ON CERTAIN CONVEX SETS IN THE SPACE OF
LOCALLY SCHLICHT FUNCTIONS

BY

Y. J. KIM AND E. P. MERKES

ABSTRACT. Let \( H = H(\ast, [+]) \) denote the real linear space of locally schlicht normalized functions in \(|z| < 1\) as defined by Hornich. Let \( K \) and \( C \) respectively be the classes of convex functions and the close-to-convex functions. If \( M \subset H \) there is a closed nonempty convex set in the \( \alpha\beta \)-plane such that for \((\alpha, \beta) \) in this set \( \alpha \ast [+] \beta \ast g \) \( \in C \) (in \( K \)) whenever \( f, g \in M \). This planar convex set is explicitly given when \( M \) is the class \( K \), the class \( C \), and for other classes. Some consequences of these results are that \( K \) and \( C \) are convex sets in \( H \) and that the extreme points of \( C \) are not in \( K \).

1. Introduction. Let \( H \) denote the class of locally schlicht analytic functions \( f \) in the open unit disk \( E \), normalized by the conditions \( f(0) = 0, \; f'(0) = 1 \). Hornich [3] provided a linear space structure to \( H \) with the operations

\[
f[+]g = \int_0^\pi f'(t)g'(t) \, dt, \quad a \ast f = \int_0^\pi (f'(t))^a \, dt,
\]

where \( f, g \in H \) and \( a \) is a real number.

This paper is primarily concerned with two convex subsets of \( H \), namely, the convex schlicht functions \( K \) and the close-to-convex functions \( C \) [4]. More specifically, the main results are summarized in the following two theorems.

**Theorem A.** Let \( f, g \in K \). Define, for real \( \alpha \) and \( \beta \),

\[
G_{\alpha, \beta}(z) = \alpha \ast [+] \beta \ast g = \int_0^\pi (f'(t))^\alpha (g'(t))^\beta \, dt.
\]

Then (i) \( G_{\alpha, \beta} \in K \) if \( \alpha > 0, \beta > 0, \alpha + \beta < 1 \). (ii) \( G_{\alpha, \beta} \in C \) if \(-1/2 \leq \alpha, \beta \leq 3/2, \; -1/2 \leq \alpha + \beta \leq 3/2 \). In each case, the result is sharp.

Sharpness here means that for each pair of real numbers \( \alpha, \beta \), not restricted as in the theorem, there exist functions \( f, g \in K \) such that the corresponding \( G_{\alpha, \beta} \) is not in the stated subclass of \( H \).

**Theorem B.** Let \( f, g \in C \). Then \( G_{\alpha, \beta} \in C \) if \(-1/3 \leq \alpha + \beta \leq 1, \; \alpha - 3\beta \leq 1, \; \beta - 3\alpha \leq 1 \). This result is sharp.
In particular these theorems prove $K$ and $C$ are convex sets in $H$. Furthermore, a line segment joining two functions in $K$ can always be extended from each end point into the convex set $C$. In particular, this implies that no extreme point of the convex set $C$ in $H$ is a convex function.

A number of authors ([5], [6], [7]) have considered the special case where only one function appears in the integrand (1). In fact, Theorem 1 and Theorem 2 reduce to theorems of Merkes and Wright [5] when $\beta = 0$. The one-dimensional results concern radial lines in $H$ joining a point of $K$ or $C$ with the origin, $f(z) = z$, in $H$. In addition to consideration of functions $G_{a,0}$ earlier papers ([2], [6]) also discuss the close-to-convexity of integrals of the form

$$g_a(z) = \int_0^\pi (f(t)/i)^a \, dt$$

where $f$ is in a given subclass of $H$. This suggests a study of functions $g \in H$ such that $g \neq 0$ in $0 < |z| < 1$. Some special cases of this problem are discussed in \S 5 of this paper.

