SU(\(n\)) ACTIONS ON DIFFERENTIABLE MANIFOLDS WITH VANISHING FIRST AND SECOND INTEGRAL PONTRJAGIN CLASSES\(^{(1)}\)

BY

EDWARD A. GROVE

ABSTRACT. In this paper we determine the connected component of the identity of the isotropy subgroups of a given action of SU(\(n\)) on a connected manifold whose first and second integral Pontrjagin classes are zero and whose dimension is less than \(n^2 - \frac{8n}{3} - 1\).

1. General introduction. Let SU(\(n\)) be a differentiable transformation group acting on a connected manifold \(M\) with \(\dim M \leq n^2 - \frac{8n}{3} - 2\), and assume that the first two integral Pontrjagin classes of \(M\) are zero. The purpose of this paper is to determine the orbit structure of \(M\) under the action of SU(\(n\)).

To be more precise, in Theorem 2.3 we determine those closed connected simple subgroups \(H\) of SU(\(n\)) such that the first integral Pontrjagin class of SU(\(n\))/\(H\), \(P_1(\text{SU}(n)/H; \mathbb{Z})\), is zero; in Theorem 2.5 we determine those closed connected simple subgroups \(H\) of SU(\(n\)) such that \(P_1(\text{SU}(n)/H; \mathbb{Z}) = 0 = P_2(\text{SU}(n)/H; \mathbb{Z})\); in Theorem 2.9 we determine the connected component of the identity of a regular isotropy subgroup of the given above actions of SU(\(n\)) on \(M\). In Theorem 3.1 we then show how the connected component of the identity of all isotropy subgroups of SU(\(n\)) is completely determined by the connected component of the identity of the principal isotropy subgroup, and we give a list of such possible subgroups.

Wu-Chung Hsiang and Wu-Yi Hsiang considered this problem in [4]. One of the key steps in their approach was their knowledge of large subgroups of SU(\(n\)). This forced them to put a severe restriction on \(M\). Our approach is more general in nature, and allows us a much less restrictive dimension requirement.

The author wishes to thank his thesis advisor Wu-Yi Hsiang for the encouragement and help he received while writing this paper. Some of these results were included in the author's Ph.D. thesis at Brown University.

In this section we recall several well-known facts. Let \(G\) be a compact con-

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connected Lie group acting differentiably on a manifold $M$. For each $x \in M$, let $G_x = \{g \in G : g \cdot x = x\}$ and let $G(x) = \{g \cdot x : g \in G\}$. Then if $f : G/G_x \cong G(x) \subseteq M$ is the natural map,

$$f_*(M) = r(G/G_x) \oplus \nu(G/G_x) = \alpha_{\xi} (\iota_{G/G_x}) \oplus \alpha_{\xi} (\phi_x)$$

where $\xi$ is the principal bundle $(G_x \to G \to G/G_x)$, $\iota_{G/G_x}$ is the isotropy representation of $G_x$ in $G$, and $\phi_x$ is the slice representation of $G_x$ at $x$. Suppose now that $x \in M$ is a regular element. Then $\phi_x$ is a trivial representation [5], and so $f_*(M) = \alpha_{\xi} (\iota_{G/G_x}) \oplus \theta$ where $\theta$ is a trivial vector bundle. The following result is trivial to prove but is frequently useful.

**Proposition 1.1.** Let $\eta : (H \to E \to B)$ and $\eta' : (G \to E' \to B')$ be principal bundles, and let

$$
\begin{array}{ccc}
G & \xrightarrow{\Delta} & G' \\
\downarrow & & \downarrow \\
E & \longrightarrow & E' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
$$

be a homomorphism of principal bundles. Then

$$
\begin{array}{ccc}
R(G) & \xrightarrow{A^1} & R(G') \\
n_{\eta} & & n'_{\eta'} \\
K(B) & \longleftarrow & K(B')
\end{array}
$$

is a commutative diagram.

So if $F : G/G_x^0 \to G/G_x$ is the natural map and $\xi_2$ is the principal bundle $(G_x^0 \to G \to G/G_x^0)$, we have

$$F^*_1 \tau(M) = \alpha_{\xi_2} (\iota_{G/G_x^0}) \oplus \theta = r(G/G_x^0) \oplus \theta,$$

and so $F^*_j (P_j (M; Z)) = P_j (G/G_x^0; Z)$ (where $P_j (N; Z)$ is the $j$th integral Pontrjagin class of $N$).

So to study $G$ actions on a manifold $M$ satisfying $P_1 (M; Z) = 0 = P_2 (M; Z)$, we must first find all those closed connected subgroups $H$ of $G$ such that $P_1 (G/H; Z) = 0 = P_2 (G/H; Z)$. 
2. Subgroups of $SU(n)$ with $P_1(SU(n)/H; Z) = 0 = P_2(SU(n)/H; Z)$. Let $G$ be a compact Lie group, $H$ a closed connected subgroup, and let $T$ be a maximal torus in $H$. Then the natural map $\pi : G/T \to G/H$ is a fibre map with fibre $H/T$, and hence the kernel of the map $\pi^* : H^*(G/H; Z) \to H^*(G/T; Z)$ consists only of torsion elements. Let $\xi$ be the principal bundle $(H \to G \to G/H)$ and $\xi'$ the principal bundle $(T \to G \to G/T)$. Then if $\phi$ is a real representation of $H$, $\pi^*(\alpha_{\xi}(\phi)) = \alpha_{\xi'}(\phi|T)$ and hence $\pi^*(P_j(\alpha_{\xi}(\phi); Z)) = P_j(\alpha_{\xi'}(\phi|T); Z)$. Let $\eta(G) = (G \to E_G \to B_G)$ be a classifying bundle for $G$. Then $\eta(T) = (T \to E_G \to B_T)$ is a classifying bundle for $T$. Letting $i : G/T \to B_T$ be the classifying map of $\xi'$, we have that $\pi^*(\alpha_{\xi}(\phi)) = i^*(\alpha_{\eta(T)}(\phi|T))$, and hence from [2] that

$$\pi^*(P_j(\alpha_{\xi}(\phi); Z)) = i^*(P_j(\alpha_{\eta(T)}(\phi|T); Z))$$

where $\sigma_{2j}$ is the $2j$-th elementary symmetric function, $\tau_{\eta(T)}$ is the transgression in the Serre spectral sequence $E(\eta(T))$ associated with $\eta(T)$, and $\{W_1, \ldots, W_N\}$ are the positive weight vectors of $\phi$.

Recall now that $r(G/H) = \alpha_{\xi}(t_{G/H}) = \alpha_{\xi}(\text{Ad}_G|H - \text{Ad}_H|H)$. By Proposition 1.1, $\alpha_{\xi}(\text{Ad}_G|H)$ is a trivial bundle, and hence $r(G/H) \oplus \alpha_{\xi}(\text{Ad}_H|H) = \theta$ where $\theta$ is a trivial vector bundle.

**Corollary 2.1 [2].** Let $T$ be a torus in $G$. Then $G/T$ is stably parallelizable.

From $\pi^*(r(G/H)) = i^*(\alpha_{\eta(T)}(t_{G/H}|T))$ and $\pi^*(\alpha_{\xi}(\text{Ad}_H|H)) = i^*(\alpha_{\eta(T)}(\text{Ad}_H|T))$, we have the following corollary.

**Corollary 2.2.** $\sum_{\alpha + \beta = j} \pi^*(P_\alpha(G/H; Z)) \cdot \pi^*(P_\beta(\alpha_{\xi}(\text{Ad}_H); Z)) = 0.$

As $E_G$ is contractible, $E_G \times_T G \to G/T$ is a homotopy equivalence, and hence we may take $\gamma = (G \to G/T \to B_T)$ to be a principal $G$ bundle. Thus we see that $i^* : H^*(B_T; Z) \to H^*(G/T; Z)$ is just the edge homomorphism

$$H^*(B_T; Z) = E_{2n}^{0,0}(\gamma) \to E_{\infty}^{0,0}(\gamma) \subset H^*(G/T; Z),$$

and hence $\ker i^* = (\text{im } d(\gamma)^+)$, the ideal in $H^*(B_T; Z)$ generated by the elements of positive degree in the image of the differential of $E(\gamma)$.

Suppose that $U$ is a closed connected normal subgroup of $H$. Then by using Proposition 1.1 together with a standard argument, it follows that $P_j(G/H; Z) = 0$ implies that $P_j(G/U; Z) = 0$.

