AXISYMMETRIC HARMONIC INTERPOLATION POLYNOMIALS IN $\mathbb{R}^N$

BY

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ABSTRACT. Corresponding to a given function $F(x, \rho)$ which is axisymmetric harmonic in an axisymmetric region $\Omega \subset \mathbb{R}^3$ and to a set of $n + 1$ circles $C_k$ in an axisymmetric subregion $\mathcal{A} \subset \Omega$, an axisymmetric harmonic polynomial $A_n(x, \rho; C_n)$ is found which on the $C_k$ interpolates to $F(x, \rho)$ or to its partial derivatives with respect to $x$. An axisymmetric subregion $B \subset \Omega$ is found such that $A_n(x, \rho; C_n)$ converges uniformly to $F(x, \rho)$ on the closure of $B$. Also a $A_n(x, \rho; \tau_0, \rho_0)$ is determined which, together with its first $n$ partial derivatives with respect to $x$, coincides with $F(x, \rho)$ on a single circle $(\tau_0, \rho_0)$ in $\Omega$ and converges uniformly to $F(x, \rho)$ in a closed torus with $(\tau_0, \rho_0)$ as central circle.

1. Introduction. In this paper we study the interpolation of an axisymmetric harmonic function by means of axisymmetric harmonic polynomials. We choose the $x$-axis as our line of symmetry and use mainly cylindrical coordinates $(x, \rho, \phi)$ related to rectangular coordinates $(x, y, z)$ by the equations $y = \rho \cos \phi$, $z = \rho \sin \phi$ and to spherical coordinates $(r, \theta, \phi)$ by the equations $x = r \cos \theta$, $\rho = r \sin \theta$. By a circle $(x_k, \rho_k)$ we mean the circle with the equations $x = x_k$, $\rho = \rho_k$. Furthermore, we define a region $\Omega$ to be axisymmetric if $(x_k, \rho_k) \in \Omega$ implies that also $(x_0, \rho_0, \phi_0) \in \Omega$ for all $0 \leq \rho \leq \rho_0$ and $0 \leq \phi \leq 2\pi$. We define a function $F$ to be axisymmetric in $\Omega$ if its values $F(x, \rho)$ in $\Omega$ are independent of $\phi$.

Specifically, we deal with the following two general problems for a given axisymmetric region $\Omega$, a given function $F(x, \rho)$ axisymmetric harmonic in $\Omega$ and a given set of circles $C_n = \{(x_k, \rho_k); k = 0, 1, \ldots, n\}$.

(1) To find an axisymmetric harmonic polynomial $A_n(x, \rho; C_n)$ of degree $n$ such that

\[ A_n(x_k, \rho_k; C_n) = F(x_k, \rho_k), \quad k = 0, 1, \ldots, n. \]

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(2) To find two axisymmetric subregions $A$ and $B$ of $\Omega$, with closures $\overline{A} \subset \Omega$ and $\overline{B} \subset \Omega$, such that, if $C_n \subset \overline{A}$, then $\Lambda_n(x, \rho; C_n)$ converges to $F(x, \rho)$ uniformly in $\overline{B}$.

Such problems may be interpreted as those of determining empirically the velocity potential $F(x, \rho)$ in an axisymmetric flow of an incompressible fluid. An example is that of a liquid or gas which flows from $x = -\infty$ in a uniform stream parallel to the $x$-axis, but which eventually streams past a smooth axisymmetric obstacle $K$. Region $\Omega$ is chosen as disjoint from $K$ and its interior. The values $F(x, \rho)$ might be found from measurements of the velocity potential taken along a set of circles $C_n = \{(x_k, \rho_k)\}$ within $\overline{A} \subset \Omega$, and the interpolating polynomial $\Lambda_n(x, \rho; C_n)$ would then be sought as satisfying equations (1.1). The subregions $A$ and $B$ of $\Omega$ must then be selected so that $\Lambda_n(x, \rho; C_n)$ would approximate to the potential $F(x, \rho)$ uniformly in $\overline{B}$ to within a prescribed degree of accuracy. (See also [4, pp. 254–256]; [6, pp. 432–467].)

Our methods for solving the first type of problem involve straightforward algebraic procedures. However, our method for solving the second type of problem is by use of the Whittaker-Bergman operator [1, pp. 43–57]

\[ G(x, y, z) = \frac{1}{2\pi i} \int_{|r|=1} g(\zeta, r) r^{-1} dr, \]

where $\zeta = x + (i/2)y(r + r^{-1}) + (1/2)z(r - r^{-1})$, which generates a harmonic function $G(x, y, z)$ as the transform of a function $g(\zeta, r)$ that is holomorphic in $\zeta$ over some region in $C$ and continuous in $r$ for $|r| = 1$. (In general, $G(x, y, z)$ is complex valued so that its real and imaginary parts are separately harmonic.) By this device, we are able to derive some results on harmonic polynomial interpolation in $R^3$ analogous to certain results about polynomial interpolation in $C$.

Finally, by similar methods, we are able to generalize our results to axisymmetric harmonic functions in $R^N$.

2. Interpolating polynomial $\Lambda_n(x, \rho; C_n)$. As an axisymmetric harmonic polynomial, $\Lambda_n(x, \rho; C_n)$ may be written as a linear combination [4, p. 254] of the zonal harmonics $r^k P_k(x/r)$ where $P_k(u)$ is the $k$th degree Legendre polynomial. As is well known [4, p. 125]

\[ r^k P_k(x/r) = \sum_{j=0}^{[k/2]} (-1)^j \gamma_{k-j} x^{k-2j} r^{2j} \]

where $[k/2]$ is the largest integer $j \leq k/2$ and

\[ \gamma_{kj} = [1 \cdot 3 \cdots (2k - 2j - 1)]/[j!(k-j)!2^j]. \]

Substituting $r^2 = x^2 + \rho^2$, we define
Thus $P_k(x, \rho)$ is an axisymmetric harmonic polynomial which is a homogeneous function of order $k$ in $x$ and $\rho$. It is an even function of $\rho$ but is an odd or even function of $x$ according as $k$ is an odd or even integer. Furthermore, since $P_k(1) = 1$, we find $P_k(1, 0) = 1$ for all $k$. In short we may write

$$P_k(x, \rho) = r^k P_k(x/r).$$

The coefficients $A_k$ are to be determined so that Equation (1.1) holds; that is, so as to satisfy the system

$$\sum_{k=0}^{n} A_k P_k(x_j, \rho_j) = F(x_j, \rho_j), \quad j = 0, 1, \ldots, n.$$ 

Eliminating the $A_k$ from (2.2) and system (2.3), we obtain for $A_k$ the equation

$$A_k(x, \rho; C_n) = \prod_{k=0}^{n} P_k(x_k, \rho_k) = 0.$$ 

Using the notation

$$V(C_n) = \det \|P_k(x_j, \rho_j)\|, \quad j, k = 0, 1, \ldots, n;$$

for $k = 0, 1, \ldots, n$, we solve (2.4) explicitly for $A_k$ to obtain

$$\Lambda_k(x, \rho; C_n) = \sum_{k=0}^{n} F(x_k, \rho_k) V_k(x, \rho; C_n) V(C_n)$$

provided that $V(C_n) \neq 0$.