2. Some lemmas. If $p(z) = 1 + a_1z + a_2z^2 + \cdots$ in $E$ and if $\Re p(z) > 0$, $z \in E$, then, by the mean value theorem of harmonic functions, we have

$$0 \leq \int_{\theta_1}^{\theta_2} \Re \{p(re^{i\theta})\} \, d\theta \leq \int_0^{2\pi} \Re \{p(re^{i\theta})\} \, d\theta = 2\pi$$

for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and for all $r$, $0 \leq r < 1$.

**Lemma 1.** If $f \in K$, then for $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$ we have

$$\frac{\theta_2 - \theta_1}{2} \leq \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} \right\} \, d\theta \leq \pi + \frac{\theta_2 - \theta_1}{2},$$

and

$$0 \leq \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} \right\} \, d\theta \leq 2\pi.$$

**Proof.** The inequalities (3) follow from (2) and the fact that $\Re |z f'/f| > 1/2$ for $f \in K$. The analytic definition of the class $K$ and (2) imply (4).

**Lemma 2.** If $f \in C$, then

$$-\pi + \frac{\theta_2 - \theta_1}{2} \leq \int_{\theta_1}^{\theta_2} \Re \left\{ \frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} \right\} \, d\theta \leq 2\pi + \frac{\theta_2 - \theta_1}{2}$$

and

$$-\pi \leq \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} \right\} \, d\theta \leq 3\pi$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 \leq r < 1$. 

Proof. For $f \in C$ there is a function $\phi$ and a real number $\beta$, $|\beta| < \pi/2$, such that $e^{-i\beta} \phi \in K$ and $\text{Re} l/(z)/\phi(z) > 0$, $z \in E$. The last condition implies $\text{Re} l/(z)/\phi(z) > 0$, $z \in E$, as well [5]. It follows that

$$\left| \arg \left[ \frac{f(re^{i\theta})}{re^{i\theta}} \right] - \arg \left[ \frac{\phi(re^{i\theta})}{re^{i\theta}} \right] \right| < \frac{\pi}{2}.$$

Now this implies

$$\left| \int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{f(re^{i\theta})}{f(re^{i\theta})} \right\} d\theta - \int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{\phi(re^{i\theta})}{\phi(re^{i\theta})} \right\} d\theta \right|$$

$$\leq \pi/2 + \pi/2 = \pi.$$

Since $e^{-i\beta} \phi \in K$ the last inequality and (3) imply (5). The left-hand side of (6) is Kaplan's characterization of close-to-convex functions [4]. The right-hand inequality in (6) follows from (4) and the fact that

$$|\arg f(re^{i\theta}) - \arg \phi(re^{i\theta})| < \pi/2.$$

The next lemma provides a class of close-to-convex functions that serve as extremal functions for the sharpness arguments in this paper.

**Lemma 3.** For real $\alpha$, the function $h_{\alpha}(z) = \int_0^z (1 + t)^{\alpha} dt$ is not univalent in $E$ if $\alpha \notin [-3, 1]$. If $-3 \leq \alpha \leq 1$, then $h_{\alpha}$ is close-to-convex in $E$. $h_{\alpha}$ is convex if and only if $-2 \leq \alpha \leq 0$.

**Proof.** $h_{\alpha}(z) = [(1 + z)^{\alpha+1} - 1]/(1 + \alpha)$ provided $\alpha \neq -1$. When $\alpha = -1$, this function is $\log(1 + z)$ which is univalent and convex in $E$. If $\alpha \neq -1$, $h_{\alpha}$ is univalent in $E$ if and only if $(1 + z)^{\alpha+1}$ is univalent. The latter is the case if and only if $-3 \leq \alpha \leq 1$ [7]. For $-3 \leq \alpha \leq -1$, furthermore, let $\phi(z) = z/(1 + z)$ and we have

$$\frac{b_{\alpha}'(z)}{\phi(z)^{\alpha+1}} = \text{Re} \frac{(1 + z)^{\alpha+2}}{1 + z} \geq 0, \ z \in E.$$