So to determine those closed connected subgroups $H$ of $G$ with $P_1(G/H; Z) = 0 = P_2(G/H; Z)$, we need only determine those closed simple subgroups $U$ of $G$ with $P_1(G/U; Z) = 0 = P_2(G/U; Z)$ and then, by using the classification theorem.
of compact Lie groups, determine whatever restrictions there are on "piecing" the various simple (we shall see later that if $P_1(SU(n)/H; Z) = 0$, then $H$ is either semisimple or a torus) groups together to form an acceptable $H$.

We now let $G = SU(n)$ and let $\lambda : H \to SU(n)$ be the embedding of $H$ as a closed subgroup, i.e. $\lambda$ is an injective homomorphism of the compact connected Lie group $H$ into $SU(n)$. By examining the spectral sequence $E(\gamma)$ of $\gamma$, we see that in low dimensions at least, $\ker i^* = (\lim r_\gamma)$ where $r_\gamma$ is the transgression of $\gamma$. Thus we have (at least in low dimensions)

$$\ker i^* = \langle \delta^* (r_{\eta(T)}(\mu_1), \ldots, r_{\eta(T)}(\mu_n)) \rangle$$

where $\{\mu_1, \ldots, \mu_n\}$ is the set of weight vectors of $\lambda$, and $\delta^* (x_1, \ldots, x_n)$ is the ring of symmetric polynomials with zero constant term.

We now apply the above to the representation $\phi = \text{Ad}_H$. Note that as $\lambda : H \to SU(n)$, $\mu_n = - (\mu_1 + \cdots + \mu_{n-1})$, and hence

$$2\sigma_2 (r_{\eta(T)}(\mu_1), \ldots, r_{\eta(T)}(\mu_n)) = - \sum_{i=1}^n (r_{\eta(T)}(\mu_i))^2.$$  

Recall that, by Corollary 2.2,

$$\pi^*(P_1(SU(n)/H; Z)) = - \pi^* P_1 (\alpha_{\phi} (\text{Ad}_H); Z)$$

$$= i^* (\sigma_2 (r_{\eta(T)}(W_1), \ldots, r_{\eta(T)}(W_N)), -r_{\eta(T)}(W_1), \ldots, -r_{\eta(T)}(W_N)))$$

$$= - i^* \sum_{j=1}^N (r_{\eta(T)}(W_j))^2.$$  

So $\pi^*(P_1(SU(n)/H; Z)) = 0$ if and only if for some integer $K$,

$$K \sum_{j=1}^n (r_{\eta(T)}(\mu_j))^2 = 2 \sum_{j=1}^n (r_{\eta(T)}(W_j))^2.$$  

So if $U$ is a closed, simple, normal subgroup of $H$, it follows that $\pi^*(P_1(SU(n)/U; Z)) = 0$ if and only if there exists an integer $K$ such that

$$K \sum_{j=1}^n |\mu_j|^2 = 2 \sum_{j=1}^N |W_j|^2$$

where $||$ is the Cartan-Killing norm.

We are now ready to determine those closed simple subgroups $G$ of $SU(n)$ for which $P_1(SU(n)/G; Z) = 0$. First note that if $\lambda : G \to SU(n)$, then $SU(n)/\lambda[G]$ is diffeomorphic to $SU(n)/\lambda^*[G]$. Hence in the following theorem if $\lambda : G \to SU(n)$ is such that $P_1(SU(n)/\lambda[G]; Z) = 0$, then we shall either include $(G, \lambda)$ or $(G, \lambda^*)$ in the list, but not both.
Theorem 2.3. Suppose $\lambda : G \to SU(n)$ is an almost faithful embedding, where $G$ is a simply connected, compact, simple Lie group such that $P_1(SU(n)/\Lambda[G]; \mathbb{Z}) = 0$. Then modulo trivial representations, $(G, \lambda)$ appears on the following list:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(4)$</td>
<td>$b \Lambda^2 \mu_4$</td>
</tr>
<tr>
<td></td>
<td>$a \mu_4 \otimes 3 \Lambda^2 \mu_4 \otimes d \mu_4^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d = 2$</td>
</tr>
<tr>
<td>$SU(5)$</td>
<td>$\mu_5 \otimes b \Lambda^2 \mu_4 \otimes c \Lambda^2 \mu_5^*$</td>
</tr>
<tr>
<td></td>
<td>$b + c = 3$</td>
</tr>
<tr>
<td>$SU(6)$</td>
<td>$b \Lambda^2 \mu_6 \otimes c \Lambda^2 \mu_6^*$</td>
</tr>
<tr>
<td></td>
<td>$b + c = 1, 3$</td>
</tr>
<tr>
<td></td>
<td>$b \Lambda^3 \mu_6$</td>
</tr>
<tr>
<td></td>
<td>$a \mu_6 \otimes b \Lambda^2 \mu_6 \otimes \Lambda^3 \mu_6 \otimes c \Lambda^2 \mu_6^* \otimes d \mu_6^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d + 4(b + c) = 6$</td>
</tr>
<tr>
<td>$SU(7)$</td>
<td>$a \mu_7 \otimes b \Lambda^2 \mu_7 \otimes c \Lambda^2 \mu_7^* \otimes d \mu_7^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d = 4, b + c = 1$</td>
</tr>
<tr>
<td>$SU(8)$</td>
<td>$\mu_8 \otimes b \Lambda^3 \mu_8 \otimes c \Lambda^3 \mu_8^*$</td>
</tr>
<tr>
<td></td>
<td>$b + c = 1$</td>
</tr>
<tr>
<td>$SU(k), k \geq 2$</td>
<td>$\text{Ad}_{SU(k)}$</td>
</tr>
<tr>
<td></td>
<td>$a + d \geq 2k$</td>
</tr>
<tr>
<td>$SU(k), k \geq 2$</td>
<td>$a \mu_k \otimes d \mu_k^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d = k - 2, b + c = 1$</td>
</tr>
<tr>
<td>$SU(k), k \geq 2$</td>
<td>$a \mu_k \otimes b S^2 \mu_k \otimes c S^2 \mu_k^* \otimes d \mu_k^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d = 2 + k, b + c = 1$</td>
</tr>
<tr>
<td>$SU(k), k \geq 4$</td>
<td>$a \mu_k \otimes b \Lambda^2 \mu_k \otimes c \Lambda^2 \mu_k^* \otimes d \mu_k^*$</td>
</tr>
<tr>
<td></td>
<td>$a + d = 4, b + c = 2$</td>
</tr>
<tr>
<td></td>
<td>$a + d = 2, b + c = 1$</td>
</tr>
<tr>
<td>$SU(k), k \geq 4$</td>
<td>$\Lambda^2 \mu_k \otimes S^2 \mu_k^*$</td>
</tr>
<tr>
<td>$Sp(3)$</td>
<td>$a \Lambda^2 \nu_6 \otimes b(\Lambda^3 \nu_6 - \nu_6)$</td>
</tr>
<tr>
<td></td>
<td>$a + b = 1, 2$</td>
</tr>
<tr>
<td></td>
<td>$4 \nu_6 \otimes (\Lambda^3 \nu_6 - \nu_6)$</td>
</tr>
<tr>
<td>$Sp(k), k \geq 3$</td>
<td>$a \nu_{2k}$</td>
</tr>
<tr>
<td></td>
<td>$a \geq 2(k + 1)$</td>
</tr>
<tr>
<td>$Sp(k), k \geq 3$</td>
<td>$4 \nu_{2k} \otimes \Lambda^2 \nu_{2k}$</td>
</tr>
<tr>
<td>$Sp(k), k \geq 3$</td>
<td>$\text{Ad}_{Sp(k)}$</td>
</tr>
<tr>
<td>$Sp(5)$</td>
<td>$a p_5 \otimes b \Delta_5$</td>
</tr>
<tr>
<td></td>
<td>$2a + b \mid 6$</td>
</tr>
<tr>
<td>$Sp(7)$</td>
<td>$a p_7 \otimes b \Delta_7$</td>
</tr>
<tr>
<td></td>
<td>$a + b \mid 5$</td>
</tr>
<tr>
<td>$Sp(9)$</td>
<td>$a p_9 \otimes b \Delta_9$</td>
</tr>
<tr>
<td></td>
<td>$a + 2b \mid 7$</td>
</tr>
<tr>
<td>$Sp(11)$</td>
<td>$a p_{11} \otimes b \Delta_{11}$</td>
</tr>
<tr>
<td></td>
<td>$a + 4b \mid 9$</td>
</tr>
</tbody>
</table>
Remark. Note that if \( \overline{G} \) is a compact simple Lie group and \( \lambda : \overline{G} \to G_1 \) is any nontrivial homomorphism of Lie groups, then \( \text{Ker } \lambda \) is a finite central subgroup. So any embedding of \( \overline{G} \) lifts to an almost faithful embedding of \( \overline{G} \)'s universal covering group \( \widetilde{\lambda} : G \to G_1 \). If \( G_1 = \text{SU}(n) \) and \( \text{P}_1(\text{SU}(n)/G) = 0 \), then \( (G, \lambda) \) will be on the above list.