The restriction $V(C_n) \neq 0$ on the choice of the circles $C_n = \{(x_k, \rho_k)\}$ implies first that the circles $(x_k, \rho_k)$ are distinct. It implies secondly that the equations

$$P_k(x_j, \rho_j) = 0$$

are not satisfied for any $k$ simultaneously for all $j = 0, 1, \ldots, n$. Factoring
the Legendre polynomial

\[ P_k(u) = \gamma_k \prod_{\nu=1}^{k} (u - \cos \alpha_{k\nu}) \]

and thus

\[ P_k(x, \rho) = \gamma_k \prod_{\nu=1}^{k} (x - \rho \cos \alpha_{k\nu}), \]

we may interpret (2.8) as requiring that each circle \((x_j, \rho_j)\) lie on some cone

\[ \rho = x \tan \alpha_{k\nu} \]

for the same value of \(k\). Also, for given \((x_j, \rho_j), j = 1, 2, \ldots, n,\) and given \(\rho_0,\)
we may determine the zeros of \(V(C_n)\) considered as an \(n\)th degree polynomial in \(x_0\) and thereby find possibly additional sets of circles \(C_n\) which fail to satisfy the condition \(V(C_n) \neq 0.\)

3. Special cases. Let us first consider the case that all the circles degenerate into points on the axis of symmetry so that \(\rho_0 = \rho_1 = \cdots = \rho_n = 0.\) Since now \(r_k^2 = x_k^2,\) we have \(P_j(x_k, 0) = x_k^j P_j(1) = x_k^j\) and thus

\[ V(C_n) = v(x_0, x_1, \ldots, x_n), \]

\[ V_k(x, \rho; C_n) = \sum_{j=0}^{n} v_j(x_0, \ldots, x_n)P_j(x, \rho), \]

where

\[ v(x_0, x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}, \]

the Vandermonde determinant and \(v \left[ x_0, x_1, \ldots, x_n \right] \) is the cofactor of \(x_k^j\) in \(v(x_0, x_1, \ldots, x_n).\) Thus

\[ \Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} \left[ \frac{v_j(x_0, \ldots, x_n)}{v(x_0, \ldots, x_n)} \right] P(x_k, 0)P_j(x, \rho). \]

On the \(x\) axis \(\Lambda_n(x, 0; C_n)\) is found from (3.2) to reduce to the Lagrange interpolation polynomial for \(F(x) = F(x, 0).\)

Let us next examine the special case \(x_0/r_0 = x_1/r_1 = \cdots = x_n/r_n = u_0 = \cos \alpha\) when the circles \(C_n\) lie on the cone \(r = (x/u_0),\) but not on any cone (2.10);
that is, $\alpha \neq \alpha_{k'}$, $\nu = 1, 2, \ldots, k$; $k = 1, \ldots, n$. In this case,

$$P_k(x, \rho) = \frac{r^k}{j^k} P_k(u_0),$$

$$V(C_n) = P_1(u_0)P_2(u_0) \cdots P_n(u_0) v(r_0, r_1, \ldots, r_n),$$

$$V_k(x, \rho; C_n) = \sum_{j=0}^{n} P_j(x, \rho)P_1(u_0) \cdots P_{j-1}(u_0)$$

$$\cdots P_{j+1}(u_0) \cdots P_n(u_0) v_{jk}(r_0, \ldots, r_n).$$

Thus,

$$(3.3) \Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} \begin{vmatrix} v_{jk}(r_0, \ldots, r_n) \\ v(r_0, \ldots, r_n) \end{vmatrix} \begin{vmatrix} P_j(x, \rho) \\ P_j(u_0) \end{vmatrix} F(x_k, x_k \tan \alpha).$$

Along a circle on the cone, $\Lambda_n(x, x \tan \alpha; C_n)$ is found from (3.3) to reduce to the Lagrange interpolation polynomial to $J(x) = F(x, x \tan \alpha)$.

Finally, let us look into the special case $r_0 = r_1 = \ldots = r_n$ when all the circles $C_n$ lie on the sphere $x^2 + \rho^2 = r_0^2$. Since now $P_k(x, \rho) = r_0^k P_k(u)$,

$$(3.4) V(C_n) = r_0^{n(n+1)/2} \det \|P_k(u)\|.$$ 

Let us write $P_n(u) = \gamma_n u^n + \sum_{\nu=1}^{n} \mu_{n\nu} P^\nu(u)$. By subtracting from the $k$th column of the determinant in (3.4) for each $k$, suitable linear combinations of the first $(k-1)$ columns, we may reduce (3.4) to the form

$$(3.5) V(C_n) = r_0^{n(n+1)/2} \gamma_1 \gamma_2 \cdots \gamma_n u(u_0, u_1, \ldots, u_n).$$

Similarly

$$V_k(x, \rho; C_n) = \gamma_1 \gamma_2 \cdots \gamma_n \sum_{i=0}^{n} \gamma_i^{-1} \left[ r_0^{i+1} v_i(u_0, \ldots, u_n) - \mu_{i+1} v_{i+1}(u_0, \ldots, u_n) \right].$$

where

$$r_{jk} = [v_{jk}(u_0, \ldots, u_n) - \mu_{j+1} v_{j+1}(u_0, \ldots, u_n)].$$

Thus

$$(3.6) \Lambda_n(x, \rho; C_n) = \sum_{j,k=0}^{n} \left[ r_0^j \gamma_j u(u_0, \ldots, u_n) - r_{jk} F(x_k, \rho_k)P_j(x, \rho) \right].$$

Along a circle on the sphere $r = r_0$, $\Lambda_n(x, \rho; C_n)$ reduces to the Lagrange interpolation polynomial for $F(x, [r_0^2 - x^2]^{1/2})$. 