For $-1 < \alpha \leq 1$, let $\phi = z$ and $\text{Re} [b_{\alpha}'(z)/\phi(z)] = \text{Re} (1 + z)^{\alpha} > 0$, $z \in E$. This proves $b_{\alpha}$ is close-to-convex for $-3 \leq \alpha \leq 1$. The convexity follows the fact, for real $\alpha$,

$$\text{Re} \left\{ \frac{zb_{\alpha}'(z)}{b_{\alpha}'(z)} + 1 \right\} = \text{Re} \frac{1 + (1 + \alpha)z}{1 + z} \geq 0$$

if and only if $-2 \leq \alpha \leq 0$.
3. Convexity results. The first question that we consider is a proof of the convexity of the sets $K$ and $C$ in the space $H$. The result for $K$ is known [2].

Theorem 1. Let $f_1, f_2$ be in $K$ (in $C$) and let $0 \leq \lambda \leq 1$. Then $G_{\lambda, 1-\lambda}$ given by (1) is in $K$ (in $C$).

Proof. If $f_1, f_2$ are in $K$, then from (1) we obtain

$$
\text{Re} \left\{ \frac{zG''_{\lambda, 1-\lambda} + 1}{G'_{\lambda, 1-\lambda}} \right\} = \lambda \text{Re} \left\{ \frac{z f''_1 + 1}{f'_1} \right\} + (1 - \lambda) \text{Re} \left\{ \frac{z f''_2 + 1}{f'_2} \right\}
$$

which is nonnegative since $f_1$ and $f_2$ are in $K$. It follows that $G_{\lambda, 1-\lambda} \in K$. Now assume $f_j \in C$. Then there exist functions $\phi_j \in K$ and real numbers $\beta_j, |\beta_j| < \pi/2$, such that

$$(7) \quad \text{Re} \left\{ e^{i\beta_j} f''_j(z)/f'_j(z) \right\} > 0, \quad z \in E \ (j = 1, 2).$$

Define

$$
\phi_{\lambda, 1-\lambda}(z) = \int_0^z (\phi'_1(t))^{\lambda} (\phi'_2(t))^{1-\lambda} dt
$$

and, by the first part of this theorem, $\phi_{\lambda, 1-\lambda} \in K$. Furthermore,

$$(8) \quad e^{i\beta} G_{\lambda, 1-\lambda} = \left( e^{i\beta_1 f'_1} \phi'_1 \right)^{\lambda} \left( e^{i\beta_2 f'_2} \phi'_2 \right)^{1-\lambda}$$

where $\beta = \lambda \beta_1 + (1 - \lambda) \beta_2$. In view of (7), it follows that the real part of the left-hand side of (8) is nonnegative and, hence, that $G_{\lambda, 1-\lambda}$ is close-to-convex [4].

Let $f_1$ and $f_2$ be two functions in $H$ and assume the corresponding $G_{\alpha, \beta}$ function (1) is in $C$ (in $K$) for $\alpha = \alpha_1, \beta = \beta_1$ and for $\alpha = \alpha_2, \beta = \beta_2$. Then $G_{\alpha, \beta}$ is in $C$ (in $K$) for all points in the $\alpha\beta$-plane on the line segment joining $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$. Indeed, the function

$$
H_\lambda(z) = \int_0^z (G'_{\alpha_1, \beta_1}(t))^{\lambda} (G'_{\alpha_2, \beta_2}(t))^{1-\lambda} dt
$$

$$
= \int_0^z (f'_1(t))^{\lambda \alpha_1 + (1-\lambda)\alpha_2} (f'_2(t))^{\lambda \beta_1 + (1-\lambda)\beta_2} dt
$$

is in $C$ (in $K$) for $0 \leq \lambda \leq 1$ by Theorem 1. It follows that for each pair $f_1, f_2 \in H$ there is a convex set in the $\alpha\beta$-plane containing $(0, 0)$ and such that the corresponding $G_{\alpha, \beta}$ function of the pair is in $C$ (in $K$). Furthermore, this set is closed in the plane since $C$ (or $K$) is a compact subclass of $H$ in the topology of uniform convergence. These remarks assist in the proof of the following useful result.