Proof of Theorem 2.3. Suppose \( G = \text{SU}(k), 2 \leq k \leq n \). If \( \lambda : \text{SU}(k) \to \text{SU}(n) \) and \( \Sigma(\lambda) = \{(j, \mu_j) : 1 \leq j \leq n\} \) is the set of weight vectors of \( \lambda \), let \( n(\lambda) = \sum_{j=1}^{n} |\mu_j|^2 \).

For \( 1 \leq p, q \leq k \), let \( e^p_q \) be the \( k \times k \) matrix \( e^p_q = [\delta^p_i \delta^q_j] \). Let \( \lambda_j = i(e^j_i - k^{-1} \sum_{i=1}^{k} e^i_j) \) for \( 1 \leq j \leq k \). Then \( \{\lambda_j : 1 \leq j \leq k-1\} \) forms a basis for \( \mathfrak{h} \), the standard Cartan subalgebra of \( \text{SU}(k) \). We may take our invariant, nondegenerate inner-product for the real Lie algebra of \( \text{SU}(k) \) so that \( (\lambda^i_j, \lambda^i_j) = (k - 1)/k \) and \( (\lambda^i_j, \lambda^j_i) = -1/k \) if \( i \neq j \). Note that

\[
n(\text{Ad}_{\text{SU}(k)}) = 2 \sum_{1 \leq i < j \leq k} |\lambda^i_j - \lambda^j_i|^2 = 2 \sum_{1 \leq i < j \leq k} 2 = 2k(k - 1).
\]

So as we have shown, \( \text{P}_1(\text{SU}(n)/\langle \lambda|\text{SU}(k) \rangle) = 0 \) implies \( 2k(k - 1) \) is an integral multiple of \( n(\lambda) \).

We let \( \pi = \{\lambda^j_j - \lambda^j_{j+1} : 1 \leq j \leq k - 1\} \) be our system of simple roots. If \( \phi : \)
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SU(k) \rightarrow SU(n) is irreducible, let \( \Lambda_\phi \) be the highest weight of \( \phi \), and let

\[
q_j = \frac{2(\Lambda_\phi, \lambda_j - \lambda_{j+1})}{(\lambda_j - \lambda_{j+1}, \lambda_j - \lambda_{j+1})} = (\Lambda_\phi, \lambda_j - \lambda_{j+1}).
\]

Then recall \( q_j \) is a nonnegative integer.

For each \( 1 \leq i \leq k-1 \), let \( \phi_i \) be the unique irreducible representation satisfying

\[
2(\Lambda_\phi, \lambda_j - \lambda_{j+1})
\]

\[
\ll_i = \cdots = 1 - (\lambda_i, \lambda_{i+1})
\]

Hence \( \ll_i \) is a nonnegative integer.

Recall that \( \phi_1 = \mu_k \) and \( \phi_j = \Lambda_j \mu_k \), \( 1 \leq j \leq k \). So

\[
\sum (\phi_i) = \{ \lambda_{i_1} + \cdots + \lambda_{i_j} : 1 \leq i_1 < i_2 < \cdots < i_j \leq k \}.
\]

Hence

\[
n(\phi_j) = \sum_{1 \leq i_1 < \cdots < i_j \leq k} |\lambda_{i_1} + \cdots + \lambda_{i_j}|^2 = \sum_{1 \leq i_1 < \cdots < i_j \leq k} j \left( \frac{k-j}{k} \right)
\]

\[
= \frac{j}{k} \frac{k(k-1) \cdots (k-j+1)}{1 \cdot 2 \cdots j}
\]

\[
= \frac{(k-1)(k-2) \cdots (k-j+1)}{2 \cdot 3 \cdots j} j(k-j) \text{ if } j > 1
\]

\[
= \frac{(k-1)(k-2) \cdots (k-j+2)}{2 \cdot 3 \cdots (j-1)} (k-j+1)(k-j) \text{ if } j > 2.
\]

Note that \( n(\phi_{j+1}) = n(\phi_j) \cdot (k-j-1)/j \). So if \( 1 \leq j \leq (k-1)/2 \), then \( n(\phi_{j+1}) \geq n(\phi_j) \). Also, \( 1 \leq j \leq (k-1)/2 \) implies \( \Sigma(\phi_j) = -\Sigma(\phi_{k-j}) \) (i.e., \( \phi_j = \phi_{k-j}^* \)) which implies \( n(\phi_j) = n(\phi_{k-j}) \). We have

\[
n(\phi_1) = k-1, \quad n(\phi_2) = (k-1)(k-2),
\]

\[
n(\phi_3) = \frac{1}{2} (k-1)(k-2)(k-3) > 2k(k-1) \text{ if } k > 9.
\]

So if \( 4 \leq j \leq \lfloor (k+1)/2 \rfloor \) and \( n(\phi_j) \leq 2k(k-1) \), it follows that \( k \leq 8 \), and hence \( k = 8 \), \( j = 4 \). By a direct check, this is impossible. So if \( 1 \leq j \leq \lfloor (k+1)/2 \rfloor \) and \( n(\phi_j) \leq 2k(k-1) \), we have \( j = 1 \), \( j = 2 \), or \( j = 3 \) and \( 6 \leq k \leq 8 \).

Lemma 2.4. Suppose \( \phi : SU(k) \rightarrow SU(n) \) is an irreducible map with \( n(\phi) \leq 2k(k-1) \). Then up to conjugation, \( \phi \) appears on the following list:

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Proof. Let 
\[ \lambda^A, \mu^A, S^2 \mu^A, \Lambda^A, \text{Ad}_{SU(k)}, \quad k \geq 2. \]

Suppose that there exist at least three distinct indices \( a, b, c \) so that \( q_a, q_b, q_c \) all are different from zero. If \( \phi_1, \phi_2 \) are two different irreducible representations, we say \( \phi_1 \geq \phi_2 \) if \( q_i(\phi_1) \geq q_i(\phi_2) \) for \( i = 1, 2, \ldots, k - 1 \). It follows easily that \( \phi_1 \geq \phi_2 \) implies \( (\Lambda^{\phi_1}, \Lambda^{\phi_1}) \geq (\Lambda^{\phi_2}, \Lambda^{\phi_2}) \).

Using this fact, one easily shows \( (\Lambda^A, A^A) > 1 \).

Since \( q_a, q_b, q_c \) are all different from zero, it follows easily by applying the Weyl group that \( \Sigma(\phi) \) contains at least \( (k - 1)(k - 2)(k - 3) \) distinct vectors \( \Lambda \) all with \( (\Lambda, \Lambda) = (\Lambda^A, \Lambda^A) \). So \( n(\phi) > (k - 1)(k - 2)(k - 3) \). So if \( k > 6 \), \( n(\phi) > 2k(k - 1) \).

Note \( 4 \leq k \) as there exist \( q_a, q_b, q_c \neq 0 \). If \( 4 \leq k \leq 6 \), one shows by direct calculation that \( (\Lambda^A, \Lambda^A) \geq 5 \) and hence \( n(\phi) > 5(k - 1)(k - 2)(k - 3) > 2k(k - 1) \) if \( 4 \leq k \leq 6 \).

Suppose next that there exist exactly 2 indices \( a, b \) with \( q_a, q_b \neq 0 \). Suppose \( k > 9 \). Further suppose that \( 3 \leq a < k - 3 \). Then \( \phi \geq \Lambda^A \mu^A = \phi_a \). So \( (\Lambda^A, \Lambda^A) \geq (\Lambda^A, \Lambda^A) \). Also, by applying the Weyl group, it is clear that we introduce at least as many new vectors \( \Lambda \) into \( \Sigma(\phi) \) with \( (\Lambda, \Lambda) = (\Lambda^A, \Lambda^A) \), as we do vectors \( \Lambda^A \) into \( \Sigma(\Lambda^A \mu^A) \) with \( (\Lambda^A, \Lambda^A) = (\Lambda^A, \Lambda^A) \), and so \( n(\phi) > n(\Lambda^A \mu^A) > 2k(k - 1) \) (as \( 3 < a < k - 3 \)).