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4. Integral representations for $A_n(x, \rho; C_n)$. In view of the Laplace formula for Legendre polynomials,

\begin{equation}
P_k(x, \rho) = \frac{1}{\pi} \int_0^{\pi} (x + \rho \cos t)^k \, dt,
\end{equation}

we may rewrite (2.2) in the form

\begin{equation}
A_n(x, \rho; C_n) = \frac{1}{\pi} \int_0^{\pi} \lambda_n(x + \rho \cos t) \, dt
\end{equation}

where the polynomial

\begin{equation}
\lambda_n(\zeta) = \sum_{k=0}^{n} A_k \zeta^k, \quad \zeta \in C,
\end{equation}

is called the associate of $A_n(x, \rho; C_n)$. More generally, if $F(x, \rho)$ is an axisymmetric harmonic function in an axisymmetric region $\Omega \subset \mathbb{R}^3$, then

\begin{equation}
F(x, \rho) = \frac{1}{\pi} \int_0^{\pi} f(x + \rho \cos t) \, dt,
\end{equation}

where the associate $f(\zeta) = F(\zeta, 0)$ is a holomorphic function of $\zeta$ in the meridian cross-section $\omega$ of $\Omega$, obtained on intersecting $\Omega$ with any plane through the $x$ axis. This meridian section $\omega$ is an axiconvex region in $C$, in the sense that $\zeta \in \omega$ implies that also $[\mu \zeta + (1 - \mu) \overline{\zeta}] \in \omega$ for all $\mu$, $0 \leq \mu \leq 1$. In fact, (4.4) is the special case of equation (1.2) with $g(\zeta, r) = f(\zeta)$.

Using (4.1), (4.2) and (4.3), we may now rewrite (2.5) in the form

\begin{equation}
V(C_n) = \frac{\pi^{n+1}}{n!} \int_0^{\pi} \cdots \int_0^{\pi} \lambda(\sigma_0, \sigma_1, \ldots, \sigma_n) \, dt_0 \, dt_1 \cdots \, dt_n
\end{equation}

where $\sigma = x + i\rho \cos t$, $\sigma_k = x_k + i\rho \cos t_k$, $k = 0, 1, \ldots, n$. Let us introduce the closed $n + 1$ and $n + 2$ dimensional cubes

\begin{align*}
T &= \{t_0, t_1, \ldots, t_n\}, \quad 0 \leq t_k \leq \pi, \quad k = 0, 1, \ldots, n, \\
T^* &= \{t, t_0, t_1, \ldots, t_n\}, \quad 0 \leq t \leq \pi, \quad 0 \leq t_k \leq \pi, \quad k = 0, 1, \ldots, n
\end{align*}

and the notation

\begin{align*}
dT &= dt_0 \cdots dt_1 \, dt_n, \quad dT^* = dt_0 \cdots dt_1 \, dt, \\
S &= \{\sigma_0, \sigma_1, \ldots, \sigma_n\}, \quad \nu(S) = \nu(\sigma_0, \sigma_1, \ldots, \sigma_n), \\
T_k &= (T)_{t_k = t}, \quad S_k = (S)_{\sigma_k = \sigma}.
\end{align*}

Thus we may write
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\[ V(C_n) = \pi^{-n-1} \int_T v(S) \, dT \]

and similarly

\[ F(x_k, \rho_k)V_k(x, \rho; C_n) = \pi^{-n-2} \int_{T^*} f(\sigma_k) v(S_k) \, dT^*. \]

Consequently, from (2.7) it follows that

\[ \Lambda_n(x, \rho; C_n) = \frac{\sum_{k=0}^n \int_{T^*} f(\sigma_k) v(S_k) \, dT^*}{\pi \int_{T^*} v(S) \, dT}. \]

5. Relation of \( \Lambda_n \) to the Lagrange interpolation polynomial for \( f(\sigma) \). Let \( l_n(\sigma; S) \) be the Lagrange polynomial which interpolates to \( f(\sigma) \) at the points \( \sigma_k = x_k + i \rho_k \cos t_k \), \( k = 0, 1, \ldots, n \). As is well known,

\[ l_n(\sigma; S) = \sum_{k=0}^n \frac{f(\sigma_k) \psi(\sigma)}{\psi(\sigma_k) (\sigma - \sigma_k)} \]

where \( \psi(\sigma) = (\sigma - \sigma_0)(\sigma - \sigma_1) \cdots (\sigma - \sigma_n) \).

Equivalently,

\[ l_n(\sigma; S) = \sum_{k=0}^n \frac{f(\sigma_k) v(S_k)}{v(S)}, \]

whereupon (4.6) becomes

\[ \Lambda_n(x, \rho; C_n) = \frac{\int_{T^*} l_n(\sigma; S) v(S) \, dT^*}{\pi \int_{T^*} v(S) \, dT}. \]

Thus we may regard \( \Lambda_n(x, \rho; C_n) \) as a certain average of the values of \( l_n(\sigma; S) \) taken over the cube \( T^* \).

6. Approximation of \( \Lambda_n(x, \rho; C_n) \) to \( F(x, \rho) \). Let us derive an integral representation for the difference

\[ \Delta_n(x, \rho; C_n) = F(x, \rho) - \Lambda_n(x, \rho; C_n). \]

Rewriting (4.4) as \( F(x, \rho) = \int_{T^*} f(\sigma) v(S) \, dT^*/\pi \int_{T^*} v(S) \, dT \), we infer from (5.2) that

\[ \Delta_n(x, \rho; C_n) = \frac{\int_{T^*} [f(\sigma) - l_n(\sigma; S)] v(S) \, dT^*}{\pi \int_{T^*} v(S) \, dT}. \]

We now state the following:

**Theorem I.** Let \( \Omega \) be an axisymmetric region and \( A \) and \( B \) axisymmetric subregions whose closures \( \overline{A} \) and \( \overline{B} \) lie in \( \Omega \). Let \( \omega, \alpha \) and \( \beta \) be respectively
the meridian sections of $\Omega$, $A$ and $B$. Let $F(x, \rho)$ be an axisymmetric harmonic function in $\Omega$ and let $C_n = \{(x_k, \rho_k)\}$ be a set of $n + 1$ circles in $\overline{A}$ with $x_0 < x_1 < \cdots < x_n$. Let

\[ \mathcal{X}(C_n) = \int_T |\nu(S)| \, dT / \int_T \nu(S) \, dT. \]

Assume further that $A$ and $B$ have the properties:

1. there exists a constant $M \geq 1$ and an integer $N > 0$ such that $\mathcal{X}(C_n) \leq M$ for all sets $C_n \subset \overline{A}$, $n \geq N$;

2. $l_n(\sigma, S)$ approximates $f(\sigma)$ uniformly for all $\sigma \in \overline{B}$ and all $T$ with $0 \leq t_k \leq \pi$, $k = 0, 1, \ldots, n$. Then $\Lambda_n(x, \rho; C_n)$ approximates uniformly to $F(x, \rho)$ for all $(x, \rho) \in B$.

