THE ORIENTED BORDISM OF $Z_{2^k}$ ACTIONS

BY

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ABSTRACT. Let $R_2$ be the subring of the rationals given by $R_2 = \mathbb{Z}[\frac{1}{2}]$. It is shown that for $G = Z_{2^k}$ the bordism group of orientation preserving $G$ actions on oriented manifolds tensored with $R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators (where $\Omega_*$ is the oriented bordism ring).

1. Introduction. Let $G$ be a group. Let $\Omega^G_*$ denote the bordism group of differentiable orientation preserving $G$ actions on closed oriented manifolds. In R. E. Stong's paper [10] $\Omega^G_*$ is understood for $G$ a $p$-group and $p$ an odd prime. In [9] H. L. Rosenzweig shows that $\Omega^{Z_2} \otimes \mathbb{Q} = 0$ if $* \neq 4k$. In this paper the module structure of $\Omega^G_*$ is determined up to 2-torsion for $G = Z_{2^k}$.

§§ 2, 3, and 4 are largely preliminary material. In §5 it is shown that $\Omega^{Z_{2^k}} \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators (where $R_n = \{a/b \mid a \text{ is an integer and } b \text{ is a power of } n\}$ is a subring of the rationals).

This paper discusses part of the research undertaken while I was a Ph. D. candidate at the University of Virginia. I would like to express my appreciation to my advisor, R. E. Stong, who directed this research in a most generous way.

2. Equivariant bordism. For a finite abelian group $G$ a family $F''$ of subgroups of $G$ is a collection of subgroups of $G$ such that if $H \in F''$ and $K < H$, then $K \in F''$. If $(M, \sigma)$ is a manifold with $G$ action, then $(M, \sigma)$ is $F''$-free if for each $x \in M$, the isotropy subgroup of $x$ is an element of $F''$.

Let $F' \subset F''$ be families of subgroups of $G$. Let $(X, A)$ be a space pair with $G$ action. Consider 5-tuples $(M, M_0, M_1, \sigma, f)$ where

1. $M, M_0, M_1$ are compact differentiable oriented manifolds with $n$ the dimension of $M$.
2. $\partial M = M_0 \cup M_1$, $\partial M_0 = \partial M_1 = M_0 \cap M_1$.
3. $\sigma: G \times M \to M$ is a differentiable $G$-action which preserves $M_0$ and $M_1$ and which preserves the orientation on $M$.
4. $(M, \sigma)$ is $F''$-free while $(M_0, \sigma/G \times M_0)$ is $F'$-free.
5. $f: (M, M_1) \to (X, A)$ is an equivariant map.

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Under the usual equivariant bordism relation (see [10, §2]) one forms a set of equivalence classes of such 5-tuples, denoted $\Omega^G_n(F'', F')(X, A)$, with an abelian group structure induced by disjoint union. The graded sum of these groups has an $\Omega_*$ module structure induced by cartesian product and is denoted by $\Omega^G_*(F'', F')(X, A)$.

Now if $h: (X, A) \to (Y, B)$ is an equivariant map between spaces with $G$ action, one has an induced homomorphism $h_*: \Omega^G_*(F'', F')(X, A) \to \Omega^G_*(F'', F')(Y, B)$ sending $[M, M_0, M_1, \sigma, f]$ into $[M, M_0, M_1, \sigma, h \circ f]$. Let $\emptyset$ denote the empty set. Then there is a degree $-1$ boundary map $\partial_*: \Omega^G_*(F'', F')(X, A) \to \Omega^G_*(F'', F')(A, \emptyset) \equiv \Omega^G_*(F'', F')(A)$ sending $[M, M_0, M_1, \sigma, f]$ into $[M_0, 0, \partial M_0, \sigma/G \times M_1, f/M_1]$. From [10, Proposition 2.1], $\Omega^G_*(F'', F')(\_)$ and $\partial_*$ define an equivariant homology theory on the category of $G$-pairs to the category of $\Omega_*$ modules. Specifically this theory satisfies equivariant homotopy, excision, and exactness axioms.

From [10, Proposition 2.2], one knows that for families of subgroups $F' \subset F''$ there is an exact triangle

$$
\Omega^G_*(F')(X, A) \xrightarrow{\alpha_*} \Omega^G_*(F'')(X, A) \xrightarrow{\beta_*} \Omega^G_{*+1}(F', F')(X, A)
$$

in which $\alpha_*$ and $\beta_*$, respectively, forget $F'$ and $F''$-freeness while $\partial_*$ sends $[M, M_0, M_1, \sigma, f]$ into $[M_0, \emptyset, \partial M_0, \sigma/G \times M_1, f/M_1]$.

**Note.** If $G$ is an abelian group and $H \subset G$, the collection of all subgroups of $H$ is a family of subgroups of $G$. If $H \subset G$, this family is denoted by $F_H$. In particular $F_e$ denotes the family consisting of the identity subgroup. Let $F$ denote the family of all subgroups of $G$.