It follows that we may assume (up to conjugation)

1. \( a = 1, \ b = 2, \)
2. \( 1 \leq a \leq 2, \ k - 2 \leq b \leq k - 1, \)
3. \( 1 \leq a \leq 2, \ b = 3, \ 6 \leq k \leq 8, \)
4. \( 1 \leq a \leq 3, \ k - 3 \leq b \leq k - 1, \ 7 \leq k \leq 8. \)

By direct computation, we now easily arrive at the result.

With this lemma, the results with \( G = SU(k) \) follow easily. One obtains the results for \( G = B_k, C_k, D_k, G_2, F_4, E_6, E_7, E_8 \) similarly.

Theorem 2.5. Suppose \( \lambda : G \rightarrow SU(n) \) is a nontrivial representation, where \( G \) is a simply connected, compact, simple Lie group. Suppose \( P_1(SU(n)/\Lambda[G]; \mathbb{Z}) = 0 = P_2(SU(n)/\Lambda[G]; \mathbb{Z}) \). Then \( (G, \lambda) \) (modulo an arbitrary trivial representation) appears on the following list:
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>$4\mu_2$</td>
</tr>
<tr>
<td>SU(3)</td>
<td>$\mu_3 \oplus b S^2 \mu_3 \oplus c S^2 \mu_3^*$</td>
</tr>
<tr>
<td>SU(4)</td>
<td>$\Lambda^2 \mu_4$</td>
</tr>
<tr>
<td></td>
<td>$2\Lambda^2 \mu_4$</td>
</tr>
<tr>
<td></td>
<td>$\Lambda^2 \mu_4 \oplus S^2 \mu_4^*$</td>
</tr>
<tr>
<td>SU(6)</td>
<td>$\Lambda^2 \mu_6$</td>
</tr>
<tr>
<td>SU($k$), $k \geq 2$</td>
<td>$\text{Ad}_{SU(k)}$</td>
</tr>
<tr>
<td>SU($k$), $k \geq 2$</td>
<td>$a \mu_k \oplus b \mu_k^*$</td>
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<tr>
<td>$SP(3)$</td>
<td>$\Lambda^2 v_6$</td>
</tr>
<tr>
<td>$SP(k)$, $k \geq 3$</td>
<td>$v_{2k}$</td>
</tr>
<tr>
<td></td>
<td>$3v_{2k}$</td>
</tr>
<tr>
<td></td>
<td>$\text{Ad}_{SP(k)}$</td>
</tr>
<tr>
<td>$SPIN(5)$</td>
<td>$a \rho_5 \oplus b \Delta_5$</td>
</tr>
<tr>
<td>$SPIN(2k+1)$, $k \geq 2$</td>
<td>$\rho_{2k+1}$</td>
</tr>
<tr>
<td></td>
<td>$3\rho_{2k+1}$</td>
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<td></td>
<td>$\text{Ad}_{SPIN(2k+1)}$</td>
</tr>
<tr>
<td>$SPIN(8)$</td>
<td>$a \rho_8 \oplus b \Delta^+_8 \oplus c \Delta^-_8$</td>
</tr>
<tr>
<td>$SPIN(10)$</td>
<td>$\Delta^+_{10}$</td>
</tr>
<tr>
<td>$SPIN(2k)$, $k \geq 4$</td>
<td>$\rho_{2k}$</td>
</tr>
<tr>
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<td>$2\rho_{2k}$</td>
</tr>
<tr>
<td></td>
<td>$3\rho_{2k}$</td>
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<td>$6\rho_{2k}$</td>
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<tr>
<td></td>
<td>$\text{Ad}_{SPIN(2k)}$</td>
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</tbody>
</table>
\[ G_2 \quad \phi_2 = \text{Ad}_{\sigma_2}, \phi_1 \]
\[ F_4 \quad \phi_4 = \text{Ad}_{F_4}, \phi_1 \]
\[ E_6 \quad \text{Ad}_{E_6} \]
\[ E_7 \quad \text{Ad}_{E_7} \]
\[ E_8 \quad \text{Ad}_{E_8} \]

**Proof.** Let \( \{ (j, \mu_j) : 1 \leq j \leq n \} = \Sigma(\lambda) \), and let \( \sigma_j(\lambda) = \sigma_j(r_{\eta(T)}(\mu_1), \ldots, r_{\eta(T)}(\mu_n)) \) for \( 3 \leq j \leq 4 \). Recall that

\[
(\ker i^*)^8 = (Zv_1 \oplus \cdots \oplus Zv_l)^2 \sigma_2(\lambda) \oplus (Zv_1 \oplus \cdots \oplus Zv_l) \sigma_3(\lambda) \oplus Z \sigma_4(\lambda) \text{ where } l = \text{rank } G.
\]

Now

\[ P_2(\text{SU}(n)/\eta \lambda; Z) + P_1(\text{SU}(n)/\eta \lambda; Z) + P_2(\text{SU}(n)/\eta \lambda; Z) = 0 \]

(modulo 2 torsion). So as \( P_1(\text{SU}(n)/\eta \lambda; Z) = 0 \) by hypothesis, we see that \( \eta^*(P_2(\text{SU}(n)/\eta \lambda; Z)) = 0 = \eta^*(P_2(\text{SU}(n)/\eta \lambda; Z)) \). Hence

\[
\left( \sum_{j=1}^{l} \sigma_j \right) \left( \sum_{1 \leq i < j \leq l} b_{ij} \right) \left( \sum_{j=1}^{l} b_{ij} \right) \sigma_2(\lambda) + \left( \sum_{j=1}^{l} \sigma_j \right) \sigma_3(\lambda) + m \cdot \sigma_4(\lambda) = \sigma_4(\text{Ad}_G)
\]

where \( a_j, b_{ij}, d_j, m \in Z \) for \( 1 \leq i \leq l, 1 \leq j \leq l \). Suppose \( G = \text{SU}(k) \). By using the two facts

\[ \sigma_4(y_1, -y_1, \ldots, y_p, -y_p) = \sigma_2(y_1^2, \ldots, y_p^2) \]

and

\[ \sigma_j(X_1, \ldots, X_p, Y_1, \ldots, Y_q) = \sum_{\alpha + \beta = j} \alpha(X_1, \ldots, X_p) \cdot \beta(Y_1, \ldots, Y_q) \]

it follows that

\[
\sigma_4(\text{Ad}_{\text{SU}(k)}) = (2k^2 - 6k - 6) \left( \sigma_1(v_1^4, \ldots, v_k^4) + 3\sigma_2(v_1^2, \ldots, v_k^2) \right) + 2 \left( \sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \ldots, v_j, \ldots, v_{k-1}) \right) + (8k^2 - 6k - 24) \left( \sum_{j=1}^{k-1} v_j \sigma_3(v_1, \ldots, v_j, \ldots, v_{k-1}) \right) + (12k^2 - 12k - 36) \sigma_4(v_1, \ldots, v_{k-1})
\]

where \( \nu_j = r_{\eta(T)}(\nu_j) \) as usual.
As an example of the method, consider \( \lambda = a \mu_k \oplus b \nu_k^* \). One shows easily that

\[
\sigma_2(\mu_k) = -(\sigma_1(v_1^2, \ldots, v_{k-1}^2) + \sigma_2(v_1, \ldots, v_{k-1}))
\]

\[
\sigma_3(\mu_k) = -\left( \sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \ldots, \tilde{v}_j, \ldots, v_{k-1}) + 2\sigma_3(v_1, \ldots, v_{k-1}) \right)
\]

\[
\sigma_4(\mu_k) = -\left( \sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \ldots, \tilde{v}_j, \ldots, v_{k-1}) + 3\sigma_4(v_1, \ldots, v_{k-1}) \right)
\]

\[
\sigma_4(\lambda) = (a + b)\sigma_4(\mu_k) + \frac{1}{2}(a + b)(a + b - 1)(\sigma_2(\mu_k))^2.
\]