Proof. From (6.2) we infer that

\[ |\Lambda_n(x, \rho; C_n)| \leq \frac{\int_{T^*} |f(\sigma) - l_n(\sigma, S)| |\nu(S)| \, dT^*}{\pi \int_T \nu(S) \, dT}. \]

By hypothesis (2), given an $\epsilon > 0$, we can find $N_0 > 0$ so that $|f(\sigma) - l_n(\sigma, S)| < \epsilon/M$ for all $n \geq N_0$, for all $\sigma \in \overline{B}$ and all $T$ with $0 \leq t_k \leq \pi$, $k = 0, 1, \ldots, n$. Using also hypothesis (1), we find

\[ |\Lambda_n(x, \rho; C_n)| < \frac{\epsilon \int_{T^*} |\nu(S)| \, dT^*}{Mn \int_T \nu(S) \, dT} < \epsilon \]

for all $(x, \rho) \in \overline{B}$ and all $n \geq \max(N_0, N)$, thereby completing the proof of Theorem I.

7. Sufficient conditions on subregions $A$ and $B$. Let us first examine some sufficient conditions on $A$ for the existence of a constant $M \geq 1$ such that $\mathcal{X}(C_n) \leq M$. It is clear from (6.3) that in any case $\mathcal{X}(C_n) \geq 1$. If $\rho_0 = \rho_1 = \cdots = \rho_n = 0$, then

\[ \nu(S) = \prod_{k=1}^{n} \prod_{j=0}^{k-1} (x_k - x_j) = |\nu(S)| > 0 \]

and $\mathcal{X}(C_n) = 1$. Thus, if we are given any constant $M \geq 1$, we can make $\mathcal{X}(C_n) \leq M$ by taking, if necessary, all the $\rho_k$ sufficiently small.

Alternatively, let us choose $C_n$ so that for some $\delta$, $0 < \delta < \pi/2$,

\[ |\arg \nu(S)| \leq \delta. \]

Then
\[ \left| \int_T v(S) \, dT \right| \geq \Re \left[ \int_T v(S) \, dT \right] \geq \int_T \Re \left[ v(S) \right] \, dT \geq \int_T |v(S)| \cos \arg v(S) \, dT \geq \left( \int_T |v(S)| \, dT \right) \cos \delta. \]

Hence, \( \mathcal{H}(C_n) \leq \sec \delta \) and we may choose \( M = \sec \delta \). Thus (7.1) is a possible sufficient condition on \( A \).

It suffices that for some constant \( b > 0 \), \( |V(C_n)| \geq b \) for circles \( C_n \) in \( A \). For then

\[ \mathcal{H}(C_n) \leq b^{-1} \int_T |v(S)| \, dT \leq b^{-1} n^{n+1} d^{n(n+1)/2} \]

where \( d \) is the transfinite diameter of region \( \alpha \). Thus

\[ M = \max \{1, b^{-1} n^{n+1} d^{n(n+1)/2}\}. \]

Let us next develop some sufficient conditions for regions \( A \) and \( B \) to have the second property required in Theorem I. These conditions are embodied in the following, cf. [8, pp. 52-57].

**Theorem II.** Let \( A \) and \( B \) be bounded axisymmetric regions with \( A \) such that there exists a constant \( M > 1 \) with \( \mathcal{H}(C_n) \leq M \) for all choices of the \( n + 1 \) circles \( C_n = \{ (x_k, \rho_k) \} \subset A \) and for all \( n \). Let

\[ D(x', \rho') = \{ (x-x')^2 + (\rho - \rho')^2 \leq [a(x', \rho')]^2 \} \]

be the smallest closed torus having as central circle \( (x', \rho') \) and containing \( B \). Let \( \Omega \) be an axisymmetric region which contains the bounded closed set \( \Gamma = \{ \bigcup D(x', \rho'): (x', \rho') \subset A \} \). Finally, let \( F(x, \rho) \) be a function which is axisymmetric, harmonic in \( \Omega \) and which is interpolated by the axisymmetric harmonic polynomial \( \Lambda_n(x, \rho; C_n) \) on the \( n + 1 \) circles \( C_n \). Then \( \Lambda_n(x, \rho; C_n) \) converges uniformly to \( F(x, \rho) \) in \( B \), as \( n \to \infty \).

**Proof.** Let us denote by \( \alpha, \beta, \gamma, \delta(x', \rho') \), and \( \omega \) the meridian sections of the sets \( A, B, \Gamma, D(x', \rho') \) and \( \Omega \) respectively. Then \( \delta_{kt} = \delta(x_k, \rho_k \cos t) \) is the smallest closed disk having its center at point \( (x_k, \rho_k \cos t) \in \overline{\alpha} \) and containing \( \overline{B} \). For all \( t, \gamma > \bigcup_{k=0}^n \delta_{kt} \) and \( \gamma \subset \omega \). Let us express \( F(x, \rho) \) by (4.4) in terms of its associate \( f(\sigma) \) which is holomorphic in \( \omega \). Formula (5.1) gives the \( n \)th degree polynomial \( l_n(\sigma; S) \) which interpolates to \( f(\sigma) \) at the points \( \sigma_k, k = 0, 1, \ldots, n \).

In order to obtain the usual integral representation for the difference \( [f(\sigma) - l_n(\sigma; S)] \), let us introduce (cf. [8, pp. 54-55]) the function \( w = \phi(\sigma) \) which is holomorphic in the complement \( \gamma' \) of \( \gamma \) and which maps the simply connected region \( \gamma' \) conformally onto the disk \( |w| > 1 \) with \( \phi(\infty) = \infty \) and \( \phi'(\infty) > 0 \). Let \( \kappa_\epsilon \) be the level curve \( |\phi(\sigma)| = 1 + \epsilon \), where \( \epsilon > 0 \) and \( \epsilon \) is chosen so small that \( \kappa_\epsilon \subset \omega \). Let \( d_\epsilon \) be the minimum distance from \( \kappa_\epsilon \) to \( \gamma \). Obviously \( d_\epsilon > 0 \).
By Hermite’s formula (a corollary of the residue theorem) [2, p. 68]

\[ f(\sigma) - l(\sigma, S) = \frac{1}{2\pi i} \int_{\delta_{\epsilon}} \frac{f(s)\psi_n(\sigma) ds}{(s-\sigma)\psi_n(s)}, \quad \sigma \in \overline{\beta}, \]

where \( \psi_n(\sigma) = \prod_{k=0}^{n} (\sigma - \sigma_k), \sigma_k \in \overline{\alpha}. \) Let us define

\[ r = \sup(|\sigma - \sigma_k|; \sigma \in \overline{\beta}, \sigma_k \in \overline{\alpha}), \]
\[ \mu = \sup(|f(s)|; s \in \kappa_{\epsilon}). \]

