L^p_{\mathrm{loc}}(\mathfrak{G})$.

Setting $k = 1$ in the second equation, and using $a\psi \in C_c^1(\mathfrak{G})$, we obtain

$$\iint_{\mathfrak{G}} [w_1(\psi_x + i\psi_y) - w_{0,x}a\psi - w_{0,y}b\psi + v_1\psi]\,dx\,dy = 0.$$

Hence $2\partial w_1/\partial\bar{z} = -aw_{0,x} - bw_{0,y} + v_1$. Because the right-hand side above is in $L^p_{\mathrm{loc}}(\mathfrak{G})$, we can again apply Vekua's theorem to show $w_{1,x}$ and $w_{1,y}$ are in $L^p_{\mathrm{loc}}(\mathfrak{G})$. We continue in this manner for $k = 2, \cdots, r-1$.

The next theorem is also from Vekua [3, p. 142].

Theorem 4.7. *If $\partial f/\partial\bar{z} \in C_\alpha(\mathfrak{G})$ for some α, $0 < \alpha < 1$, then $f \in C^1_\alpha(\mathfrak{G})$ (after modification on a set of measure zero, of course).*

Theorem 4.8. *If $\mathfrak{D}w = 0$ in $\mathfrak{G}$, then w is continuously differentiable in $\mathfrak{G}$ (and therefore hyperanalytic).*

Proof. By Theorem 4.6, if $\mathfrak{D}w = 0$ then $\partial w_0/\partial\bar{z} = 0$. Thus $w_0 \in C^1_\alpha(\mathfrak{G})$, for any α, $0 < \alpha < 1$. Furthermore, $2\partial w_1/\partial\bar{z} = -aw_{0,x} - bw_{0,y}$. Since the right-hand side is in $C_\alpha(\mathfrak{G})$, $w_1 \in C^1_\alpha(\mathfrak{G})$. Continuing in this manner we can show $w_k \in C^1_\alpha(\mathfrak{G})$, $k = 0, \cdots, r-1$, and thus w is continuously differentiable.

5. The operator J. In this section are discussed properties of the operator $J_{\mathfrak{G}}$, associated with a domain $\mathfrak{G}$, given by

$$(J_{\mathfrak{G}}w)(z) = \frac{1}{2\pi i}\iint_{\mathfrak{G}} \frac{t_\xi(\zeta)}{i + eb(\zeta)}\,\frac{w(\zeta)}{t(\zeta) - t(z)}\,d\xi\,d\eta. \tag{5.1}$$

For sufficiently smooth functions w in a domain $\mathfrak{G}$, Douglis proved the formula $\mathfrak{D}(J_{\mathfrak{G}}w) = w$; we will find more general conditions for which the formula is true.

Theorem 5.1. *Let $\mathfrak{G}$ be any domain. If $v \in L^1(\mathfrak{G})$, then $J_{\mathfrak{G}}v \in L^1_{loc}(\mathfrak{G})$, and $v = \mathfrak{D}(J_{\mathfrak{G}}v)$,*

Proof. Let $\mathfrak{G}_0$ be a bounded domain inside $\mathfrak{G}$, and denote the characteristic function on $\mathfrak{G}_0$ by $X_{\mathfrak{G}_0}$. We can establish the bound

$$\iint_{\mathfrak{G}}\left(\iint_{\mathfrak{G}}\left|X_{\mathfrak{G}_0}(z)\frac{t_\xi(\zeta)}{i+eb(\zeta)}\,\frac{v(\zeta)}{t(\zeta)-t(z)}\right|dx\,dy\right)d\xi\,d\eta$$
$$\leq M(a,b)\iint_{\mathfrak{G}}\left(\iint_{\mathfrak{G}_0}\frac{1}{|\zeta-z|}\,dx\,dy\right)|v(\zeta)|\,d\xi\,d\eta$$
$$\leq M(a,b)M(\mathfrak{G}_0)\iint_{\mathfrak{G}}|v(\zeta)|\,d\xi\,d\eta \;\leq M(a,b,\mathfrak{G}_0)|v,\mathfrak{G}|_1.$$

By Fubini's theorem, the following integral is finite:

$$\iint_{\mathfrak{G}}\left(\iint_{\mathfrak{G}}X_{\mathfrak{G}_0}(z)\,\frac{t_\xi(\zeta)}{i+eb(\zeta)}\,\frac{v(\zeta)}{t(\zeta)-t(z)}\,d\xi\,d\eta\right)dx\,dy$$
$$=2\pi i\iint_{\mathfrak{G}_0}(J_{\mathfrak{G}}v)(z)\,dx\,dy.$$

Therefore $J_{\mathfrak{G}}v \in L^1_{loc}(\mathfrak{G})$.

Next suppose $\phi \in C^1_c(\mathfrak{G})$. Apply (3.3) to ϕ to obtain

$$\iint_{\mathfrak{G}}\frac{t_x}{i+eb}\,v\phi\,dx\,dy=\iint_{\mathfrak{G}}\frac{t_x}{i+eb}\,vJ_{\mathfrak{G}}(\mathfrak{D}\phi)\,dx\,dy$$
$$=\iint_{\mathfrak{G}}\frac{t_x(z)}{i+eb(z)}v(z)\left(\frac{1}{2\pi i}\iint_{\mathfrak{G}}\frac{t_\xi(\zeta)}{i+eb(\zeta)}\,\frac{(\mathfrak{D}\phi)(\zeta)}{t(\zeta)-t(z)}\,d\xi\,d\eta\right)dx\,dy$$
$$=-\iint_{\mathfrak{G}}\frac{t_\xi(\zeta)}{i+eb(\zeta)}\,(J_{\mathfrak{G}}v)(\zeta)(\mathfrak{D}\phi)(\zeta)\,d\xi\,d\eta.$$

The interchange of orders of integration is justified by the bound

$$\iint_{\mathfrak{G}}\left(\iint_{\mathfrak{G}}\left|\frac{t_x(z)}{i+eb(z)}\,v(z)\,\frac{t_\xi(\zeta)}{i+eb(\zeta)}\,\frac{(\mathfrak{D}\phi)(\zeta)}{t(\zeta)-t(z)}\right|d\xi\,d\eta\right)dx\,dy$$
$$\leq M(a,b)\iint_{\mathfrak{G}}|v(z)|\left(\iint_{\operatorname{supp}\phi}\frac{|\mathfrak{D}\phi(\zeta)|}{|\zeta-z|}\,d\xi\,d\eta\right)dx\,dy$$
$$\leq M(a,b,\phi)\,|v,\mathfrak{G}|_1.$$

Corollary 5.2. *If $w \in L^1_{loc}(\mathfrak{G})$, $v \in L^1(\mathfrak{G})$, and $v = \mathfrak{D}w$ in $\mathfrak{G}$, then $w = \Phi + J_{\mathfrak{G}}v$ where Φ is hyperanalytic in $\mathfrak{G}$.*

Conversely, if Φ is hyperanalytic in $\mathfrak{G}$, $v \in L^1(\mathfrak{G})$, then $\Phi + J_{\mathfrak{G}}v \in L^1_{loc}(\mathfrak{G})$, and $\mathfrak{D}(\Phi + J_{\mathfrak{G}}v) = v$.

Proof. If $w \in L^1_{\mathrm{loc}}(\mathfrak{G})$, $v \in L^1(\mathfrak{G})$, and $v = \mathfrak{D}w$ in $\mathfrak{G}$, then by the previous theorem $\mathfrak{D}(w - J_{\mathfrak{G}}v) = v - v = 0$. By Theorem 4.8, $w - J_{\mathfrak{G}}v$ is hyperanalytic in $\mathfrak{G}$.

The converse part of the corollary follows immediately from the previous theorem.

Theorem 5.3. *Let $\mathfrak{G}$ be a bounded domain. If $v \in L^p(\mathfrak{G})$, $2 < p < \infty$, then the function $w = J_{\mathfrak{G}}v$ satisfies*

(i) $|w(z)| \leq M(a, b, p, \mathfrak{G})|v, \mathfrak{G}|_p$, $z \in \mathbf{C}$,

(ii) $|w(z_1) - w(z_2)| \leq M(a, b, p) \leq |v, \mathfrak{G}|_p |z_1 - z_2|^{(p-2)/p}$, $z_1, z_2 \in \mathbf{C}$.