So

\[
\sigma_4(Ad_{SU(n)}(\lambda)) = -\left( \sum_{j=1}^{k-1} a_j v_j^2 + \sum_{1 \leq i < j \leq k-1} b_{ij} v_i v_j \right)
\]

\[
\cdot (a + b)(\sigma_1(v_1^2, \ldots, v_{k-1}^2) + \sigma_2(v_1, \ldots, v_{k-1}))
\]

\[
-\left( a + b \right) \sum_{j=1}^{k-1} d_j v_j \left( \sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \ldots, \tilde{v}_j, \ldots, v_{k-1}) + 2\sigma_3(v_1, \ldots, v_{k-1}) \right)
\]

\[
+ m \left( \frac{1}{2} (a + b)(a + b - 1)\sigma_1(v_1, \ldots, v_{k-1})
\right.
\]

\[
+ \frac{3}{2} (a + b)(a + b - 1)\sigma_2(v_1^2, \ldots, v_{k-1})
\]

\[
+ (a + b)(a + b - 1) \sum_{j=1}^{k-1} v_j^3 \sigma_1(v_1, \ldots, \tilde{v}_j, \ldots, v_{k-1})
\]

\[
+ (2(a + b)(a + b - 1) - (a + b)) \sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \ldots, \tilde{v}_j, \ldots, v_{k-1})
\]

\[
+ (3(a + b)(a + b - 1) - 3(a + b))\sigma_4(v_1, \ldots, v_{k-1})\right).
\]

Clearly, \( a_i = a_j \) and \( d_i = d_j \) if \( 1 \leq i, j \leq k - 1 \). Also, \( b_{ij} = b_{\alpha \beta} \) if \( 1 \leq i < j \leq k - 1, 1 \leq \alpha < \beta \leq k - 1 \). Note \(-a + b)a_1 + \frac{1}{2}(a + b)(a + b - 1)m = 2k^2 - 6k - 18 \). Suppose \( k = 2 \). Then this is the only condition. So \(-a + b)a_1 + \frac{1}{2}(a + b)(a + b - 1)m = 0 \). This is clearly acceptable for \( a + b \mid 4 \).

Suppose \( k > 2 \).

\[-(a + b)(a_1 + a_2 + b_{12}) - (a + b)(d_1 + d_2) + 3(a + b)(a + b - 1)m/2 = 6k^2 - 3k - 18.
\]

So \(-a + b)b_{12} - (a - b)(d_1 + d_2) + \frac{1}{2}(a + b)(a + b - 1)m = 2k^2 - 6k - 18 \). Also, \[-(a + b)(a_1 + a_2) - (a - b)d_1 + (a + b)(a + b - 1)m = 4k^2 - 2k - 12 \]. So \(-a - b)d_2 = 0 \).
Hence \(-(a + b)b_{12} + \frac{1}{2}(a + b)(a + b - 1)m = 2k^2 - k - 6.

Suppose \(k = 3\). Then this is the only condition. \(-a + b)a_1 + \frac{1}{2}a + b(a + b - 1)m = 9.\)
So \(a + b|6\) is acceptable.

Suppose \(k \geq 4\). \(-a + b)||(a + b_1 + b_2 + b_1 + 2b_2) + (2a + b)(a + b - 1) - (a + b)m = 8k^2 - 6k - 24\) or \((a + b)m = 2k - 2k - 6\). So
\[-(a + b)a_1 + (a + b - 1)k = 2k^2 - k - 6,\]
\[-(a + b)a_1 = 2k^2 - (a + b)k - 6 = 2(k^2 - 3) - (a + b).\]
Hence \((a + b)|2k^2 - 6\). Recall \((a + b)|2k\). Hence \((a + b)|2k^2\). So \((a + b)|6\). So \(a\mu_k \mathcal{H} b\mu^*\), \(k \geq 4\) is acceptable when \((a + b = 1)\), \((a + b = 2)\), \((a + b = 3 \text{ and } 3|a)\), \((a + b = 6 \text{ and } 3|a)\). This concludes the proof.

Remark. Suppose \(\lambda: G_1 \times G_2 \to SU(n)\) so that \(P_1(SU(n)/\mathrm{im} \lambda; Z) = 0\). Recall \(\lambda = \sum_{j=1}^t \phi_j \otimes \psi_j\) where \(\phi_1, \ldots, \phi_t, \psi_1, \ldots, \psi_t\) are either trivial or irreducible.

Let \(T_1\) be a maximal torus for \(G_1\), and \(T_2\) for \(G_2\). Let \(v_1, \ldots, v_{rk G_1}\) be a base for \(H^2(B_{T_1}; Z)\), and \(\overline{v}_1, \ldots, \overline{v}_{rk G_2}\) for \(H^2(B_{T_2}; Z)\). For some nonzero integer \(M\), we have
\[\sigma_2(Ad_{G_1}) + \sigma_2(Ad_{G_2}) = \sigma_2(Ad_{G_1} \times G_2) = M \cdot \sum_{j=1}^t \sigma_2(\phi_j \otimes \psi_j)\]
\[= M \cdot \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j) + (\dim \phi_j)\sigma_2(\psi_j)\]
\[= M \cdot \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j) + M \sum_{j=1}^t (\dim \phi_j)\sigma_2(\psi_j).\]

So as \(\sigma_2(Ad_{G_1})\) and \(\sigma_2(\phi_j)\) \((1 \leq j \leq t)\) are polynomials in \(v_1, \ldots, v_{rk G_1}\), and \(\sigma_2(Ad_{G_2})\) and \(\sigma_2(\psi_j)\) \((1 \leq j \leq t)\) are polynomials in \(\overline{v}_1, \ldots, \overline{v}_{rk G_2}\), we have
\[\sigma_2(Ad_{G_1}) = M \sum_{j=1}^t (\dim \psi_j)\sigma_2(\phi_j), \quad \text{and} \]
\[\sigma_2(Ad_{G_2}) = M \sum_{j=1}^t (\dim \phi_j)\sigma_2(\psi_j).\]

So as \(n(Ad_G) = 0\) if and only if \(G\) is a torus, we have the following.

**Lemma 2.7.** Let \(H\) be a compact connected subgroup of \(SU(n)\) so that \(P_1(SU(n)/H) = 0\). Then either \(H\) is a torus or \(H\) is semisimple.

So we may assume that \(G_1\) and \(G_2\) are simple. So if \(P_1(SU(n)/\mathrm{im} \lambda; Z) = 0 = P_2(SU(n)/\mathrm{im} \lambda; Z)\), then as before
\[ (2.8) \quad \sigma_4(\text{Ad}_{G_1}) + \sigma_2(\text{Ad}_{G_1})\sigma_1(\text{Ad}_{G_2}) + \sigma_4(\text{Ad}_{G_2}) = \sigma_4(\text{Ad}_{G_1 \times G_2}) \in \ker i^*. \]

**Theorem 2.9.** Suppose that SU(n) acts as a differentiable transformation group on a connected manifold \( M \), where \( p_1(M; Z) = 0 = p_2(M; Z) \). Suppose also that \( \dim M \leq n^2 - 8n/3 - 2 \). Let \( G \) be the connected component of the identity of "the" principal isotropy subgroup of the action. Then (possibly modulo a low-dimensional trivial representation), \( G \) together with its embedding appears on the following list.

\[
\begin{array}{c|c}
G & \lambda \\
--- & --- \\
SU(k), & k \geq 4 \quad \mu_k \\
 & k \geq 6 \quad a \mu_k \oplus b \mu_k^* \quad a + b = 2 \\
 & k \geq 9 \quad a \mu_k \oplus b \mu_k^* \quad a + b = 3, \ 3 \mid k \\
 & k \geq 18 \quad a \mu_k \oplus b \mu_k^* \quad a + b = 6, \ 3 \mid k \\
SO(2k + 1), & k \geq 2 \quad \rho_{2k+1} \\
 & k \geq 11 \quad 3\rho_{2k+1} \quad 3 \mid 2k - 1 \\
S_p(k), & k \geq 3 \quad \gamma_{2k} \\
 & k \geq 8 \quad 3\gamma_{2k} \quad 3 \mid k + 1 \\
SO(2k), & k \geq 4 \quad \rho_{2k} \\
 & k \geq 6 \quad 2\rho_{2k} \\
 & k \geq 10 \quad 3\rho_{2k} \quad 3 \mid k - 1 \\
 & k \geq 19 \quad 6\rho_{2k} \quad 3 \mid k - 1 \\
\end{array}
\]

**Proof.** Note that \( n^2 - 8n/3 - 2 \geq \dim M \geq \dim (SU(n) / \text{im} \lambda) = \dim SU(n) - \dim G = n^2 - 1 - \dim G \) and so \( \dim G > 8n/3 \); that is, the dimension of \( G \) as a Lie group is greater than \( 8/3 \) times the dimension of \( \lambda \) as a representation.