Since \( \delta_{\epsilon} \subset \gamma, \) for all \( \sigma \in \delta_{\epsilon} \) and all \( s \in \kappa_{\epsilon}, |\sigma - \sigma_k| + d_\epsilon \leq |s - \sigma_k|. \) Since

\[ (|\sigma - \sigma_k| + d_\epsilon)/|\sigma - \sigma_k| = 1 + d_\epsilon/|\sigma - \sigma_k|^{-1} \geq 1 + (d_\epsilon/r), \]

(7.3) \[ |f(\sigma) - l(\sigma, S)| \leq \frac{\mu}{2\pi d_\epsilon} \left( \frac{r^2}{r + d_\epsilon} \right)^{n+1} \int_{\kappa_{\epsilon}} |ds| = a\nu^{n+1}, \]

where \( a > 0 \) and \( 0 < \nu < 1. \) If we now substitute from (7.3) into (6.4), we find

\[ |\Delta_n(x, \rho; C_n)| \leq a\nu^{n+1}. \]

Clearly, given any \( \epsilon > 0, \) we can choose \( N \) so large that \( |\Delta_n(x, \rho; C_n)| < \epsilon \) for all circles \( C_n \subset \overline{\Lambda}, n \geq N, \) and for any point on any circle \( (x, \rho) \subset \overline{B}. \)

Therefore, \( \Lambda_n(x, \rho; C_n) \) converges to \( F(x, \rho) \) uniformly on \( \overline{B}, \) as stated in Theorem II.

Theorem II may be restated with \( \Lambda \) and \( \Omega \) given and \( B \) to be specified, or \( B \) and \( \Omega \) given and \( \Lambda \) to be specified.

An immediate consequence of Theorem II is the following simpler result.

Cf. [2, p. 81].

Corollary. Let \( \Lambda \) and \( \Omega \) be defined as in Theorem II and let \( \Delta \) and \( \Omega \) be bounded axisymmetric regions with \( \overline{\Lambda} \subset B, \overline{\Omega} \subset \overline{\Delta}, \overline{\Lambda} \subset \overline{\Omega}. \) Assume that \( 0 < b < c \)

where

\[ b = \max \{|(x' - x)^2 + (\rho' - \rho)^2|^{1/2}; (x', \rho') \subset \partial A, (x, \rho) \subset \partial B\}, \]
\[ c = \min \{|(x' - x)^2 + (\rho' - \rho)^2|^{1/2}; (x', \rho') \subset \partial A, (x, \rho) \subset \partial \Delta\}. \]

Then, if \( F(x, \rho) \) and \( \Lambda_n(x, \rho; C_n) \) be specified as in Theorem II, \( \Lambda_n(x, \rho; C_n) \)

converges uniformly to \( F(x, \rho) \) in \( \overline{B} \) as \( n \to \infty. \)

Proof. The assumption \( b < c \) ensures that \( \partial(x', \rho') \subset \overline{\Lambda} \) for every circle \( (x', \rho') \subset \overline{\Lambda} \) and therefore that \( \Gamma \subset \overline{\Delta}. \) Thus, if \( \Omega \) is chosen so that \( \overline{\Delta} \subset \overline{\Omega}, \) then also \( \Gamma \subset \overline{\Omega} \) and Theorem II leads at once to the corollary.

Example. Let \( \Lambda \) be the ellipsoid of revolution and \( B \) the ball defined by the inequalities:

\[ \overline{\Lambda} = \{(x^2 + \rho^2)^{1/2} + (x - c)^2 + \rho^2)^{1/2} \leq b\}, \quad b > c > 0, \]
\[ \overline{B} = \{(x - c)^2 + \rho^2 \leq b^2\}. \]
Then $\Gamma$ is the ball $x^2 + \rho^2 < (b + b)^2$ since $\{b + [(x - c)^2 + \rho^2]^{1/2}\}$ is the radius of the smallest circle having its center at any boundary point $(x, \rho)$ of the elliptic meridian section of $\overline{A}$ and containing the meridian section of $\overline{B}$. The minor axis of ellipsoid $A$ is $m = (b^2 - c^2)^{1/2}$, which can be taken small if necessary to satisfy the condition $\lambda(C_n) \leq M$ for some $M > 1$. Finally, $\Omega$ can be taken as any ball $x^2 + \rho^2 < k^2$ with $k > (b + b)$.

8. Another interpolation problem. Instead of (1.1), let us impose upon $\Lambda_n(x, \rho; C_n)$ the conditions that

$$\Lambda_n^{(k)}(x, \rho; C_n) = F^{(k)}(x, \rho), \quad k = 0, 1, 2, \ldots, n,$$

where superscript $k$ signifies a $k$th partial derivative with respect to $x$. In eliminating the $A_k$ from (2.2) subject to (8.1), we may use the identity obtained by differentiating (4.1) with respect to $x$:

$$P_n(x, \rho) = \frac{n!}{n!} \int_0^\pi (x + i\rho \cos t)^{n-1} dt = nP_{n-1}(x, \rho),$$

so that by induction

$$P_n^{(k)}(x, \rho) = n(n-1) \cdots (n-k+1)P_{n-k}(x, \rho).$$

If $E$ denotes the determinant (2.4), we have now for $\Lambda_n(x, \rho; C_n)$ the equation:

$$\frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^n}{\partial x_n^n} E = 0.$$

The cofactor of $\Lambda_n$ in the determinant on the left side of (8.3) is

$$W(C_n) = \frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^n}{\partial x_n^n} V(C_n) = \det \left| \begin{array}{c} \frac{\partial}{\partial x_1} P_{k-1}(x, \rho) \\
\frac{\partial}{\partial x_2} P_{k-1}(x, \rho) \\
\vdots \\
\frac{\partial}{\partial x_n} P_{k-1}(x, \rho) 
\end{array} \right| = \det \left| k(k-1) \cdots (k-j+1)P_{k-j}(x, \rho) \right| = 1!2! \cdots n!.$$