Lemma 4. Let $M$ be a nonempty subclass of $H$. Then the set of points in the
CONVEX SETS IN THE SPACE OF SCHLICHT FUNCTIONS 221

\(\alpha \beta\)-plane such that \(G_{\alpha, \beta}\) is in \(C\) (in \(K\)) for each pair \(f_1, f_2\) in \(M\) is nonempty, closed, and convex.

Proof. For each pair \(f, g \in M\) there is a maximal closed convex set in the \(\alpha \beta\)-plane such that the functions \(G_{\alpha, \beta}\) in (1) are in \(C\) (in \(K\)). The intersection of these sets, taken over all pairs \(f, g \in M\), is a closed, convex set containing the origin.

4. Proofs of the principal theorems. We begin with a proof of part (i) of Theorem A. Let \(f_1, f_2\) be in \(K\). Then from (1)

\[
R_{\alpha, \beta}(z) = \text{Re} \left\{ 1 + \frac{zG_{\alpha, \beta}'(z)}{G_{\alpha, \beta}(z)} \right\}
\]

(9)

\[
= (1 - \alpha - \beta) + \alpha \text{Re} \left\{ 1 + \frac{zf_1''(z)}{f_1'(z)} \right\} + \beta \text{Re} \left\{ 1 + \frac{zf_2''(z)}{f_2'(z)} \right\}
\]

which is clearly nonnegative if \(\alpha \geq 0\), \(\beta \geq 0\), and \(\alpha + \beta \leq 1\). For the sharpness, let \(f_1 = f_2 = z/(1 + z)\). Then

\[
G_{\alpha, \beta}(z) = \int_0^z \frac{dt}{(1 + t)^{2(\alpha + \beta)}},
\]

and, by Lemma 3, this function is in \(K\) if and only if \(0 \leq \alpha + \beta \leq 1\). The fact that \(\alpha \geq 0\) follows from consideration of \(G_{\alpha, \beta}\) for \(f_1 = z/(1 + z)\) and \(f_2 = z\). We obtain \(\beta \geq 0\) by symmetry.

To prove (ii) of Theorem A, we first observe that the closed convex region

\[-1/2 \leq \alpha, \beta, \alpha + \beta \leq 3/2\]

is the convex hull of the points \((3/2, 0), (0, 3/2), (-1/2, 0), (0, -1/2), (3/2, -1/2), \text{ and } (-1/2, 3/2)\). That the first four of these points are contained in the convex set \(D\), that exists by Lemma 4 with \(M = K\), is proved in [5]. For \(\alpha = 3/2\), \(\beta = -1/2\) we have from (9) that

\[
R_{3/2, -1/2}(z) = \frac{3}{2} \text{Re} \left\{ 1 + \frac{zf_1''(z)}{f_1'(z)} \right\} - \frac{1}{2} \text{Re} \left\{ 1 + \frac{zf_2''(z)}{f_2'(z)} \right\}
\]

\[
\geq -\frac{1}{2} \text{Re} \left\{ 1 + \frac{zf_2''(z)}{f_2'(z)} \right\}.
\]

Since \(f_2 \in K\), this inequality and Lemma 1 imply

\[
\int_{\theta_1}^{\theta_2} R_{3/2, -1/2}(re^{i\theta}) \, d\theta \geq -\frac{1}{2}(2\pi) = -\pi,
\]

from which we conclude that \(G_{3/2, -1/2} \in C\) [4].

To verify the sharpness of the region in (ii) of Theorem A, consider the con-
vex functions $z$ and $z/(1 + z)$. With $f_1 = z$, $f_2 = z/(1 + z)$ we obtain $-1/2 \leq \beta \leq 3/2$ and $-1/2 \leq \alpha \leq 3/2$ follows by symmetry. The remaining restrictions on $\alpha$ and $\beta$ are established by setting $f_1 = f_2 = z/(1 + z)$.