By Lemma 2.7, \( G \) is either a torus or \( G \) is semisimple. Suppose \( G \) is a torus, say \( G = T^k \). Then \( k = \dim G > 8n/3 \). This is impossible as \( \text{rk} SU(n) = n - 1 \). So \( G \) is semisimple.

We first suppose \( G = G_1 \times G_2 \) where \( G_1 \) and \( G_2 \) are compact simple Lie groups. Recall \( \lambda = \sum_{j=1}^l \phi_j \otimes \psi_j \). By 2.6 if for some \( j \) both \( \phi_j \) and \( \psi_j \) are non-trivial, then there exists a positive integer \( M \) such that \( n(\text{Ad}_{G_1}) \geq M \cdot \dim \phi_j \cdot n(\phi_j) \) and \( n(\text{Ad}_{G_2}) \geq M \cdot \dim \psi_j \cdot n(\psi_j) \). This turns out to be a severe restriction on \( \phi_j \) and \( \psi_j \). Using this fact together with 2.6 and 2.8 in the usual way, we arrive at the list for \( G_1 \times G_2 \). We then use this list together with the additional
requirement \( \dim G > 8n/3 \) to prove that \( G \) must be simple, and that, in fact, \((G,\lambda)\) lies on the above list.

3. The orbit types of \( SU(n) \) actions on manifolds with vanishing first and second integral Pontrjagin classes.

Theorem 3.1. Let \( SU(n) \) act as a differentiable transformation group on a connected manifold \( M \). Suppose that \( P_1(M; \mathbb{Z}) = 0 = P_2(M; \mathbb{Z}) \), and that \( \dim M \leq n^2 - 8n/3 - 2 \). Let \( y \in M \) be a regular element and let \( x \in M \). Then after conjugating \( G_y \) if necessary, the inclusion \( G_y^0 \subseteq G_x^0 \subseteq SU(n) \) appears on the following list:

1. If

\[
G_y^0 = \left( SU(k) \xrightarrow{a\mu_k \oplus b\mu_k^* \oplus (n-(a+b))\theta} SU(n) \right)
\]

where \( a + b = 1, 2, 3, 6 \) and \( 3|k \) if \( a + b = 3, 6 \), then

\[
G_y^0 \subseteq G_x^0 \subseteq SU(n) = \left( SU(k) \xrightarrow{\mu_k \oplus (l-k)\theta} SU(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-(a+b))\theta} SU(n) \right).
\]

2. If

\[
G_y^0 = \left( SO(2k+1) \xrightarrow{a\rho_{2k+1} \oplus (n-(a+2k+1))\theta} SU(n) \right),
\]

where \( a = 1, 3 \) and \( 3|(2k+1) \) if \( a = 3 \), then

\[
G_y^0 \subseteq G_x^0 \subseteq SU(n) = \left( SO(2k+1) \xrightarrow{\rho_{2k+1} \oplus (l-(2k+1))\theta} SO(2l+1) \xrightarrow{a\rho_{2k+1} \oplus (n-(a+2l+1))\theta} SU(n) \right).
\]

3. If

\[
G_y^0 = \left( S_p(k) \xrightarrow{\sigma \gamma_{2k} \oplus (n-2ak)\theta} SU(n) \right)
\]

where \( a = 1, 3 \) and \( 3|(k-1) \) if \( a = 3 \), then

\[
G_y^0 \subseteq G_x^0 \subseteq SU(n) = \left( S_p(k) \xrightarrow{\gamma_{2k} \oplus (2l-k)\theta} S_p(l) \xrightarrow{a\gamma_{2l} \oplus (n-2al)\theta} SU(n) \right).
\]

4. If

\[
G_y^0 = \left( SO(2k) \xrightarrow{a\rho_{2k} \oplus (n-2ak)\theta} SU(n) \right)
\]

where \( a = 1, 2, 3, 6 \) and \( 3|(k-1) \) if \( a = 3, 6 \), then

\[
G_y^0 \subseteq G_x^0 \subseteq SU(n) = \left( SO(2k) \xrightarrow{\rho_{2k} \oplus (n-2k)\theta} SO(2l) \xrightarrow{a\rho_{2l} \oplus (n-2al)\theta} SU(n) \right).
\]

Proof. As an example of the method, we shall give the proof for
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\((G^0_y) = \left[ \frac{\lambda \mu_k \oplus \beta \mu_k \oplus (n-2k)\theta}{\text{SU}(k) \oplus \text{SU}(k) \oplus \text{SU}(n)} \right] \)

where \(a + b = 2\), \(a \geq 1\), and \(k \geq 6\).

As \(y\) is regular, after conjugating \(G_x\) by a suitable element of \(\text{SU}(n)\), we may suppose \(G^0_y \subseteq G^0_x\). Hence we first must study the lattice of closed connected subgroups \(G^0_y \subseteq H \subseteq \text{SU}(n)\).

If \(V\) is a subspace of \(\mathbb{C}^n\), let \(\text{SU}(V)\) be the subgroup of \(\text{SU}(n)\) which leaves \(V^\perp\) fixed pointwise. Then in [5], the following theorem is proven.

**Theorem.** Let \(V\) be a subspace of \(\mathbb{C}^n\) with \(\dim V \geq 3\). Let \(H\) be a subgroup of \(\text{SU}(n)\) which contains \(\text{SU}(V)\). Then there exists a subspace \(V_1\) of \(\mathbb{C}^n\) with \(V \subseteq V_1\) so that \(\text{SU}(V_1)\) is a normal subgroup of \(H\).

We shall determine a similar result for \(G^0_y\). Suppose \(\phi: G^0_y \to H\), \(\psi: H \to \text{SU}(n)\), and \(\lambda = \phi \circ \psi\). We may take \(H = T^c \times U_1 \times \cdots \times U_d\) where \(T^c\) is a torus of rank \(c\), \(U_j\) is a compact, simply connected, simple Lie group, and where \(\phi\) and \(\psi\) have finite kernels. Then \(\phi = \phi_T \oplus \phi_1 \oplus \cdots \oplus \phi_d\). Clearly \(\phi_T\) is trivial as \(\text{SU}(k)\) is simple and \(T^c\) is abelian. So we may take \(V = U_1 \times \cdots \times U_d\), and \(\phi = \phi_1 \oplus \cdots \oplus \phi_d\). Now \(\psi = \sum_{i=1}^c \psi_{i_1} \oplus \cdots \oplus \psi_{i_d}\) where each \(\psi_{i_j}\) is either trivial or irreducible.

Let \((\psi \circ \phi)_j = (\psi_{i_1} \circ \phi_1) \oplus \cdots \oplus (\psi_{i_d} \circ \phi_d)\). Note that \((\psi \circ \phi)_j\) is a nontrivial representation for some \(j\).

For some \(1 \leq i \leq m\), \(\psi_i \circ \phi_i: \text{SU}(k) \to \text{SU}(n)\) is nontrivial. Note that the nontrivial components of \((\psi_i \circ \phi_i)_j\) have dimension \(\geq k^2\). Also if there exists \(1 \leq \alpha \leq m\) with \(i \neq \alpha\) and \(\psi_i \circ \phi_\alpha\) nontrivial, then the nontrivial components of \((\psi \circ \phi)_j\) have dimension \(\geq k^2\) which is greater than \(2k\).

So as \(\psi \circ \phi = a \mu_k \oplus b \mu_k \oplus (n-2k)\theta\) and \(a + b = 2\), it is clear that we may assume

1. \((\psi \circ \phi)_j\) is trivial if \(j > 2\).
2. If \(j = 1, 2\), then there exists at most one integer \(i_j\) with \(1 \leq i_j \leq m\) so that \(\psi_i \circ \phi_j\) is nontrivial.

So \(G_y\) is contained in a normal subgroup \(V\) of \(H\), where \(V = U_1 \times U_2\).

Suppose \(V = \text{SU}(l)\). Then we may write \(\phi: \text{SU}(k) \to \text{SU}(l), \psi: \text{SU}(l) \to \text{SU}(n)\), and \(\psi \circ \phi = \lambda = a \mu_k \oplus b \mu_k \oplus (n-2k)\theta\), where \(a + b = 2\). So if we write \(\psi = \psi_1 \oplus \cdots \oplus \psi_t \oplus \lambda : \text{SU}(l) \to \text{SU}(n)\), where \(\psi_j\) is irreducible for \(1 \leq j \leq t\), then \((\psi_1 \circ \phi) \oplus \cdots \oplus (\psi_t \circ \phi) \oplus \lambda = \psi \circ \phi = a \mu_k \oplus b \mu_k \oplus (n-2k)\theta\).