Defining

$$W_k(x, \rho; C_n) = \frac{\partial}{\partial x_1} \frac{\partial^2}{\partial x_2^2} \cdots \frac{\partial^{k-1}}{\partial x_{k-1}^{k-1}} \frac{\partial^{k+1}}{\partial x_{k+1}^{k+1}} \cdots \frac{\partial^n}{\partial x_n^n} V_k(x, \rho; C_n),$$

we obtain

$$\Lambda_n(x, \rho; C_n) = F(x_0, \rho_0) + [1!2! \cdots n!]^{-1} \sum_{k=1}^n F^{(k)}(x, \rho) W_k(x, \rho; C_n).$$
Similarly to (4.5), we may write

\[ W_k(x, \rho; C_n) = \pi^{-n-1} \int_{T_k} w_k(S_k) dT_k \]

where

\[ w_0(S_0) = \frac{\partial}{\partial \sigma_1} \frac{\partial^2}{\partial \sigma_2^2} \cdots \frac{\partial^n}{\partial \sigma_n^n} v(S_0), \]

\[ w_k(S_k) = \frac{\partial}{\partial \sigma_k} \cdots \frac{\partial^{k-1}}{\partial \sigma_{k-1}^{k-1}} \frac{\partial^{k+1}}{\partial \sigma_{k+1}^{k+1}} \cdots \frac{\partial^n}{\partial \sigma_n^n} v(S_k), \quad k = 1, 2, \ldots, n. \]

Thus we may rewrite (8.4) as

\[ \Lambda_n(x, \rho; C_n) = F(x_0, \rho_0) + \prod_{i=1}^{n+2} \int_{T^*}^{n} \sum_{k=1}^{n} f^{(k)}(\sigma_k) w_k(S_k) dT^*. \]

The quantities (8.6) also occur in the polynomial \( l_n(\sigma; S) = \sum_{k=0}^{n} a_k \sigma^k \) which satisfies the conditions:

\[ l^{(k)}(\sigma_k; S) = f^{(k)}(\sigma_k; S), \quad k = 0, 1, \ldots, n. \]

On eliminating the \( a_k \), we find that

\[ l_n(\sigma, S) = f(\sigma_0) + \prod_{k=1}^{n} f^{(k)}(\sigma_k) w_k(S_k). \]

This permits us to rewrite (8.7) as

\[ \Lambda_n(x, \rho; C_n) = \pi^{-n-2} \int_{T^*}^{n} l_n(\sigma, S) dT^*. \]

A representation for \( l_n(\sigma, S) \) alternative to (8.9) is

\[ l_n(\sigma, S) = f(\sigma_0) + \sum_{k=1}^{n} f^{(k)}(\sigma_k) q_k(\sigma) \]

where

\[ q_k(\sigma) = \int_{\sigma_0}^{\sigma} d\nu_1 \int_{\nu_1}^{\nu_2} d\nu_2 \int_{\nu_2}^{\nu_3} d\nu_3 \cdots \int_{\nu_{k-1}}^{\nu_k} d\nu_k. \]

The \( k \)th degree polynomial \( q_k(\sigma) \) is essentially the "Abel-Gontscharoff polynomial", defined by the equations [2, pp. 46-47]

\[ q_k(\sigma_0) = q'_k(\sigma_1) = \cdots = q^{(k-1)}(\sigma_{k-1}) = 0, \quad q^{(k)}(\sigma) = 1. \]
Consequently, for \( j = 0, 1, \ldots, n \)

\[
\int_{S}^{j} f(\sigma, \delta) S = \sum_{k=j}^{n} \int_{S}^{j} f(\sigma, \delta) q_{k}^{j} (\sigma) = f^{j}(\sigma)
\]
as required by (8.8).

Let us define

\[
Q_{k}(x, \rho) = \pi^{-n-1} \int_{T_{k}} q_{k}(x + i \rho \cos \delta) dT_{k}.
\]

From (8.10) and (8.12), we conclude that

\[
\Lambda_{n}(x, \rho; C_{n}) = F(x_{0}, \rho_{0}) + \sum_{k=1}^{n} F^{(k)}(x_{k}, \rho_{k}) Q_{k}(x, \rho).
\]

Regarding the convergence of \( \Lambda_{n}(x, \rho; C_{n}) \) to \( F(x, \rho) \), we may state the following theorem:

**Theorem III.** Let \( A \) and \( B \) be bounded axisymmetric regions and the circles \( C_{n} = \{(x_{k}, \rho_{k}) \in A \}. \) Let \( \Gamma, \Omega \) and \( F(x, \rho) \) be defined as in Theorem II. Let \( \Lambda_{n}(x, \rho; C_{n}) \) be the \( n \)th degree polynomial such that \( \Lambda_{n}^{(k)}(x_{k}, \rho_{k}; C_{n}) = F^{(k)}(x_{k}, \rho_{k}), \) \( k = 0, 1, \ldots, n. \) Then \( \Lambda_{n}(x, \rho; C_{n}) \) converges to \( F(x, \rho) \) uniformly in \( B, \) as \( n \to \infty. \)

**Proof.** From (8.10) and (4.4), we observe that

\[
F(x, \rho) - \Lambda_{n}(x, \rho; C_{n}) = \pi^{-n-2} \int_{T_{n}} \left[ f(\sigma) - l_{n}(\sigma, S) \right] dT^{*}.
\]

By the residue theorem

\[
\int_{T_{n}} \left[ f(\sigma) - l_{n}(\sigma, S) \right] dT^{*} = \frac{1}{2\pi i} \int_{K_{\epsilon}} \frac{f(s)(\sigma - \sigma_{0})(\sigma - \sigma_{1})^{2} \cdots (\sigma - \sigma_{n})^{n}}{(s - \sigma)(s - \sigma_{0})(s - \sigma_{1})^{2} \cdots (s - \sigma_{n})^{n}} ds.
\]

where \( K_{\epsilon} \) has the same meaning as for (7.2). Using the same notation as in (7.3), we infer that

\[
|f(\sigma) - l_{n}(\sigma, S)| \leq \frac{\mu}{2\pi d_{\epsilon}} \left( \frac{r}{r + d_{\epsilon}} \right)^{N} \int_{K_{\epsilon}} \int ds = a
\]

where \( a > 0, \ 0 \leq \nu < 1 \) and \( N = n(n + 1)/2. \) On using (8.18) in conjunction with (8.16), we infer that \( |F(x, \rho) - \Lambda_{n}(x, \rho; C_{n})| \leq a \) and thus complete the proof of Theorem III. [A corollary similar to that for Theorem II is also valid.]