Proof. Using

$$w(z) = \frac{1}{2\pi i} \iint_{\mathfrak{G}} \frac{t_{\xi}(\zeta)}{i + eb(\zeta)} \frac{v(\zeta)}{t(\zeta) - t(z)}\, d\xi\, d\eta$$

we obtain the bound (where $1/p + 1/q = 1$)

$$\begin{aligned} |w(z)| &\leq M(a, b) \iint_{\mathfrak{G}} \frac{|v(\zeta)|}{|\zeta - z|}\, d\xi\, d\eta \\ &\leq M(a, b)\, |v, \mathfrak{G}|_p \left(\iint_{\mathfrak{G}} |\zeta - z|^{-q}\, d\xi\, d\eta \right)^{1/q} \leq M(a, b, p, \mathfrak{G})|v, \mathfrak{G}|_p. \end{aligned}$$

To show (ii) we use

$$w(z_1) - w(z_2) = \frac{t(z_1) - t(z_2)}{2\pi i} \iint_{\mathfrak{G}} \frac{t_{\xi}(\zeta)}{i + eb(\zeta)} \frac{v(\zeta)}{(t(\zeta) - t(z_1))(t(\zeta) - t(z_2))}\, d\xi\, d\eta.$$

Thus

$$\begin{aligned} |w(z_1) - w(z_2)| &\leq M(a, b)|z_1 - z_2| \iint_{\mathfrak{G}} \frac{|v(\zeta)|}{|\zeta - z_1||\zeta - z_2|}\, d\xi\, d\eta \\ &\leq M(a, b)|z_1 - z_2||v, \mathfrak{G}|_p \left(\iint_{\mathfrak{G}} |\zeta - z_1|^{-q} |\zeta - z_2|^{-q}\, d\xi\, d\eta \right)^{1/q}. \end{aligned}$$

But it is known [2, p. 39], that for $1 < q < 2$,

$$\iint_{\mathfrak{G}} |\zeta - z_1|^{-q} |\zeta - z_2|^{-q}\, d\xi\, d\eta \leq M(p)|z_1 - z_2|^{2-2q}.$$

Since $1 + (2 - 2q)/q = (p - 2)/p$, (ii) is proved.

Now, following Vekua, we introduce the space of functions $L^{p,\nu}(\mathbf{C})$. We use the notation $\mathbf{C}_0 \equiv$ unit disk in $\mathbf{C}$.

Definition 5.4. $w \in L^{p,\nu}(\mathbf{C})$ if w is defined in all of $\mathbf{C}$ and $w, w^{(\nu)} \in L^p(\mathbf{C}_0)$, where $w^{(\nu)}$ is defined by the formula $w^{(\nu)}(z) = |z|^{-\nu} w(1/z)$, $z \in \mathbf{C}_0$.

The norm of w in this space is given by

$$|w, \mathbf{C}|_{p,\nu} = |w, \mathbf{C}_0|_p + |w^{(\nu)}, \mathbf{C}_0|_p.$$

Theorem 5.5. *Let $v \in L^{p,2}(\mathbf{C})$, $2 < p < \infty$. Then the function $w = J_{\mathbf{C}} v = Jv$ satisfies*

(i) $|w(z)| \leq M(a, b, p)|v, \mathbf{C}|_{p,2}$, $z \in \mathbf{C}$.

(ii) $|w(z_1) - w(z_2)| \leq M(a, b, p)|v, \mathbf{C}|_{p,2}|z_1 - z_2|^{(p-2)/p}$, $z_1, z_2 \in \mathbf{C}$.

(iii) *For any $R > 1$, a constant $M(a, b, p, R)$ exists such that, for $|z| \geq R$,*

$$|w(z)| \leq M(a, b, p, R)|v, \mathbf{C}|_{p,2}|z|^{(2-p)/p}.$$

(iv) $\mathfrak{D}w = \mathfrak{D}(Jv) = v$ *in* $\mathbf{C}$.

Proof. We can write $w(z) = \tilde{w}(z) + \hat{w}(z)$, where

$$\tilde{w}(z) = \frac{1}{2\pi i} \iint_{\mathbf{C}_0} \frac{t_\xi(\zeta)}{i + eb(\zeta)} \frac{v(\zeta)}{t(\zeta) - t(z)}\, d\xi d\eta,$$

(5.2)

$$\hat{w}(z) = \frac{1}{2\pi i} \iint_{\mathbf{C}_0} \frac{t_\xi(1/\zeta)}{i + eb(1/\zeta)} \frac{v(1/\zeta)}{t(1/\zeta) - t(z)} \frac{1}{|\zeta|^4}\, d\xi d\eta.$$

By Theorem 5.3, for $z \in \mathbf{C}$, $|\tilde{w}(z)| \leq M(a, b, p)|v, \mathbf{C}_0|_p$. Also,

$$|\hat{w}(z)| \leq M(a, b) \iint_{\mathbf{C}_0} \frac{|v(1/\zeta)|}{|1/\zeta - z|\,|\zeta|^4}\, d\xi d\eta.$$

Thus, if $1/p + 1/q = 1$,

$$|\hat{w}(z)| \leq M(a, b)|v^{(2)}, \mathbf{C}_0|_p (I_1(z))^{1/q} \tag{5.3}$$

where

$$I_1(z) = \iint_{\mathbf{C}_0} |\zeta|^{-q}|1 - \zeta z|^{-q}\, d\xi d\eta.$$

If $|z| \geq ½$, then

$$I_1(z) = |z|^{-q} \iint_{\mathbf{C}_0} |\zeta|^{-q}\left|\zeta - \frac{1}{z}\right|^{-q} d\xi d\eta.$$

Using a known result for an integral of this type [3, p. 39], we obtain

(5.4)

$$I_1(z) \leq |z|^{-q} M(P)|1/z|^{2-2q},$$

$$I_1(z) \leq M(P)(1/|z|)^{2-q} \leq M(P)2^{2-q} = M(P).$$

If $|z| \leq ½$, then $|1 - z\zeta| \geq ½$, and $I_1(z) \leq 2^q \iint_{\mathbf{C}_0} |\zeta|^{-q} d\xi d\eta = M(P)$. Thus

$|\hat{w}(z)| \leq M(a, b, p)|v^{(2)}, C_0|_p$, and (i) is proved.

For (ii), apply Theorem 5.3 to obtain, for $z_1, z_2 \in C$,

$$|\tilde{w}(z_1) - \tilde{w}(z_2)| \leq M(a, b, p)\,|v, C_0|_p |z_1 - z_2|^{(p-2)/p}.$$

Also,

$$\hat{w}(z_1) - \hat{w}(z_2)$$
$$= \frac{t(z_1) - t(z_2)}{2\pi i} \iint_{C_0} \frac{t_\xi(1/\zeta)}{i + eb(1/\zeta)} \frac{v(1/\zeta)}{|\zeta|^4} \frac{1}{t(1/\zeta) - t(z_1)} \frac{1}{t(1/\zeta) - t(z_2)}\, d\xi\, d\eta$$

and

$$|\hat{w}(z_1) - \hat{w}(z_2)| \leq M(a, b)|z_1 - z_2| \iint_{C_0} \frac{|v(1/\zeta)|}{|\zeta|^4 |1/\zeta - z_1|\,|1/\zeta - z_2|}\, d\xi\, d\eta$$
$$\leq M(a, b)\,|z_1 - z_2| I_2(z_1, z_2)$$

where

$$I_2(z_1, z_2) = \iint_{C_0} \frac{|v(1/\zeta)|}{|\zeta|^2} \frac{1}{|1 - \zeta z_1|\,|1 - \zeta z_2|}\, d\xi\, dn.$$

But Vekua has shown [3, pp. 44–45] that I_2 satisfies

$$I_2(z_1, z_2) \leq M(p)|v^{(2)}, C_0|_p |z_1 - z_2|^{(p-2)/p}.$$

Thus (ii) is proved.