We turn now to the proof of Theorem B. The convex set in the $\alpha\beta$-plane in this theorem is simply the closed convex hull of the points $(1, 0)$, $(0, 1)$, $(-1/3, 0)$ and $(0, -1/3)$. For each of these points it has been proved [5] that $G_{\alpha, \beta} \in \mathcal{C}$ whenever $f_1, f_2 \in \mathcal{C}$. Hence, by Lemma 4, $G_{\alpha, \beta} \in \mathcal{C}$ for all $\alpha, \beta$ in this convex hull. The sharpness of this closed convex set remains to be established.

The function $s(z) = z/(1 + z)^2$ is close-to-convex relative to the convex function $\phi(z) = z/(1 + z)$. Let $f_1(z) = f_2(z) = s(z)$ in (1). Then $G_{\alpha, \beta}(z) = \int_0^z (1 + t)^{-3\alpha - 3\beta} \, dt$. By Lemma 3, this function is univalent in $E$ if and only if $-1/3 \leq \alpha + \beta \leq 1$ and is close-to-convex for $\alpha$ and $\beta$ in this interval. Thus $G_{\alpha, \beta}$ is not univalent for the given pair of functions whenever $\alpha + \beta > 1$ or $\alpha + \beta < -1/3$. If we set $f_1(z) = z$ and $f_2(z) = s(z)$ then $G'_{\alpha, \beta} = (1 + z)^{-3\beta}$ from which we conclude $G_{\alpha, \beta}$ is not univalent for $\beta > 1$ or $\beta < -1/3$. The restrictions on $\alpha$ are obtained by symmetry and the sharpness of the closed region in Theorem B is proved.

A curious result is obtained when $f_1$ is in $\mathcal{C}$ while $f_2$ is restricted to $\mathcal{K}$.

Theorem C. If $f_1 \in \mathcal{C}$, $f_2 \in \mathcal{K}$, then $G_{\alpha, \beta} \in \mathcal{C}$ when

$$-1/3 \leq \alpha, \quad -1 \leq 3\alpha + 2\beta \leq 3, \quad -3 \leq \alpha - 2\beta \leq 1.$$ 

This result is sharp.

Proof. The convex set (10) is the closed convex hull of the points $(1, 0)$, $(-1/3, 0)$, $(0, -1/2)$, $(0, 3/2)$ and $(-1/3, 4/3)$. All, except the last, are known [5] to be points for which $G_{\alpha, \beta} \in \mathcal{C}$ whenever $f_1 \in \mathcal{C}$ and $f_2 \in \mathcal{K}$. The proof that $(-1/3, 4/3)$ is also such a point parallels the proof of the analogous result for $(3/2, -1/2)$ in the justification of Theorem A. Sharpness of the first inequality follows by letting $f_1 = z/(1 + z)$ and $f_2 = z$. The sharpness of the next two sets of inequalities in (10) are respectively established by setting $f_1(z) = z(1 + 1/2z)/(1 + z)^2$, $f_1(z) = z(1 + 1/2z)$ and setting $f_2 = z/(1 + z)$ in each case. The conclusion is obtained from these functions and Lemma 3.

5. Related results.

Theorem 2. Let $f_1, f_2$ be in $\mathcal{K}$. Define

$$g_{\alpha, \beta}(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^\alpha \left( \frac{f_2(t)}{t} \right)^\beta \, dt.$$ 