**9. Expansion about a single circle.** Let us finally seek the axisymmetric harmonic polynomial \( \Lambda_{n}(x, \rho; x_{0}, \rho_{0}) \) which in a single circle \( (x_{0}, \rho_{0}) \) has \( (n' + 1) \)-fold coincidence with an axisymmetric harmonic function \( F(x, \rho) \) in the sense that
This corresponds to (8.1) with all \( n + 1 \) circles \( C_n \) coalescing in a single circle \( (x_0, p_0) \). Consequently, from (8.4)

\[
A_n(x; p; x_0, p_0) = F(x_0, p_0) + \frac{1}{1!2! \cdots n!} F'(x_0, p_0)l_i^{(x_0, p_0)},
\]

where \( l_i^{(x_0, p_0)} = \sum_{k=0}^n c_{jk} P_k(x, p) \),

one finds that \( c_{jk} = 0 \) for \( k > j \). That is, \( U_j(x; p; x_0, p_0) \) is a polynomial of degree \( j \). By differentiating the determinant \( \nu \) times with respect to \( x \), we verify immediately that

\[
U_j^{(\nu)}(x_0, p_0; x_0, p_0) = 0, \quad \nu = 0, 1, 2, \ldots, j - 1,
\]

Thus we verify at once that \( A_n(x; p; x_0, p_0) \) as given by (9.2) does satisfy all the conditions (9.1).

Furthermore, since each \( P_k(x, p) \) is an even function of \( p \), we infer that

\( A_j^{(\nu)}(x_0, p; x_0, p_0) \) is divisible by \( (p^2 - p_0^2) \) for \( \nu = 0, 1, \ldots, j \). Taken with (9.3) and (9.4), this information means that we can write \( U_j(x; p; x_0, p_0) \) in the form

\[
U_j(x; p; x_0, p_0) = \frac{1}{1!2! \cdots n!} (x - x_0)^j + \sum_{\nu=1}^{[j/2]} d_j^{(\nu)} (x - x_0)^j - 2\nu(p^2 - p_0^2)^{\nu}.
\]

A more direct approach to an expansion about the circle \( (x_0, p_0) \) is, using (4.4) and assuming \( f(\sigma) \) to be holomorphic in the closure of \( \omega \), to write

\[
F(x_0 + h, p_0 + k) = \frac{1}{\pi} \int_0^{\pi} f([x_0 + ip_0 \cos t] + (b + ik \cos t)] dt
\]

with \( c^2 = b^2 + k^2 < a_0^2 \) where \( a_0 \) is the shortest distance of points \( \sigma_0 = x_0 + ip_0 \cos t, \quad 0 \leq t \leq \pi, \) to the boundary \( \partial \omega \) of \( \omega \). Setting \( r = b + ik \cos t \), we have

\[
f(\sigma_0 + r) = f(\sigma_0) + f'(\sigma_0) r + \cdots + f^{(n)}(\sigma_0) r^n/n! + r_n(\sigma_0, r)^{n+1}
\]

where \( r_n(\sigma_0, r) = (1/2\pi i) \int_{k_\varepsilon} f(s)(s - \sigma_0)^{-n-2}(s - \sigma_0 - r)^{-1} ds \) and \( k_\varepsilon \) is a circle \( |s - \sigma_0| = b_0 \) with \( c < b_0 < a_0 \).
Let us define the product $H_1(x_1, \rho_1) \ast H_2(x_2, \rho_2)$ of two axisymmetric harmonic functions, for which the associates are $h_1(\sigma_1)$ and $h_2(\sigma_2)$ respectively, as

\begin{equation}
(9.8) \quad H_1(x_1, \rho_1) \ast H_2(x_2, \rho_2) = \frac{1}{\pi} \int_0^\pi b_1(x_1 + i\rho_1 \cos t)b_2(x_2 + i\rho_2 \cos t)\,dt.
\end{equation}

This product is an axisymmetric harmonic function if $x_1 = x_2$ and $\rho_1 = \rho_2$, but in general it is not a harmonic function.

Using (9.6), (9.7) and (9.8), we may now write

\begin{equation}
(9.9) \quad F(x_0 + h, \rho_0 + k) = F_n(x_0, \rho_0) + \sum_{j=0}^n (j!)^{-1} F^{(j)}(x_0, \rho_0) \ast P_{j+1}(b, k)
\end{equation}

where

\begin{equation}
(9.10) \quad F_n(x_0 + b, \rho_0 + k) = F(x_0, \rho_0) + \sum_{j=0}^n (j!)^{-1} F^{(j)}(x_0, \rho_0) \ast P_{j+1}(b, k)
\end{equation}

and

\begin{equation}
(9.11) \quad \Lambda_n(x_0 + b, \rho_0 + k; x_0, \rho_0) = \int_0^\pi \int_0^\pi f(s)(s - x_0 - i\rho_0 \cos t)^{-n-2} \cdot [s - (x_0 + b) - i(\rho_0 + k) \cos t]^{-1} \,ds\,dt.
\end{equation}

Since $P^{(j)}(0, 0) = 0$ or $j!$ according as $\nu < j$ or $\nu = j$, it follows that $F_n(x_0, \rho_0 + 0) = F^{(j)}(x_0, \rho_0)$ and thus $F_n(x_0 + b, \rho_0 + k) = \Lambda_n(x_0 + b, \rho_0 + k; x_0, \rho_0)$.

Let us next evaluate $|\Lambda_n(x_0 + b, \rho_0 + k; x_0, \rho_0)|$, setting $\mu = \max |(\sigma)|: \sigma \in \partial \Omega$. Thus

$$|\Lambda_n(x_0 + b, \rho_0 + k; x_0, \rho_0)| \leq \mu b_0^{-n+1}(b_0 - c)^{-1}.$$

Also

$$|P_{n+1}(b, k)| \leq c^{n+1}|P_{n+1}(b/c)| \leq c^{n+1}.$$

In view of these results and (9.9), we may now state the following:

**Theorem IV.** Let $\Omega$ be an axisymmetric region in whose closure $F(x, \rho)$ is an axisymmetric harmonic function. Let $(x_0, \rho_0)$ be any circle in $\Omega$. Then the $n$th degree axisymmetric harmonic polynomial $\Lambda_n(x, \rho; x_0, \rho_0) = F_n(x_0 + b, \rho_0 + k)$ given by (9.2) or (9.10) has the property:

$$\Lambda_n^{(j)}(x_0, \rho_0; x_0, \rho_0) = F^{(j)}(x_0, \rho_0), \quad j = 0, 1, \ldots, n,$$

where the superscript $^{(j)}$ denotes the $j$th derivative with respect to $x$. Furthermore, $\Lambda_n(x, \rho; x_0, \rho_0)$ converges uniformly to $F(x, \rho)$ in any torus $(x - x_0)^2 + (\rho - \rho_0)^2 \leq c^2$ that is contained in $\Omega$, as $n \to \infty$. 
The $n$th degree polynomial $F_n(x_0 + h, \rho_0 + k)$ in $h$ and $k$ is clearly the analogue of the Taylor polynomial for a holomorphic function $f(\sigma), \sigma \in \mathbb{C}$. It is however not harmonic in $h$ and $k$ in general, but it is axisymmetric harmonic in $x = x_0 + b$ and $\rho = \rho_0 + k$.