To prove (iii), we note that for $|z| > 1$,

$$|\tilde{w}(z)| \leq M(a, b) \iint_{C_0} \frac{|v(\zeta)|}{|\zeta - z|}\, d\xi\, d\eta$$
$$\leq M(a, b)\, \frac{1}{|z| - 1} \iint_{C_0} |v(\zeta)|\, d\xi\, d\eta$$
$$\leq M(a, b, p)|v, C_0|_p\ (1/(|z| - 1).$$

Furthermore, with use of (5.3) and (5.4) we obtain, for $|z| > 1$,

$$|\hat{w}(z)| \leq M(a, b, p)|v^{(2)}, C_0|_p\, |z|^{(2-p)/p}.$$

Hence for $|z| > 1$,

$$|w(z)| \leq M(a, b, p)|v, C|_{p,2}[|z|^{(2-p)/p} + 1/(|z| - 1)].$$

But if $R > 1$, $|z| \geq R$, then

$$1/(|z| - 1) \leq K(R)|z|^{(2-p)/p}$$

where $K(R) = (R^{1-2/p})/(R-1)$. Thus (iii) follows.

The proof of (iv) is the same as that of the last part of Theorem 5.1.

Corollary 5.6. *If* $w \in L^1_{loc}(\mathrm{C})$, $v \in L^{p,2}(\mathrm{C})$, $2 < p < \infty$, *and* $v = \mathfrak{D}w$ *in* C, *then* $w = \Phi + Jv$ *where* Φ *is hyperanalytic in* C.

Proof. $\mathfrak{D}(w - Jv) = v - v = 0$ in C. Therefore $w - Jv$ is hyperanalytic in C.

6. Properties of solutions. It is now possible to discuss certain properties of solutions of the equation

$$\mathfrak{D}w + \sum_{k=0}^{r-1} e^k \sum_{l=0}^{k} (A_{kl}w_l + B_{kl}\bar{w}_l) = 0 \tag{6.1}$$

where w is a hypercomplex function, and the A_{kl} and B_{kl} are complex valued. First we must prove a "product rule" and a "chain rule" for the operator $\mathfrak{D}$.

Theorem 6.1. *Let* $w, v \in L^1_{loc}(\mathfrak{G})$, *with* $\mathfrak{D}w$, $\mathfrak{D}v \in L^p(\mathfrak{G})$, $2 < p < \infty$. *Then in* $\mathfrak{G}$, $\mathfrak{D}(wv) = (\mathfrak{D}w)v + w(\mathfrak{D}v)$.

Proof. Since the product rule is easily verified if either w or v is in $C^1(\mathfrak{G})$, we assume it is true in these cases. Let $\mathfrak{G}_1$ be a bounded subdomain, $\overline{\mathfrak{G}}_1 \subset \mathfrak{G}$. By Corollary 5.2, we have in $\mathfrak{G}_1$

$$w = \Phi + J_{\mathfrak{G}_1}(\mathfrak{D}w), \qquad v = \Psi + J_{\mathfrak{G}_1}(\mathfrak{D}v),$$

where Φ and Ψ are hyperanalytic in $\mathfrak{G}_1$, and, by Theorem 5.3, $J_{\mathfrak{G}_1}(\mathfrak{D}w)$ and $J_{\mathfrak{G}_1}(\mathfrak{D}v)$ are in $B_\alpha(\mathrm{C})$, $\alpha = (p-2)/p$. Therefore inside $\mathfrak{G}_1$,

$$\mathfrak{D}(wv) = \Phi(\mathfrak{D}v) + \Psi(\mathfrak{D}w) + \mathfrak{D}[J_{\mathfrak{G}_1}(\mathfrak{D}w) \cdot J_{\mathfrak{G}_1}(\mathfrak{D}v)].$$

Thus it suffices to prove the product rule for the product $J_{\mathfrak{G}_1}(\mathfrak{D}w) \cdot J_{\mathfrak{G}_1}(\mathfrak{D}v)$. Let $\phi \in C^1_c(\mathfrak{G}_1)$, $\psi_n \in C^\infty_c(\mathfrak{G}_1)$, with $\psi_n \to \mathfrak{D}w$ in $L^p(\mathfrak{G}_1)$. Since Douglis showed [1, p. 273] that $J_{\mathfrak{G}_1}\psi_n \in C^1(\mathfrak{G}_1)$, we have

$$\iint_{\mathfrak{G}_1} \frac{t_x}{i + eb} [\phi(\psi_n J_{\mathfrak{G}_1}(\mathfrak{D}v) + (J_{\mathfrak{G}_1}\psi_n)\mathfrak{D}v) + \mathfrak{D}\phi(J_{\mathfrak{G}_1}\psi_n \cdot J_{\mathfrak{G}_1}(\mathfrak{D}v))]\, dx\, dy = 0.$$

By Theorem 5.3, $J_{\mathfrak{G}_1}\psi_n \to J_{\mathfrak{G}_1}(\mathfrak{D}w)$ uniformly in C. Thus after taking limits, we have the result

$$\mathfrak{D}[J_{\mathfrak{G}_1}(\mathfrak{D}w) \cdot J_{\mathfrak{G}_1}(\mathfrak{D}v)] = \mathfrak{D}w \cdot J_{\mathfrak{G}_1}(\mathfrak{D}v) + J_{\mathfrak{G}_1}(\mathfrak{D}w) \cdot \mathfrak{D}v.$$

Theorem 6.2. *Let $w \in L^1_{loc}(\mathfrak{G})$, $\mathfrak{D}w \in L^p(\mathfrak{G})$, $2 < p < \infty$, and $w(z) = w_0(z) + N(z)$ where w_0 is complex and N nilpotent. Let the values of w_0 lie inside a bounded domain $\mathfrak{G}_*$, and let f be a complex function which is analytic in some domain containing $\overline{\mathfrak{G}}_*$. Then in $\mathfrak{G}$, $\mathfrak{D}f(w(z)) = f'(w(z))(\mathfrak{D}w)(z)$.*

Proof. Recall that $f(w(z))$ is given by

$$f(w(z)) = \sum_{k=0}^{r-1} \frac{1}{k!} (N(z))^k f^{(k)}(w_0(z)).$$

Let $\phi \in C^1_c(\mathfrak{G})$, and let $\mathfrak{G}_1$ be a bounded subdomain of $\mathfrak{G}$ containing the support of ϕ. Then inside $\mathfrak{G}_1$, $w = \Phi + J_{\mathfrak{G}_1}(\mathfrak{D}w)$ where Φ is hyperanalytic in $\mathfrak{G}_1$. Let $\Psi_n \in C^1_\infty(\mathfrak{G})$, $\Psi_n \to \mathfrak{D}w$ in $L^p(\mathfrak{G}_1)$. By Theorem 5.3, the sequence $w_n = \Phi + J_{\mathfrak{G}_1}\Psi_n$ converges uniformly in $\mathfrak{G}_1$ to w. Writing $w_n = (w_n)_0 + N_n$ where N_n is nilpotent, we have

$$f^{(k)}((w_n)_0(z)) \to f^{(k)}(w_0(z)),$$
$$(N_n(z))^k \to (N(z))^k$$

uniformly in $\mathfrak{G}_1$. Therefore,

$$\iint_{\mathfrak{G}_1} \frac{t_x(z)}{i + eb(z)} f(w(z))(\mathfrak{D}\phi)(z)\,dx\,dy$$
$$= \lim_n \iint_{\mathfrak{G}_1} \frac{t_x(z)}{i + eb(z)} f(w_n(z))(\mathfrak{D}\phi)(z)\,dx\,dy$$
$$= \lim_n -\iint_{\mathfrak{G}_1} \frac{t_x(z)}{i + eb(z)} f'(w_n(z))(\mathfrak{D}w_n)(z)\phi(z)\,dx\,dy$$
$$= \lim_n -\iint_{\mathfrak{G}_1} \frac{t_x(z)}{i + eb(z)} f'(w_n(z))\Psi_n(z)\phi(z)\,dx\,dy$$
$$= -\iint_{\mathfrak{G}_1} \frac{t_x(z)}{i + eb(z)} f'(w(z))(\mathfrak{D}w)(z)\phi(z)\,dx\,dy.$$

The next result was proved by Vekua [3, p. 154].

Theorem 6.3. *Let A, B and w be complex functions in $\mathbf{C}$ with $A, B \in L^{p,2}(\mathbf{C})$, $2 < p < \infty$. If w is continuous and bounded on the whole plane $\mathbf{C}$, and satisfies $\partial w/\partial \bar{z} + Aw + B\bar{w} = 0$ then w has the form $w(z) = C \exp \omega(z)$ where C is a complex constant, and ω is a complex function continuous and bounded* *on the whole plane. (Thus w is bounded away from zero provided it is not identically zero.)*

Following is an analogous result for solutions of (6.1).