Then (i) $g_{\alpha, \beta} \in \mathcal{K}$ provided $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta \leq 2$ and (ii) $g_{\alpha, \beta} \in \mathcal{C}$ provided $-1 \leq \alpha \leq 3$, $-1 \leq \beta \leq 3$, and $-1 \leq \alpha + \beta \leq 3$. These results are sharp.
Proof. The convexity of the set \( D \) in the \( \alpha \beta \)-plane such that \( g_{\alpha, \beta} \in K \) for all \( f_1, f_2 \in K \) follows from Lemma 4 by setting \( M = \{ \varphi \in H : zg' \in K \} \) \((i)\) is a simple consequence of this since \((0, 0), (2, 0), (0, 2)\) are in \( D \) by [5]. The closed convex region in \((ii)\) is the closed convex hull of the points \((-1, 0), (0, -1), (3, 0), (0, 3), (3, -1), (-1, 3)\). The first four are known [5] to be in the set \( D \) such that \( g_{\alpha, \beta} \in C \) for all \( f_1, f_2 \in K \). To prove \((3, -1)\) is also in \( D \), we observe that

\[
M_{3, -1}(z) = \text{Re} \left\{ 1 + \frac{zg_{3, -1}'(z)}{g_{3, -1}(z)} \right\}
\]

\[
= -1 + 3 \text{Re} \left\{ \frac{z/f_1'(z)}{f_1(z)} \right\} - \text{Re} \left\{ \frac{z/f_2'(z)}{f_2(z)} \right\}
\]

\[
\geq \frac{1}{2} - \text{Re} \left\{ \frac{z/f_2'(z)}{f_2(z)} \right\}
\]

since \( \text{Re}[z/f_1'f_1] \geq 1/2 \) for \( f_1 \in K \). Let \( 0 \leq \theta_1 < \theta_2 < 2\pi \) and integrate

\[
M_{3, -1}(re^{i\theta})
\]

with respect to \( \theta \) from \( \theta_1 \) to \( \theta_2 \) for a fixed \( r, 0 < r < 1 \). By (3) in Lemma 1 and the above inequality, we have

\[
\int_{\theta_1}^{\theta_2} M_{3, -1}(re^{i\theta}) d\theta \geq \frac{\theta_2 - \theta_1}{2} - \left( \frac{\theta_2 - \theta_1}{2} \right) = -\pi.
\]

This proves \( g_{3, -1} \in C \) [4]. The fact that \( g_{-1, 3} \in C \) follows by symmetry. An application of Lemma 4 now completes the proof of \((ii)\). Sharpness in \((i)\) and in \((ii)\) follows by considering the functions \((11)\) obtained by the pairing \( z/(1 + z) \) with itself or with \( z \) for the choices of \( f_1 \) and \( f_2 \).

A similar argument to the proof of Theorem B can be used to prove the next result.

**Theorem 3.** If \( f_1, f_2 \in C \), then the function \((11)\) is in \( C \) if \((\alpha, \beta)\) belongs to the closed convex region \( D \) given by \(-1/2 \leq \alpha + \beta \leq 1, \alpha - 2\beta \leq 1 \) and \( \beta - 2\alpha \leq 1 \). The result is sharp.

The extremal functions for the first inequalities are

\[
f_1(z) = f_2(z) = \frac{z(1 + \mu z)}{(1 + z)^2}, \quad \mu = \cos \gamma e^{i\gamma}, \quad 0 < \gamma < \pi
\]

(see [5]). For the next set of inequalities in the definition of \( D \) the extremal functions are \( f_1 \) as given above and \( f_2 = z/(1 + \mu z)^2, \mu = \cos \gamma e^{i\gamma}, 0 < \gamma < \pi \).

Finally, the analog of Theorem C for functions of the form \((11)\) can be proved by the methods in this paper.

**Theorem 4.** If \( f_1 \in C \) and \( f_2 \in K \), then the function \( g_{\alpha, \beta} \), defined by \((11)\), be-
longs to \( C \) if \((\alpha, \beta)\) satisfies all the following inequalities:
\[-1/2 < \alpha < 1, \quad -1 < 2\alpha + \beta < 3, \quad -1 < \beta - \alpha < 3.\]
The result is sharp.

Sharpness is obtained from various pairings of the functions \( z, z/(1 + \mu z) \) in \( K \) and \( z(1 + \mu z)/(1 + z)^2 \) in \( C \) where \( \mu = \cos \gamma e^{i\gamma}, 0 \leq \gamma < \pi. \)

REFERENCES