10. Generalization to $\mathbb{R}^N$. We now propose to generalize the preceding results to $N$ dimensions. Let us consider functions $F$ of $N$ real variables $\xi_1, \xi_2, \ldots, \xi_N$ that depend only upon the two quantities $x$ and $\rho$ where

$$x = \xi_1, \quad \rho^2 = \xi_2^2 + \xi_3^2 + \cdots + \xi_N^2, \quad N \geq 3.$$  

The corresponding Laplace’s differential equation $\nabla^2 F = 0$ is then (cf. [5, p. 167])

$$\left(\frac{\partial}{\partial x}\right)(\rho^{N-2}\frac{\partial F}{\partial x}) + \left(\frac{\partial}{\partial \rho}\right)(\rho^{N-2}\frac{\partial F}{\partial \rho}) = 0$$

and its solutions are the axisymmetric harmonic functions in $\mathbb{R}^N$. By a circle $(x_k, \rho_k)$ we mean the locus of points in $\mathbb{R}^N$ that satisfy simultaneously the two equations $x = x_k, \rho = \rho_k$ (cf. [5, pp. 151–152]), and by an axisymmetric region $\Omega \subset \mathbb{R}^N$ we mean one such that, if circle $(x_0, \rho_0) \subset \Omega$, then also circle $(x_0, \rho) \subset \Omega$ for $0 \leq \rho \leq \rho_0$. We propose the same problems in $\mathbb{R}^N$ as were stated in §1 for $\mathbb{R}^3$.

It is well known that every axisymmetric harmonic polynomial in $\mathbb{R}^N$ may be written as a linear combination of the polynomials $P_k(x, \rho)$ defined by the equation

$$P_k(x, \rho) = j^{k} P_{k}^\mu(x/r), \quad \mu = (N - 2)/2,$$

where $r^2 = x^2 + \rho^2$.

The function $P_k^\mu(\mu)$ is the so-called Gegenbauer or ultraspherical harmonic polynomial given by [7, pp. 81–85]

$$P_k^\mu(\mu) = \sum_{j=0}^{[k/2]} (-1)^j \gamma_{k-j} \mu^{k-2j}$$

with coefficients expressed in terms of the gamma function as $\gamma_{k-j} = 2^{k-2j}(k-j+\mu)(-1)^j \Gamma(\mu+j+1) \Gamma(k-j+1) \Gamma(k+1)$. Corresponding to equation (4.1), we now have (cf. [7, p. 97])

$$P_k(x, \rho) = p_k \int_0^\pi (x \cos t + \rho \sin t)^k \sin^{N-3} t dt$$

where

$$p_k = 2^3 \Gamma(\mu)^{-2} \left[(k+N-3)k/k!\right].$$
We note that \( P^{(\mu)}(1) = P_{1}(1, 0) = (k + N - 3)/(k!(N - 3)!). \) When \( N = 3 \) and thus \( \mu = \frac{1}{3}, \) \( f^{(\mu)}(u) \) clearly reduces to the Legendre polynomial of degree \( k. \) From (2.2), (10.3) and (10.5) we derive the relation

\[
(10.7) \quad A_n(x, \rho; C_n) = \int_{0}^{\pi} \lambda_n(x + i\rho \cos t) \sin^{N-3} t dt
\]

where the polynomial \( \lambda_n(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^k, \) \( \zeta \in C, \) is called the associate of \( A_n(x, \rho; C_n). \) This suggests the representation

\[
(10.8) \quad F(x, \rho) = \int_{0}^{\pi} f(x + i\rho \cos t) \sin^{N-3} t dt
\]

where \( f(\zeta) \) is a holomorphic function of the complex variable \( \zeta = \xi + i\eta \) in the meridian section \( \omega \) of \( \Omega; \) that is, on the set of the intersection points of \( \Omega \) with a plane

\[
\xi_j = \rho k_j, \quad j = 2, 3, \ldots, N, \quad k_2 + k_3 + \cdots + k_N = 1.
\]

Indeed, if the origin is contained in \( \Omega, \) the series \( F(x, \rho) = \sum_{k=0}^{\infty} c_k P^k(x, \rho) \) is valid uniformly within \( N \) dimensional ball \( x^2 + \rho^2 \leq r_0^2 \) contained in \( \Omega. \) Thus the series \( f(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k \) is valid uniformly within the disk \( |\zeta| \leq r_0 \) contained in \( \omega, \) since the \( c_k \) given by (10.6) are uniformly bounded for all \( k. \) The extension of the representation (10.8) for \( F(x, \rho) \) in all of \( \Omega \) and for \( f(\zeta) \) in all of \( \omega \) is by harmonic and analytic continuations respectively.

Furthermore, from (10.8) it follows that

\[
(10.9) \quad f(\zeta) = F(0, 0) \int_{0}^{\pi} \sin^{N-3} t dt.
\]

That is, we may regard \( f(\zeta) \) as the analytic continuation of the function given by the right side of (10.9), from the real \( \zeta \) to complex \( \zeta. \)

The determination of the \( A_k \) in (2.2) now proceeds as in §2, with the result given by equation (2.7). In order to generalize equation (4.6), we add the following to the notation given in §4:

\[
A_k = \sum_{k=0}^{n} \rho_k \sin^{N-3} t_k, \quad M(T_k) = [M(T)]_{t_k = T}, \quad M(T*) = M(T) \sin^{N-3} t.
\]

Then since \( V(C_n) = \det |P_{C_n}(x, \rho)| = \int_{T} v(S)M(T) dT, \) we find

\[
A_n(x, \rho; C_n) = \frac{\sum_{k=0}^{n} \int_{T} f(\sigma_k) v(S_k) M(T^*) dT^*}{\int_{T} v(S)M(T) dT}.
\]

Finally, if we introduce the Lagrange interpolation polynomial \( l_n(\sigma; S) \) given by
equation (5.1), we obtain the expression

\[ \Lambda_n(x, \rho; C_n) = \frac{\int_{T^*} l_n(\sigma; S) v(S) M(T^*) dT^*}{\int_T v(S) M(T) dT}. \]

The details regarding the convergence of \( \Lambda_n(x, \rho; C_n) \) to \( F(x, \rho) \) carry over to \( \mathbb{R}^N \) essentially as in §6. In fact, if we now define, instead of (6.3),

\[ \mathcal{X}(C_n) = \int_T |v(S)| M(T) dT / \int_T v(S) M(T) dT, \]

the theorems given in §6 and §7 remain valid for axisymmetric harmonic functions in \( \mathbb{R}^N \). Similarly, we may modify the material in §8 and §9 so that it also remains valid for \( \mathbb{R}^N \).

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