Theorem 6.4. *Let w be continuous and bounded and satisfy* (6.1) *in the whole plane, where each A_{kl}, $B_{kl} \in L^{p,2}(\mathbf{C})$, $2<p<\infty$. Then w has the form*

$$w(z) = C \exp \omega(z) \tag{6.2}$$

where C is a hypercomplex constant, and ω is a hypercomplex function in $B_\alpha(\mathbf{C})$, $\alpha = (p-2)/p$. Moreover, near infinity, $\omega(z) = O(|z|^{-\alpha})$.

Proof. If w is identically zero, we can set $C = 0$, $\omega = 0$. In general, let p be the smallest value of k such that w_k is not identically zero. Since w is continuous and bounded, (6.1) shows that $\mathfrak{D}w \in L^{p,2}(\mathbf{C}) \subset L^p(\mathbf{C})$. We can use Theorem 4.6 and set the coefficient of e^p in (6.1) equal to zero to obtain

$$\partial w_p/\partial x + i\partial w_p/\partial y + A_{p,p}w_p + B_{p,p}\bar{w}_p = 0.$$

By the preceding theorem, w_p is bounded away from zero, and therefore the function

$$\left(\sum_{k=p}^{r-1} w_k e^{k-p}\right)^{-1}$$

is bounded and continuous (see (1.14)). Define v to be the function

$$v = \left(\sum_{k=p}^{r-1} w_k e^{k-p}\right)^{-1} \sum_{k=p}^{r-1} e^{k-p} \sum_{l=0}^{k} (A_{kl}w_l + B_{kl}\bar{w}_l).$$

Then $v \in L^{p,2}(\mathbf{C})$, and

$$wv = e^p\left(\sum_{k=p}^{r-1} w_k e^{k-p}\right) \cdot v = \sum_{k=p}^{r-1} e^k \sum_{l=0}^{k} (A_{kl}w_l + B_{kl}\bar{w}_l) = -\mathfrak{D}w.$$

Now define $\omega = -Jv$, $\Phi = w\exp(-\omega) = w\exp(Jv)$. By Theorems 6.1, 6.2, $\mathfrak{D}\Phi = \exp(Jv)[wv + \mathfrak{D}w] = 0$, and Φ is hyperanalytic. The stated properties of ω are a result of Theorem 5.5.

Finally, since ω is bounded, $\exp(-\omega)$ is also bounded. By Theorem 3.9, Φ is constant.

Corollary 6.5. *If w satisfies the conditions of Theorem* 6.4, *and $w(z_0) = 0$ for some z_0 in $\mathbf{C}$, then w is identically zero.*

Proof. Since $\exp(z+E)$ is never nilpotent for any hypercomplex number $z+E$, $w(z_0)=0$ implies $C=0$ in the representation (6.2).

It is now convenient to introduce an operator Q, given in terms of the operator J and the coefficients in (6.1).

Definition 6.6. The operator Q is given by

$$Qw = J\left[\sum_{k=0}^{r-1} e^k \sum_{l=0}^{k} (A_{kl}w_l + B_{kl}\bar{w}_l)\right].$$

Theorem 6.7. *If each A_{kl}, B_{kl} is in $L^{p,2}(\mathbf{C})$, $2<p<\infty$, then Q is compact in the space $B(\mathbf{C})$ and maps this space into $B_\alpha(\mathbf{C})$, $\alpha=(p-2)/p$. Moreover, near infinity, $|Qw(z)| = O(|z|^{-\alpha})$.*

Proof. If $w \in B(\mathbf{C})$, then the function

$$v = \sum_{k=0}^{r-1} e^k \sum_{l=0}^{k} (A_{kl}w_l + B_{kl}\bar{w}_l)$$

is in $L^{p,2}(\mathbf{C})$. By Theorem 5.5, $Qw = Jv$ is in $B_\alpha(\mathbf{C})$. Furthermore, properties (i), (ii), and (iii) of Theorem 5.5 yield

(i) $|Qw(z)| \le M(a,b,p,A_{kl},B_{kl})|w,\mathbf{C}|_\infty$,

(ii) $|Qw(z_1)-Qw(z_2)| \le M(a,b,p,A_{kl},B_{kl})|w,\mathbf{C}|_\infty |z_1-z_2|^\alpha$,

(iii) for $|z|\ge 2$, $|w(z)| \le M(a,b,p,A_{kl},B_{kl})|w,\mathbf{C}|_\infty |z|^{-\alpha}$

(The dependence of M on the A_{kl} and B_{kl} arises of course from bounds on the $L^{p,2}(\mathbf{C})$ norms of these functions.) Thus a family in $B(\mathbf{C})$ which is uniformly bounded is mapped onto a family in $B_\alpha(\mathbf{C})$ which is uniformly bounded and equicontinuous, and uniformly $O(|z|^{-\alpha})$ at infinity. Hence by Arzela's theorem, Q is compact in the space $B(\mathbf{C})$.

Theorem 6.8. *If each A_{kl}, B_{kl} is in $L^{p,2}(\mathbf{C})$, $2<p<\infty$, and v is in $B(\mathbf{C})$, then the equation $w+Qw=v$ has a unique solution in $B(\mathbf{C})$.*

Proof. Since Q is compact in $B(\mathbf{C})$, it is sufficient to show the homogeneous equation $w+Qw=0$ has only the zero solution. But if $w=-Qw$, then w is in $B(\mathbf{C})$ and vanishes at infinity. Furthermore, differentiation shows that w satisfies (6.1). Hence w has the representation (6.2), and since w vanishes at infinity, $C=0$ and $w=0$.

We can now give an analogue to the "generating pairs" developed by Bers [2].

Suppose w is continuous and bounded in $\mathbf{C}$ and satisfies (6.1), with A_{kl}, $B_{kl} \in L^{p,2}(\mathbf{C})$, $2<p<\infty$. By Corollary 5.6, $w + Qw = \Phi$ where Φ is hyperanalytic in $\mathbf{C}$. But w, $Qw \in B(\mathbf{C})$, and thus Φ is a constant. Therefore

$$w + Qw = c = \tilde{c} + i\hat{c} \tag{6.3}$$

where $\tilde{c}$ and $\hat{c}$ are real hypercomplex numbers (i.e., each coefficient of e^k is real). Let $\tilde{w}$ be the unique solution in $B(\mathbf{C})$ of $\tilde{w} + Q\tilde{w} = 1$, and let $\hat{w}$ be the unique solution in $B(\mathbf{C})$ of $\hat{w} + Q\hat{w} = i$. Then w, the unique solution in $B(\mathbf{C})$ to (6.3), is clearly $w = \tilde{c}\tilde{w} + \hat{c}\hat{w}$. Therefore any solution of (6.1) for the same functions A_{kl}, B_{kl} is of this form. The solutions $\tilde{w}$ and $\hat{w}$ are called the *generating pair* associated with the coefficients A_{kl}, B_{kl} and the equation (6.1).

7. **Further properties of solutions.** We now direct our attention to the equation

$$\mathfrak{D}w + Aw + B\bar{w} = 0 \tag{7.1}$$

where w, A, and B are hypercomplex functions. For this special case of (6.1), we can obtain integral representations for solutions w in a regular domain $\mathfrak{G}$. The next lemma and theorem establish a Green's identity for solutions of (7.1).

Lemma 7.1. *Let $\mathfrak{G}$ be a regular domain, with $u \in C(\bar{\mathfrak{G}})$, $\mathfrak{D}u \in L^p(\mathfrak{G})$ where $2 < p < \infty$. Then*

$$\int_\Gamma u(z)\,dt(z) = -\iint_{\mathfrak{G}} \frac{t_x(z)}{i + eb(z)} (\mathfrak{D}u)(z)\,dx\,dy.$$

Proof. Let $\{\psi_n\}$ be a sequence in $C_c^1(\mathfrak{G})$ such that $\psi_n \to \mathfrak{D}u$ in $L^p(\mathfrak{G})$. By Theorem 5.3, $J_{\mathfrak{G}}\psi_n \to J_{\mathfrak{G}}(\mathfrak{D}u)$ pointwise uniformly in the whole plane $\mathbf{C}$. In Theorem 3.2, we set $w \equiv J_{\mathfrak{G}}\psi_n$, $v \equiv 1$, to obtain

$$\int_\Gamma (J_{\mathfrak{G}}\,\psi_n)(z)\,dt(z) = -\iint_{\mathfrak{G}} \frac{t_x(z)}{i + eb(z)}\, \psi_n(z)\,dx\,dy.$$

Letting $n \to \infty$, we have

$$\int_\Gamma J_{\mathfrak{G}}(\mathfrak{D}u)(z)\,dt(z) = -\iint_{\mathfrak{G}} \frac{t_x(z)}{i + eb(z)} (\mathfrak{D}u)(z)\,dx\,dy.$$

By Corollary 5.2, in $\mathfrak{G}$, $J_{\mathfrak{G}}(\mathfrak{D}u) = u - \Phi$, where Φ is hyperanalytic in $\mathfrak{G}$. Furthermore, $J_{\mathfrak{G}}(\mathfrak{D}u)$, $u \in C(\bar{\mathfrak{G}})$, and thus $\Phi \in C(\bar{\mathfrak{G}})$. We may use Theorem 3.2 with $w \equiv \Phi$, $v \equiv 1$, to conclude

$$\int_\Gamma \Phi(z)\,dt(z) = 0,$$

and the theorem is proved.