Hence \(\psi = \psi_1 \oplus \lambda\) or \(\psi = \psi_1 \oplus \psi_2 \oplus \lambda\).

Now \(l^2 - 1 \geq k^2 - 1 > 8n/3\). Hence \(\psi_1\) and \(\psi_2\) have drastic limitations on their dimensions. Using the fact that \(\phi > \psi\) implies \(\dim \phi > \dim \psi\), it follows
that the only possible irreducible components of \( \psi \) are \( \mu_1 \) and \( \mu_1^* \). Arguing similarly for \( \phi \), we see that the only possible irreducible components of \( \phi \) are \( \mu_k \) and \( \mu_k^* \).

Hence, it is clear that one of the following cases holds:

1. \( \phi = a\mu_k \oplus b\mu_k^* \oplus (l-2k)\theta, \psi = \mu_1 \oplus (n-1)\theta \).
2. \( \phi = b\mu_k \oplus a\mu_k^* \oplus (l-2k), \psi = \mu_1^* \oplus (n-1)\theta \).
3. \( \phi = \mu_k \oplus (l-k)\theta, \psi = a\mu_1 \oplus b\mu_1^* \oplus (n-2l)\theta \).
4. \( \phi = \mu_1^* \oplus (l-k)\theta, \psi = b\mu_1 \oplus a\mu_1^* \oplus (n-2l)\theta \).

Arguing similarly for the case when \( V \) is some other simple compact Lie group (and using the fact that \( \dim V > k^2 - 1 > 8n/3 \)), we get the following additional possibilities, where \( \phi : SU(k) \to V \), \( \psi : V \to SU(n) \), and \( \psi \circ \phi = \lambda \).

\[
\begin{align*}
V & \quad \phi \quad \psi \\
Sp(l) (a = 1 = b) & \quad \mu_k \oplus \mu_k^* \oplus 2(l-k)\theta \quad \gamma_{2l} \oplus (n-2l)\theta \\
SO(l) (a = 1 = b) & \quad \mu_k \oplus \mu_k^* \oplus (l-2k)\theta \quad \rho_l \oplus (n-l)\theta
\end{align*}
\]

Now suppose \( V = U_1 \times U_2 \). Then using arguments similar to the above, one shows that \( V = SU(1) \times SU(1) \), and that we can write \( \phi = \phi_1 \oplus \phi_2 \) and \( \psi = (\psi_1 \otimes \theta) \oplus (\theta \otimes \psi_2) \) where modulo trivial copies,

\[
\begin{align*}
\phi_1 &= \mu_k^*, \mu_k^* \\
\phi_2 &= \mu_k, \mu_k^* \\
\psi_1 &= \mu_1^*, \mu_1^* \\
\psi_2 &= \mu_1, \mu_1^*.
\end{align*}
\]

We now specialize to the case \( H = G_x^0 \).

We shall show that \( H = G_x^0 \) implies \( V = SU(l) \), and that in fact

\[
(G_y^0 \subseteq V \subseteq SU(n)) = \left( SU(k) \rightarrow SU(l) \rightarrow SU(n) \right) \left( \mu_k \oplus (l-k)\theta \rightarrow a\mu_1 \oplus b\mu_1^* \oplus (n-2k)\theta \right).
\]

Let \( \xi_1 = (G_x \rightarrow G \rightarrow G/G_x) \). Then for \( j = 1, 2 \),

\[0 = P_j(r(G/G_x) \oplus \nu(G/G_x)) = P_j(\alpha_{\xi_1}^\phi |G_x - \text{Ad}_{G_x}) \oplus \alpha_{\xi_1}^\phi(\phi_x)).\]

Now \( V \) is a connected normal subgroup of \( G_x^0 \) so that

\[
G_x^0 \phi \subseteq G_x \subseteq SU(n) = G
\]

i.e. \( \psi \circ \phi = \lambda \).

Let \( \pi : G/V \to G/G_x \) be the natural map.

Let \( \xi_2 = (V \psi \to G \rightarrow G/V) \). Then if \( \Gamma \) is a real representation of \( G_x \),

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\[ \pi^1(\alpha_{G_1}(\Gamma)) = \alpha_{G_2}(\Gamma|V). \] So

\[ \pi^1(\alpha_{\xi_1}(Ad G_x - Ad G_x) \otimes \alpha_{\xi_2}(\phi_x)) = \alpha_{\xi_2}(Ad G_x - Ad G_x) \otimes \alpha_{\xi_2}(\phi_x | V) \]

\[ = \alpha_{\xi_2}(Ad G_x - Ad G_x - \theta) \otimes \alpha_{\xi_2}(\phi_x | V) \]

\[ = r(G/V) \otimes \alpha_{\xi_2}(\phi_x | V) \in K_0(G/V), \]

where \( \theta \) is some trivial representation of \( V \). So modulo torsion,

\[ P_1(G/V) + P_1(\alpha_{\xi_2}(\phi_x | V)) = 0, \]

and

\[ P_2(G/V) + P_2(G/V) + P_2(\alpha_{\xi_2}(\phi_x | V)) + P_2(\alpha_{\xi_2}(\phi_x | V)) = 0. \]

Recall that as \( G_x \) is the principal isotropy subgroup of \( \phi_x, \phi_x|G_x = Ad G_x|G_x = Ad_{G_x} \otimes \text{trivial copies} \). So

\[ <\phi IG_x = Ad IG_x = Ad IG_x = Ad IG_x \otimes \text{trivial copies} \]

\[ = ((Ad G_x|\phi_x - Ad G_x|\phi_x)|\phi_x) \otimes \text{trivial copies} \]

\[ = Ad IG_x = Ad IG_x = Ad IG_x \otimes \text{trivial copies}. \]

Note also that \( \dim(\text{tsu}(n)/G_x \otimes \phi_x) = \dim M \leq n^2 - 8n/3 - 2 \), and hence \( \dim \phi_x \leq n^2 - 8n/3 - 2 \).

We shall now examine the several possibilities for \( V, \phi, \) and \( \psi \).

Suppose \( V = SU(l) \), and \( \phi = a\mu_k \otimes b\mu_k^* \otimes (l - 2k)\theta, \psi = \mu_k \otimes (n - l)\theta \). Then

\[ \phi_x|SU(k) = \phi_x \circ \phi = (\mu_k \otimes \mu_l^* \otimes \theta) \circ \phi - \mu_k \otimes \mu_k^* \otimes \text{trivial copies} \]

\[ = ab(\mu_k \otimes \mu_k^* \otimes \mu_k^* \otimes (a^2 + b^2 - 1)\mu_k \otimes \mu_k^* \]

\[ \otimes (a + b)(l - (a + b)k)(\mu_k \otimes \mu_k^*) \otimes \text{trivial copies}. \]

Recall \( k^2 - 1 > 8n/3 \), and \( l \geq 2k \). So \( \dim \phi_x < n^2 < 9k^4/64 < 9l^4/1024 \).

This is a severe limitation on \( \phi_x \). By listing all the irreducible representations \( \Gamma \) of \( SU(l) \) with \( \dim \Gamma < 9l^4/1024 \) and then using our determination of \( \phi_x \circ \phi \), we see easily that \( \phi_x|SU(l) \) can contain copies only of \( \mu_l, \mu_l^*, \mu_l \otimes \mu_l, \mu_l \otimes \mu_l^*, \mu_l^* \otimes \mu_l, \mu_l^* \otimes \mu_l^* \). Hence
\[ \phi_x |SU(l) = c(\mu_1 \otimes \mu_1 \otimes \mu_1^* \otimes \mu_1^*) \otimes d \mu_1 \otimes \mu_1^* \otimes e(\mu_1 \otimes \mu_1^*) \oplus F_\theta, \]

and so \( \phi_x |SU(k) = \phi_x \circ \phi = (4abc + (a^2 + b^2) d) \mu_k \otimes \mu_k^* \oplus \) a sum of representations not containing \( \mu_k \otimes \mu_k^* \). So \( 4abc + (a^2 + b^2)d = a^2 + b^2 - 1 \). This is impossible.