Definition 7.2. For fixed A and B, we define the operator

$$Cw \equiv \mathfrak{D}w + Aw + B\bar{w} \tag{7.2}$$

and an associated operator

$$\tilde{C}v \equiv \mathfrak{D}v - Av + B^*\bar{v} \tag{7.3}$$

where B^* is defined by

$$B^* \equiv \frac{i+eb}{t_x}\,\frac{\overline{t_x}}{\overline{i+eb}}\,\bar{B}. \tag{7.4}$$

Theorem 7.3. *Let $\mathfrak{G}$ be a regular domain, and $A, B \in L^p(\mathfrak{G})$, with $2 < p < \infty$. If $w, v \in C(\overline{\mathfrak{G}})$, and satisfy, in $\mathfrak{G}$, $Cw = 0$, $\tilde{C}v = 0$, then the integral $\int_\Gamma w(z)v(z)dt(z)$ is a real hypercomplex number.*

Proof. Since $w, v \in C(\overline{\mathfrak{G}})$,

$$\mathfrak{D}w = -Aw - B\bar{w} \in L^p(\mathfrak{G}), \qquad \mathfrak{D}v = Av - B^*\bar{v} \in L^p(\mathfrak{G}).$$

(We remark that the quantity $((i+eb)/t_x)(\overline{t_x}/\overline{(i+eb)})$ is bounded in the whole complex plane, as can be seen by (1.13).) By Lemma 7.1 and Theorem 6.1,

$$\begin{aligned}
\int_\Gamma w(z)v(z)\,dt(z) &= -\iint_{\mathfrak{G}} \frac{t_x}{i+eb}\,\mathfrak{D}(wv)\,dx\,dy \\
&= -\iint_{\mathfrak{G}} \frac{t_x}{i+eb}\,(w\mathfrak{D}v + v\mathfrak{D}w)\,dx\,dy \\
&= \iint_{\mathfrak{G}} \left(\frac{t_x}{i+eb}\,B\bar{w}v + \frac{\overline{t_x}}{\overline{i+eb}}\,\bar{B}w\bar{v}\right)dx\,dy
\end{aligned}$$

which is a real hypercomplex number.

Following Vekua's techniques, we now introduce some special solutions of (7.1).

Theorem 7.4. *Let A and B be hypercomplex functions in $L^{p,2}(\mathbf{C})$, where $2 < p < \infty$. Then there exist hypercomplex functions of two complex variables, $X^{(1)}(z,\zeta)$ and $X^{(2)}(z,\zeta)$, with the properties*

(1) *In $\mathbf{C} - \{\zeta\}$, for $j = 1, 2$,*

$$\mathfrak{D}_z X^{(j)}(z,\zeta) + A(z)X^{(j)}(z,\zeta) + B(z)\overline{X^{(j)}(z,\zeta)} = 0.$$

(Here $\mathfrak{D}_z$ denotes our usual differential operator $\mathfrak{D}$ where differentiation is with respect to the variable z rather than ζ.)

(2) $$X^{(1)}(z,\zeta) = \frac{\exp[\omega^{(1)}(z) - \omega^{(1)}(\zeta)]}{2(t(\zeta) - t(z))}, \qquad X^{(2)}(z,\zeta) = \frac{\exp[\omega^{(2)}(z) - \omega^{(2)}(\zeta)]}{2i(t(\zeta) - t(z))}$$

where for $j = 1, 2$, $\omega^{(j)} \in B_\alpha(\mathbf{C})$, $\alpha = (p-2)/p$, *and* $\omega^{(j)}(z) = O(|z|^{-\alpha})$ *as* $|z| \to \infty$.

Proof. Since the proof for $X^{(1)}$ and $X^{(2)}$ are nearly identical, we give the proof only for $X^{(1)}$.(2) We temporarily fix a point ζ in $\mathbf{C}$, and define a function $\hat{B}$ by

$$\hat{B}(z) \equiv B(z)\frac{t(z) - t(\zeta)}{\overline{t(z) - t(\zeta)}}.$$

We have $\hat{B} \in L^{p,2}(\mathbf{C})$, since

$$|\hat{B}(z)| \le |B(z)| \cdot |t(z) - t(\zeta)| \cdot \left|\frac{1}{t(z) - t(\zeta)}\right|$$
$$\le |B(z)| \cdot M(a, b) \cdot |z - \zeta| \frac{M(a, b)}{|z - \zeta|} \le M(a, b) \cdot |B(z)|.$$

Now consider the functional equation

$$w(z) + (Qw)(z) - (Qw)(\zeta) = 1, \quad z \in \mathbf{C}, \tag{7.5}$$

where Q is the operator defined by $Qw \equiv J(Aw + \hat{B}\bar{w})$. By Theorem 6.7, Q is compact in the space $B(\mathbf{C})$. If we define an operator P by

$$(Pw)(z) = (Qw)(z) - (Qw)(\zeta), \quad z \in \mathbf{C},$$

then (7.5) may be written as

$$w(z) + (Pw)(z) = 1, \quad z \in \mathbf{C}. \tag{7.6}$$

Moreover, since Q maps a bounded sequence in $B(\mathbf{C})$ onto a sequence in $B(\mathbf{C})$ with a convergent subsequence, P has the same property. Thus P is compact in $B(\mathbf{C})$. Therefore, in order to show (7.5) has a solution in $B(\mathbf{C})$, it is sufficient to show the homogeneous equation has only the zero solution. Suppose then that $v \in B(\mathbf{C})$ and satisfies

$$v(z) + (Qv)(z) - (Qv)(\zeta) = 0, \quad z \in \mathbf{C}. \tag{7.7}$$

Differentiating this equation, we obtain

$$\mathfrak{D}v + Av + \hat{B}\bar{v} = 0.$$

Since $v(\zeta) = 0$, by Corollary 6.5, $v = 0$.

Thus we may let w be the unique solution in $B(\mathbf{C})$ to (7.5). Differentiating (7.5), we obtain

(2) In the case of $X^{(2)}$ we replace the 1 on the right-hand sides of (7.5) and (7.6) by $-i$. This serves to define the functions $w_2(z, \zeta) \equiv -i \exp[\omega_2(z) - \omega_2(\zeta)]$, and $X^2(z, \zeta) \equiv \frac{1}{2}w_2(z, \zeta)[t(\zeta) - t(z)]^{-1}$.

$$\mathfrak{D}w + Aw + \hat{B}\overline{w} = 0.$$

According to Theorem (6.4), w has the form $w(z) = C \exp \omega(z)$, where C is a hypercomplex constant, $\omega \in B_\alpha(\mathbf{C})$, $\alpha = (p-2)/p$, and $\omega(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$. But since $w(\zeta) = 1$, we conclude $C = \exp[-\omega(\zeta)]$, and

$$w(z) = \exp[\omega(z) - \omega(\zeta)] \equiv w(z, \zeta).$$

We now set

$$X^{(1)}(z, \zeta) \equiv \frac{w(z, \zeta)}{2(t(\zeta) - t(z))} = \frac{\exp[\omega(z) - \omega(\zeta)]}{2(t(\zeta) - t(z))}.$$

Then, for $z \in \mathbf{C} - \{\zeta\}$,

$$\begin{aligned}\mathfrak{D}_z X^{(1)}(z, \zeta) &= \frac{1}{2(t(\zeta) - t(z))} \mathfrak{D}_z w(z, \zeta) \\ &= \frac{1}{2(t(\zeta) - t(z))} \cdot (-A(z)w(z, \zeta) - \hat{B}(z)\overline{w(z, \zeta)}) \\ &= -A(z)X^{(1)}(z, \zeta) - B(z)\overline{X^{(1)}(z, \zeta)}.\end{aligned}$$

We now define *fundamental kernels* associated with fixed A, B in $L^{p,2}(\mathbf{C})$ and the equation (7.1).

Definition 7.5. The fundamental kernels $\Omega^{(1)}$ and $\Omega^{(2)}$, associated with A and B in $L^{p,2}(\mathbf{C})$, are

$$\Omega^{(1)}(z, \zeta) \equiv X^{(1)}(z, \zeta) + iX^{(2)}(z, \zeta), \tag{7.8}$$

$$\Omega^{(2)}(z, \zeta) \equiv X^{(1)}(z, \zeta) - iX^{(2)}(z, \zeta), \tag{7.9}$$

where $X^{(1)}$ and $X^{(2)}$ are the functions described in Theorem 7.4.

Theorem 7.6. *The fundamental kernels $\Omega^{(1)}$ and $\Omega^{(2)}$ satisfy:*

(1) *For each ζ in $\mathbf{C}$, in $\mathbf{C} - \{\zeta\}$*

$$\mathfrak{D}_z\Omega^{(1)}(z, \zeta) + A(z)\Omega^{(1)}(z, \zeta) + B(z)\overline{\Omega^{(2)}(z, \zeta)} = 0, \tag{7.10}$$

$$\mathfrak{D}_z\Omega^{(2)}(z, \zeta) + A(z)\Omega^{(2)}(z, \zeta) + B(z)\overline{\Omega^{(1)}(z, \zeta)} = 0. \tag{7.11}$$

(2) *For fixed ζ, and $j = 1, 2$,*

$$|\Omega^{(j)}(z, \zeta)| = O(|z|^{-1}) \quad \text{as } |z| \to \infty.$$

(3) *As $|z - \zeta| \to 0$,*

$$\left|\Omega^{(1)}(z, \zeta) - \frac{1}{t(\zeta) - t(z)}\right| = O(|z - \zeta|^{-2/p}), \tag{7.12}$$

$$|\Omega^{(2)}(z,\zeta)| = O(|z-\zeta|^{-2/p}). \tag{7.13}$$

Proof. Property (1) is readily verified from (1) of Theorem 7.4. Property (2) follows from the relations

$$\Omega^{(1)}(z,\zeta) = \frac{\exp[\omega^{(1)}(z)-\omega^{(1)}(\zeta)] + \exp[\omega^{(2)}(z)-\omega^{(2)}(\zeta)]}{2(t(\zeta)-t(z))},$$

$$\Omega^{(2)}(z,\zeta) = \frac{\exp[\omega^{(1)}(z)-\omega^{(1)}(\zeta)] - \exp[\omega^{(2)}(z)-\omega^{(2)}(\zeta)]}{2(t(\zeta)-t(z))},$$

because each $\omega^{(j)}$ is bounded in $\mathbf{C}$, and by (2.6).

To show (3), first we remark that from (3.11) it is easily seen that the hypercomplex function $\exp(z+E)$ is uniformly Lipschitz continuous wherever $z+E$ remains bounded. Since each $\omega^{(j)}$ is bounded, there is a positive constant K such that

$$\left|\Omega^{(1)}(z,\zeta) - \frac{1}{t(\zeta)-t(z)}\right|$$

$$= \left|\frac{\exp[\omega^{(1)}(z)-\omega^{(1)}(\zeta)] - \exp[0] + \exp[\omega^{(2)}(z)-\omega^{(2)}(\zeta)] - \exp[0]}{2(t(\zeta)-t(z))}\right|$$

$$\leq (K|\omega^{(1)}(z)-\omega^{(1)}(\zeta)| + K|\omega^{(2)}(z)-\omega^{(2)}(\zeta)|)\frac{M(a,b)}{2|\zeta-z|}.$$

Because each $\omega^{(j)} \in B_\alpha(\mathbf{C})$, $\alpha=(p-2)/p$, we obtain the desired result for $\Omega^{(1)}$. To obtain the result for $\Omega^{(2)}$, we apply the same analysis to the equation

$$\Omega^{(2)}(z,\zeta)$$

$$= \frac{\exp[\omega^{(1)}(z)-\omega^{(1)}(\zeta)] - \exp[0] + \exp[0] - \exp[\omega^{(2)}(z)-\omega^{(2)}(\zeta)]}{2(t(\zeta)-t(z))}.$$

The next three theorems develop an integral formula for solutions of (7.1). The proofs are variations on those of Vekua for the case when (7.1) is a complex equation.

Theorem 7.7. *Let $\mathfrak{G}$ be a regular domain, and let A and B be in $L^{p,2}(\mathbf{C})$ where $2<p<\infty$. Furthermore let w be in $C(\overline{\mathfrak{G}})$ and satisfy in $\mathfrak{G}$*

$$Cw = \mathfrak{D}w + Aw + B\overline{w} = 0.$$

If $\tilde{\Omega}^{(1)}$ and $\tilde{\Omega}^{(2)}$ are the fundamental kernels for the associated equation

$$\tilde{C}v = \mathfrak{D}v - Av + B^*\overline{v} = 0,$$

then

$$-\frac{1}{2\pi i}\int_{\Gamma}\tilde{\Omega}^{(1)}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\tilde{\Omega}^{(2)}(\zeta, z)w(\zeta)\,dt(\zeta)}$$

$$= \begin{cases} w(z), & \textit{if } z \in \mathfrak{G}, \\ 0, & \textit{if } z \notin \overline{\mathfrak{G}}. \end{cases}$$

Proof. Let $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ be the corresponding solutions of $\tilde{C}v = 0$ as described in Theorem 7.4. Using Theorem 7.3, we obtain the formulas, for $j = 1, 2$,

$$\int_{\Gamma}\widetilde{X^{(j)}}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\widetilde{X^{(j)}}(\zeta, z)w(\zeta)\,dt(\zeta)},$$

$$= \begin{cases} \displaystyle\int_{|\zeta - z| = \epsilon}\widetilde{X^{(j)}}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\widetilde{X^{(j)}}(\zeta, z)w(\zeta)\,dt(\zeta)}, & \text{if } z \in \mathfrak{G}, \\ 0, \quad \text{if } z \notin \overline{\mathfrak{G}}. \end{cases}$$

where ϵ is a sufficiently small positive number. We multiply by i the equation for $j = 2$ and add to the equation for $j = 1$ to obtain

$$\int_{\Gamma}\widetilde{\Omega^{(1)}}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\widetilde{\Omega^{(2)}}(\zeta, z)w(\zeta)\,dt(\zeta)}$$

$$= \begin{cases} \displaystyle\int_{|\zeta - z| = \epsilon}\widetilde{\Omega^{(1)}}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\widetilde{\Omega^{(2)}}(\zeta, z)w(\zeta)\,dt(\zeta)}, & \text{if } z \in \mathfrak{G}, \\ 0, \quad \text{if } z \notin \overline{\mathfrak{G}}. \end{cases}$$

Using (7.12), (7.13) we obtain for z in $\mathfrak{G}$,

$$\lim_{\epsilon \to 0}\int_{|\zeta - z| = \epsilon}\widetilde{\Omega^{(1)}}(\zeta, z)w(\zeta)\,dt(\zeta) - \overline{\widetilde{\Omega^{(2)}}(\zeta, z)w(\zeta)\,dt(\zeta)}$$

$$= \lim_{\epsilon \to 0}\int_{|\zeta - z| = \epsilon}\frac{w(\zeta)}{t(z) - t(\zeta)}\,dt(\zeta).$$

But Douglis showed [1, pp. 271–272], using the continuity of w, that the latter limit is $-2\pi i\, w(z)$. Thus the theorem is proved.