In the other cases for \( V, \phi, \psi \), we argue similarly to the above. In some cases there will exist a representation \( \phi_x |V \) so that \( \phi_x \circ \phi = \lambda \). In these cases, by using \( P_1(G/V) + P_1(a_x \phi_x |V) = 0 \), we arrive at the conclusion that \( V = SU(l) \), \( \phi = \mu_k \otimes (l - k) \theta \), and \( \psi = a \mu_l \oplus b \mu_l^* \), i.e.

\[
(G_0 \subseteq V \subseteq SU(n)) = \left( \begin{array}{c}
(SU(k) \mu_k \otimes (l - k) \theta) \\
(SU(l) a \mu_l \otimes b \mu_l^* \otimes (n - 2k) \theta) \\
(SU(n))
\end{array} \right)
\]

as was to be shown.

Finally we shall show \( G_x^0 = V \).

So suppose \( G_x^0 \neq V \). We shall derive a contradiction. We give here a modification of an argument found in [4].

We may suppose \( x \) was chosen so that \( \dim G^0_x \) is minimal among these \( G_t \) with \( G_t^0 \) not simple. We see that after conjugating if necessary, we have \( y \in S_x \), the slice at \( x \). Let \( \phi_x^t \) be the slice representation of \( G_x^0 \) at \( x \). Then \( \phi_x^t |G_x^0 = \phi_1 \otimes \beta \theta \), where \( \phi_1 \) is a nontrivial representation without trivial copies.

We may write \( y = \bar{y} + n \) where \( \bar{y} \in W_\phi_1 \), the representation space of \( \phi_1 \), and where \( n \in (W_\phi_1)^L \) i.e. \( n \in W_\beta \theta^\perp \).

Clearly \( G_x^0 = G_2^0 \) and so \( \bar{y} \) is a regular element.

Let \( \bar{z} \) be an element of the unit ball of \( W_\phi_1 \subseteq S_x \subseteq M \). To show: \( (G_0^0(\phi_1)_{\bar{z}})^0 \subseteq V \). By the minimality condition

\[
(G^0_x) = \left( \begin{array}{c}
(SU(l') a \mu_l \otimes b \mu_l^* \otimes (n - 2l') \theta) \\
(SU(n))
\end{array} \right).
\]

So as \( G_x^0 \subseteq G_x^0 \subseteq N_{SU(n)}(v) \), clearly \( G_x^0 \subseteq V \). Hence \( (G_0^0(\phi_1)_{\bar{z}})^0 = G_x^0 \subseteq V \). So we may apply the following proposition found in [4].

Proposition. Let \( H \) be a closed, connected, proper normal subgroup of a compact, connected, Lie group \( K \).

Let \( \psi : K \rightarrow O(m) \) be a representation, such that for each \( x \in W_\psi - \{0\} \), \( K_x^0 \subseteq H \). Then

1. \( H \subseteq \ker \psi \).
2. \( \rank K/H = 1 \).
Hence $V \subseteq \ker \phi_1 = \ker \phi_x \mid G^0_x$ and rank $G^0_x/V = 1$. So $\phi_x \mid G^0_y$ is trivial. But note that $\phi_x \circ \phi = (I - k)(\mu_k \otimes \mu_k^*) \otimes$ trivial copies, as

$$\phi_x \mid G^0_y = (\phi_x \mid G_y)G^0_y = (\text{Ad}_{G_x} \mid G_y - \text{Ad}_{G_y})G^0_y$$

$$= (\text{Ad}_{G_x} \mid V)G^0_y - \text{Ad}_{G_y} \mid G^0_y$$

$$= \text{Ad}_V \mid G^0_y - \text{Ad}_{G_0} \otimes \theta.$$

Hence $l = k$, and so $V = G^0_y$. So $G^0_y \sim G^0_x \times L$, where either $L = \text{SU}(2)$ or $L = T^1$.

Note that $\phi_x \mid G^0_x = \theta \otimes y$. Let $\{a_j\}$ be the standard base for "the" Cartan subalgebra of $L$. Suppose

$$(G^0_x \rightarrow \text{SU}(n)) = \left(\text{SU}(k) \times L \xrightarrow{(a \mu_k \otimes b \mu_k^* \otimes \theta \otimes \theta) \otimes (\theta \otimes \delta)} \text{SU}(n)\right).$$

Then $\dim \Gamma = 1$. So $\Gamma = \emptyset$ or $L = T^1$. Suppose $\Gamma = \emptyset$. Let $\xi_3 = (G^0_x \rightarrow \text{SU}(n) \rightarrow \text{SU}(n)/G^0_x)$. Then

$$P_1(\alpha_3 \mid \text{SU}(n)/G^0_x \otimes \phi_x \mid G^0_x) = 0.$$ 

So

$$M(2\sigma_2(\mu_k) + \sigma_2(\delta)) = \sigma_2(\phi_x \mid G^0_x)$$

$$= \sigma_2((a \mu_k \otimes b \mu_k^* \otimes \theta) \otimes (\theta \otimes \delta))$$

$$\otimes ((a \mu_k^* \otimes b \mu_k \otimes \theta) \otimes (\theta \otimes \delta^*)) - \mu_k \otimes \mu_k^* - \text{Ad}_L) + \sigma_2(\gamma)$$

$$= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + (4k + 2t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\text{Ad}_L) + \sigma_2(\gamma).$$

Hence $2M = 6k + 4t + 4 \dim \delta$. So

$$(3k + 2t + 2 \dim \delta)\sigma_2(\delta) = (6k + 4t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\text{Ad}_L) + \sigma_2(\gamma)$$

or

$$-k\sigma_2(\delta) = \sigma_2(\gamma) - \sigma_2(\text{Ad}_L).$$

So as $\sigma_2(\delta)$, $\sigma_2(\gamma)$, $\sigma_2(\text{Ad}_L)$ are all polynomials in $\tau(a_1)$ with negative coefficients, we see $L = \text{SU}(2)$. So $\sigma_2(\text{Ad}_{\text{SU}(2)}) = \sigma_2(\gamma) + k\sigma_2(\delta)$. An easy check of $R_0(\text{SU}(2))$ shows this is impossible.

So $\Gamma \neq \emptyset$. Hence $L = T^1$. By considering $L \subseteq \text{SU}(n)$, we see $(2k + t)\Gamma \otimes \delta$: 
$L \rightarrow \text{SU}(n)$. We easily have
\[
M(2\sigma_2(\mu_k) + \sigma_2((2k + t)\Gamma \ominus \delta)) = M\sigma_2((a\mu_k \otimes b\mu_k \otimes t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta)) = 
\]
\[
= \sigma_2((a\mu_k \otimes b\mu_k \otimes t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta)) - \mu_k \otimes \mu_k^* + \sigma_2(y)
\]
\[
= \sigma_2((a^2 + b^2 - 1)\mu_k \otimes \mu_k^* \otimes ab(\mu_k \otimes \mu_k \otimes \mu_k^* \otimes \mu_k^*) \oplus (a + b)t(\mu_k \otimes \mu_k^*)
\]
\[
= (2k((a + b)^2 - 1) + 2(a + b)t + 2(a + b) \dim \delta)\sigma_2(\mu_k)
\]
\[
+ \sigma_2((2k + t)(\Gamma \otimes \delta^* \otimes \Gamma^* \otimes \delta) \oplus \theta \otimes (\delta \otimes \delta^*)) + \sigma_2(y)
\]
\[
= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + \sigma_2((2k + t)\Gamma \otimes \delta) \oplus ((2k + t)\Gamma^* \otimes \delta^*))
\]
\[
= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + (4k + 2t + 2 \dim \delta)\sigma_2((2k + t)\Gamma \otimes \delta) + \sigma_2(y).
\]
So $2M = 6k + 4t + 4 \dim \delta$, or $M = 3k + 2t + 2 \dim \delta$. Hence
\[
(3k + 2t + 2 \dim \delta)\sigma_2((2k + t)\Gamma \otimes \delta) = (4k + 2t + 2 \dim \delta)\sigma_2((2k + t)\Gamma \otimes \delta) + \sigma_2(y).
\]
So $0 = k\sigma_2((2k + t)\Gamma \otimes \delta) + \sigma_2(y)$. But as $\phi_x|G_0^0 = \theta \otimes \gamma$, we see that $\gamma \otimes C : L \rightarrow \text{SU}(n)$, and hence $k\sigma_2((2k + t)\Gamma \otimes \delta)$ and $\sigma_2(y)$ are polynomials in $\tau(a_1)$ with negative coefficients. This is impossible.

The other cases follow similarly.

REFERENCES


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