Theorem 7.8. *Let $A, B \in L^{p,2}(\mathrm{C})$, where $2 < p < \infty$. Let $\Omega^{(1)}, \Omega^{(2)}$ be fundamental kernels for the equation $Cw = \mathfrak{D}w + Aw + B\overline{w} = 0$ and $\widetilde{\Omega^{(1)}}, \widetilde{\Omega^{(2)}}$ fundamental kernels for the equation $\tilde{C}v = \mathfrak{D}v - Av + B^{*}\overline{v} = 0$. Then, for $z \neq \zeta$,*

$$\Omega^{(1)}(z, \zeta) = -\widetilde{\Omega^{(1)}}(\zeta, z), \qquad \Omega^{(2)}(z, \zeta) = -\overline{\widetilde{\Omega^{(2)}}(\zeta, z)}.$$

Proof. Let z, ζ be fixed, $z \neq \zeta$, and let ϵ be small enough that $0 < \epsilon < |z - \zeta| < 1/\epsilon$. Then by the previous theorem, for $j = 1, 2$,

$$
\begin{aligned}
X^{(j)}(z, \zeta) = &-\frac{1}{2\pi i} \int_{|s-\zeta|=1/\epsilon} \tilde{\Omega}^{(1)}(s, z) X^{(j)}(s, \zeta)\, dt(s) \\
&- \overline{\tilde{\Omega}^{(2)}(s, z) X^{(j)}(s, \zeta)\, dt(s)} + \frac{1}{2\pi i} \int_{|s-\zeta|=\epsilon} \tilde{\Omega}^{(1)}(s, z) X^{(j)}(s, \zeta)\, dt(s) \\
&- \overline{\tilde{\Omega}^{(2)}(s, z) X^{(j)}(s, \zeta)\, dt(s)}.
\end{aligned}
$$

Using Theorem 7.6 and the relations (7.8), (7.9), we obtain the estimates

$$
\begin{aligned}
|\tilde{\Omega}^{(j)}(s, z)|,\ |X^{(j)}(s, \zeta)| &= O(|s|^{-1}) && \text{as } |s| \to \infty, \\
\left| X^{(1)}(s, \zeta) - \frac{1}{2(t(\zeta) - t(s))} \right| &= O(|s - \zeta|^{-2/p}) && \text{as } |s - \zeta| \to 0, \\
\left| X^{(2)}(s, \zeta) - \frac{1}{2i(t(\zeta) - t(s))} \right| &= O(|s - \zeta|^{-2/p}) && \text{as } |s - \zeta| \to 0.
\end{aligned}
$$

Letting $\epsilon \to 0$, we therefore obtain

$$
\begin{aligned}
X^{(1)}(z, \zeta) &= \lim_{\epsilon \to 0} \frac{1}{4\pi i} \left[\int_{|s-\zeta|=\epsilon} \frac{\tilde{\Omega}^{(1)}(s, z)}{(t(\zeta) - t(s))}\, dt(s) - \overline{\frac{\tilde{\Omega}^{(2)}(s, z)}{(t(\zeta) - t(s))}\, dt(s)} \right], \\
X^{(2)}(z, \zeta) &= \lim_{\epsilon \to 0} \frac{-1}{4\pi} \left[\int_{|s-\zeta|=\epsilon} \frac{\tilde{\Omega}^{(1)}(s, z)}{(t(\zeta) - t(s))}\, dt(s) - \overline{\frac{\tilde{\Omega}^{(2)}(s, z)}{(t(\zeta) - t(s))}\, dt(s)} \right].
\end{aligned}
$$

As in the proof of the previous theorem, we remark that Douglis has shown [1, pp. 271–272] that the above limits are

$$
\begin{aligned}
X^{(1)}(z, \zeta) &= \frac{1}{4\pi i} \left[-2\pi i \tilde{\Omega}^{(1)}(\zeta, z) + \overline{2\pi i \tilde{\Omega}^{(2)}(\zeta, z)} \right] \\
&= -\frac{1}{2} \left[\tilde{\Omega}^{(1)}(\zeta, z) + \overline{\tilde{\Omega}^{(2)}(\zeta, z)} \right], \\
X^{(2)}(z, \zeta) &= -\frac{1}{4\pi} \left[-2\pi i \tilde{\Omega}^{(1)}(\zeta, z) - \overline{2\pi i \tilde{\Omega}^{(2)}(\zeta, z)} \right] \\
&= -\frac{1}{2i} \left[\tilde{\Omega}^{(1)}(\zeta, z) - \overline{\tilde{\Omega}^{(2)}(\zeta, z)} \right].
\end{aligned}
$$

The relations (7.8), (7.9) complete the proof.

The next result is merely stated. It is an immediate consequence of the preceding two theorems.

Theorem 7.9. *Let $\mathfrak{G}$ be a regular domain, and let A and B be in $L^{p,2}(\mathbf{C})$ where $2 < p < \infty$. Furthermore, let w be in $C(\overline{\mathfrak{G}})$ and satisfy in $\mathfrak{G}$*

$$Cw = \mathfrak{D}w + Aw + B\bar{w} = 0.$$

Then

$$\frac{1}{2\pi i}\int_\Gamma \Omega^{(1)}(z,\zeta)w(\zeta)\,dt(\zeta) - \Omega^{(2)}(z,\zeta)\overline{w(\zeta)}\,\overline{dt(\zeta)} = \begin{cases} w(z), & \text{if } z \in \mathfrak{G}, \\ 0, & \text{if } z \notin \overline{\mathfrak{G}}. \end{cases} \tag{7.14}$$

Theorem 7.10. *Let $\mathfrak{G}$ be a regular domain, and let ψ be a function which is hyperanalytic outside $\overline{\mathfrak{G}}$, continuous up to the boundary Γ, and vanishes at infinity. Then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{\psi(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta) = \begin{cases} 0, & \text{if } z \in \mathfrak{G}, \\ -\psi(z), & \text{if } z \notin \overline{\mathfrak{G}}. \end{cases} \tag{7.15}$$

Proof. For given z in $\mathbf{C}$, choose R positive such that $R > 2|z|$ and $\mathfrak{G} \subset \{\zeta : |\zeta| \leq R\}$. Then applying Theorems 3.2 and 3.4 we have

$$\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\psi(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta) - \frac{1}{2\pi i}\int_\Gamma \frac{\psi(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta) = \begin{cases} 0, & \text{if } z \in \mathfrak{G}, \\ \psi(z), & \text{if } z \notin \overline{\mathfrak{G}}. \end{cases}$$

(We have used an elementary consequence of Theorem 3.2, given in Douglis [1, p. 276], stating that if Φ is hyperanalytic in a regular domain and continuous up to the boundary Γ, then $\int_\Gamma \Phi(\zeta)dt(\zeta) = 0$.) Now, setting $\zeta = Re^{i\theta}$ on $|\zeta| = R$, and using $|\zeta - z| \geq R/2$,

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{\psi(\zeta)}{t(\zeta)-t(z)}\,dt(\zeta)\right| \leq \frac{M(a,b)}{2\pi}\int_0^{2\pi} \frac{|\psi(Re^{i\theta})|}{|\zeta - z|}\,R\,d\theta$$
$$\leq \frac{M(a,b)}{2\pi}\sup_{|\zeta|=R}|\psi(\zeta)| \cdot (4\pi).$$

Since $\sup_{|\zeta|=R}|\psi(\zeta)| \to 0$ as $R \to \infty$, the theorem is proved.

Theorem 7.11. *Let $\mathfrak{G}$ be a regular domain, and let ψ be a function which is hyperanalytic outside $\overline{\mathfrak{G}}$, continuous up to the boundary Γ, and vanishes at infinity. Let A and B be in $L^{p,2}(\mathbf{C})$ where $2 < p < \infty$, and outside $\overline{\mathfrak{G}}$, $A = B \equiv 0$. Then if $\Omega^{(1)}$ and $\Omega^{(2)}$ are the fundamental kernels for A and B,*

$$\frac{1}{2\pi i}\int_\Gamma \Omega^{(1)}(z,\zeta)\psi(\zeta)\,dt(\zeta) - \Omega^{(2)}(z,\zeta)\overline{\psi(\zeta)}\,\overline{dt(\zeta)} = \begin{cases} 0, & \text{if } z \in \mathfrak{G}, \\ -\psi(z), & \text{if } z \notin \overline{\mathfrak{G}}. \end{cases} \tag{7.16}$$

Proof. For given z in $\mathbf{C}$, choose R as in the proof of the previous theorem. Since $A = B \equiv 0$ outside $\overline{\mathfrak{G}}$, ψ satisfies (7.1) outside $\overline{\mathfrak{G}}$. Thus we can apply Theorem 7.9 to the domain $\{\zeta: |\zeta| \leq R\} - \overline{\mathfrak{G}}$ to obtain

$$\frac{1}{2\pi i}\int_{|\zeta|=R} \Omega^{(1)}(z,\zeta)\psi(\zeta)\,dt(\zeta) - \Omega^{(2)}(z,\zeta)\overline{\psi(\zeta)}\,\overline{dt(\zeta)}$$

$$-\frac{1}{2\pi i}\int_{\Gamma} \Omega^{(1)}(z,\zeta)\psi(\zeta)\,dt(\zeta) - \Omega^{(2)}(z,\zeta)\overline{\psi(\zeta)}\,\overline{dt(\zeta)}$$

$$= \begin{cases} 0, & \text{if } z \in \mathfrak{G}, \\ \psi(z), & \text{if } z \notin \overline{\mathfrak{G}}. \end{cases}$$

Using the same estimates as in the proof of the previous theorem, along with the estimates, for $j = 1, 2$,

$$|\Omega^{(j)}(z,\zeta)| = O(|\zeta|^{-1}) \quad \text{as } |\zeta| \longrightarrow \infty,$$

we can show that the integral on $|\zeta| = R$ approaches zero as $R \longrightarrow \infty$.

Theorem 7.12. *Let $\mathfrak{G}$ be a regular domain, and let A and B be in $L^{p,2}(\mathbf{C})$, where $2 < p < \infty$, and outside $\overline{\mathfrak{G}}$. Suppose $A = B \equiv 0$. Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be the fundamental kernels for A and B. If w is in $C(\overline{\mathfrak{G}})$ and satisfies in $\mathfrak{G}$*

$$Cw = \mathfrak{D}w + Aw + B\overline{w} = 0$$

then w has the representation, for z in $\mathfrak{G}$,

$$w(z) = \frac{1}{2\pi i}\int_{\Gamma} \Omega^{(1)}(z,\zeta)\phi(\zeta)\,dt(\zeta) - \Omega^{(2)}(z,\zeta)\overline{\Phi(\zeta)}\,\overline{dt(\zeta)} \tag{7.17}$$

$$\stackrel{\text{def}}{\equiv} \mathcal{K}(\Phi, \mathfrak{G})$$

where Φ is the function, hyperanalytic in $\mathfrak{G}$ and continuous in $\overline{\mathfrak{G}}$, given in $\mathfrak{G}$ by the formula

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{w(\zeta)}{t(\zeta) - t(z)}\,dt(\zeta). \tag{7.18}$$

Proof. Since $\mathfrak{D}w = -Aw - B\overline{w} \in L^p(\mathfrak{G})$, we can use Corollary 5.2 to conclude that in $\mathfrak{G}$

$$\begin{aligned} w &= \Phi + J_{\mathfrak{G}}^{-}(-Aw - B\overline{w}) \\ &= \Phi + J(-Aw - B\overline{w}) \quad (\text{since } A = B = 0 \text{ outside } \overline{\mathfrak{G}}) \end{aligned} \tag{7.19}$$

where Φ is hyperanalytic in $\mathfrak{G}$. By Theorem 5.5, the function $J(-Aw - B\overline{w})$ is continuous in $\mathbf{C}$ and vanishes at infinity. Moreover, because $A = B = 0$ outside $\overline{\mathfrak{G}}$, $J(-Aw - B\overline{w})$ is hyperanalytic outside $\overline{\mathfrak{G}}$. Since w is in $C(\overline{\mathfrak{G}})$, Φ must also be in $C(\overline{\mathfrak{G}})$, and thus we can substitute (7.19) for w in formula (7.14) of Theorem 7.9, and use Theorem 7.11 applied to $J(-Aw - B\bar{w})$, to obtain (7.17). To obtain (7.18), we substitute $\Phi = w + J(Aw + B\bar{w})$ into the representation (3.4) for Φ, and apply Theorem 7.10 to the function $J(Aw + B\bar{w})$.

Theorem 7.13. *Under the same hypothesis as in Theorem 7.12, for z in $\mathfrak{G}$ the following representation holds,*

$$w(z) = \Phi(z) + \iint_{\mathfrak{G}} \Gamma^{(1)}(z, \zeta)\Phi(\zeta)\,d\xi\,d\eta + \iint_{\mathfrak{G}} \Gamma^{(2)}(z, \zeta)\overline{\Phi(\zeta)}\,d\xi\,d\eta \tag{7.20}$$

where Φ is as in Theorem 7.12, and

$$\Gamma^{(1)}(z, \zeta) = -\frac{1}{2\pi i}\left[\frac{t_\xi(\zeta)}{i + eb(\zeta)}A(\zeta)\Omega^{(1)}(z, \zeta) - \frac{\overline{t_\xi(\zeta)}}{i + eb(\zeta)}\overline{B(\zeta)}\Omega^{(2)}(z, \zeta)\right],$$

$$\Gamma^{(2)}(z, \zeta) = \frac{1}{2\pi i}\left[\frac{\overline{t_\xi(\zeta)}}{\overline{i + eb(\zeta)}}\overline{A(\zeta)}\Omega^{(2)}(z, \zeta) - \frac{t_\xi(\zeta)}{i + eb(\zeta)}B(\zeta)\Omega^{(1)}(z, \zeta)\right].$$

Proof. For z in $\mathfrak{G}$ and ϵ a sufficiently small positive number, we can apply Lemma 7.1 to the domain $\mathfrak{G}_\epsilon = \mathfrak{G} - \{\zeta : |\zeta - z| \le \epsilon\}$, and use the previous theorem to obtain

$$\begin{aligned} w(z) - \frac{1}{2\pi i}\int_{|\zeta - z| = \epsilon} \Omega^{(1)}(z, \zeta)\Phi(\zeta)\,dt(\zeta) - \Omega^{(2)}(z, \zeta)\overline{\Phi(\zeta)}\,\overline{dt(\zeta)} \\ = -\frac{1}{2\pi i}\iint_{G_\epsilon}\frac{t_\xi(\zeta)}{i + eb(\zeta)}\Phi(\zeta)\mathfrak{D}_\zeta\Omega^{(1)}(z, \zeta)\,d\xi\,d\eta \\ + \frac{1}{2\pi i}\iint_{G_\epsilon}\frac{\overline{t_\xi(\zeta)}}{\overline{i + eb(\zeta)}}\overline{\Phi(\zeta)}\,\overline{\mathfrak{D}_\zeta\overline{\Omega^{(2)}(z, \zeta)}}\,d\xi\,d\eta. \end{aligned} \tag{7.21}$$

With the relations (7.12), (7.13) and the analysis used in the proof of Theorem 7.7 we can show that as $\epsilon \to 0$ the left side of (7.21) becomes $w(z) - \Phi(z)$. Furthermore, from Theorem 7.8 and the relations (7.10), (7.11) we obtain the formulas

$$\mathfrak{D}_\zeta\Omega^{(1)}(z, \zeta) = A(\zeta)\Omega^{(1)}(z, \zeta) - B^*(\zeta)\Omega^{(2)}(z, \zeta),$$

$$\mathfrak{D}_\zeta\overline{\Omega^{(2)}(z, \zeta)} = A(\zeta)\overline{\Omega^{(2)}(z, \zeta)} - B^*(\zeta)\overline{\Omega^{(1)}(z, \zeta)}.$$

Substituting these expressions into the right side of (7.21), and using (7.4), we obtain the desired result.

We close this section with the remark that the representations (7.17) and (7.20) give a method of obtaining approximating families of solutions to (7.1). It is immediate from Theorem 7.6 that, for any function Φ hyperanalytic in $\mathfrak{G}$ and continuous in $\overline{\mathfrak{G}}$, the representations (7.17) and (7.20) give a solution in $\mathfrak{G}$ to (7.1). Thus, by approximating a hyperanalytic function Φ by a sequence of hyperanalytic functions Φ_n, we can approximate the solution $\mathcal{K}(\Phi, \mathfrak{G})$ (see (7.17)) by the sequence of solutions $\mathcal{K}(\Phi_n, \mathfrak{G})$. Furthermore, it is possible to obtain approximating families of hyperanalytic functions from approximating families of analytic functions through the representation (3.9).

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DEPARTMENT OF MATHEMATICS AND RESEARCH CENTER FOR APPLIED SCIENCE, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401 (Current address of R. P. Gilbert)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720 (Current address of Gerald Hile)