3. Classifying spaces for bundles with $G$ action. Let $G$ be a finite abelian group with exactly $r$ distinct irreducible complex representations. Let $C^\infty = C_1^\infty \oplus C_2^\infty \oplus \cdots \oplus C_r^\infty$. Define a $G$ action on $C^\infty$ by considering $C_i^\infty$ as a countable direct sum of the $i$th irreducible representation. Now let $BU_s$ be the Grassmannian of complex $s$-planes in $C^\infty$ and $\gamma_s$ be the universal complex $s$-plane bundle over $BU_s$. Since the elements of $G$ act on $C^\infty$ via complex linear transformations, there is an induced $G$ action on $BU_s$ and $\gamma_s$ (see [10, §3]). One learns from Atiyah [2, §1.6] that $\gamma_s \to BU_s$ is the universal complex $n$-plane bundle in the category of $G$-spaces.

One can perform essentially the same construction in the real case by taking the Grassmannian of real $n$-planes in $C^\infty$. In this way one gets $BO_s$ together with its canonical bundle $\gamma_s$, the universal real $s$-plane bundle in the category of $G$-spaces. Note that in what follows these $G$-spaces are called $BU_s$ and $BO_s$.
except in cases where the context does not make the meaning clear. In these cases the notation \((BU_s, G)\) and \((BO_s, G)\) is used.

In the process of defining \(BU_s\) and \(BO_s\) together with their canonical bundles one may place a metric on the \(\gamma_s\) such that the \(G\) action is orthogonal with respect to this metric. Further, for any \(G\)-bundle \(E \to X\) of dimension \(s\) over a compact Hausdorff space \(X\), one may assume there is a metric on \(E\) such that

(a) the \(G\) action on \(E\) is orthogonal with respect to this metric,
(b) the bundle map covering the classifying map takes

\[(D(E), S(E)) \to (D(\gamma_s), S(\gamma_s))\]

where \(D(-)\) denotes the unit disc bundle and \(S(-)\) denotes the unit sphere bundle.

Now consider the \(G\)-spaces \(BO_s\) and \(BU_s\) and the fixed sets of subgroups of \(G\) acting on \(BO_s\) and \(BU_s\). Let \(H < G\) and \(X\) be a compact Hausdorff \(G\)-space. The isomorphism classes of \(G\)-bundles over \(X\) of real dimension \(s\), \(\text{vect}_s^G(X)\), are in 1-1 correspondence with the \(G\)-homotopy classes of equivariant maps from \(X\) into \([X, BO_s]_G\). Now if \(H < G\) fixes \(X\), any equivariant map \(X \to BO_s\) goes into the fixed set of \(H\) acting on \(BO_s, F_H(BO_s)\). Hence if \(H\) fixes \(X, \text{vect}^G_s(X) \leftrightarrow [X, F_H(BO_s)]_G\). It follows that \(F_H(BO_s)\) is the classifying space of \(G\) bundles of dimension \(s\) over base spaces \(X\) such that \(H\) fixes \(X\). Exactly the same analysis is true for complex \(s\)-bundles over \(X\) and \(F_H(BU_s)\).

Further, if \(E \to X\) is a complex \(G\) bundle and \(H < G\) fixes \(X, E\) splits into \(G\) subbundles according to the nontrivial irreducible complex representations of \(H\) [2, §1.6]. The classifying space for \(G\)-bundles over a base which \(H\) fixes can be understood in terms of these subbundles. Using this information one can compute explicitly the fixed sets \(F_H(BU_s)\). Using similar techniques one can understand \(F_H(BO_s)\). In particular, for the purposes of this paper one records the following computations.

**Proposition 3.1.** If \(H < G\) with \(d = \text{the order of } H\), then \(F_H(BU_s, G)\) is \(G\) homotopy equivalent to \(\bigcup BU_{t_1} \times \cdots \times BU_{t_d}\) where \(\Sigma t_i = s\). \(\square\)

Since the real irreducible representations of \(Z_2\) are multiplication by \(+1\) and by \(-1\) on one-dimensional vector spaces, a \(Z_{2k}\) bundle \(E\) over a \(Z_{2k}\) space which is fixed by \(Z_2\) decomposes into \(E_1 \oplus E_{-1}\) where \(Z_2\) acts in the fibers of \(E_i\) by multiplication by \(i\). Thus the classifying space for \(s\)-dimensional real vector bundles over \(Z_{2k}\) spaces fixed by \(Z_2\) is \(\bigcup BO_{t_1} \times BO_{t_{-1}}\) where \(t_{-1} + t_1 = s\). Thus
Proposition 3.2. \( F_{Z_2}(BO_s, Z_{2k}) \) is \( Z_{2k} \) homotopy equivalent to \( \bigcup BO_{t_1} \times BO_{t_1} \). □

It is evident that the component of \( F_{Z_2}(BO_s, Z_{2k}) \) above which \( Z_2 \) acts as \(-1\) in the fibers of the canonical bundle is a \( BO_s \). Denote this component by \( F_{Z_2}^{-}(BO_s, Z_{2k}) \). The \( Z_{2k} \) action restricted to \( F_{Z_2}(BO_s, Z_{2k}) \) can be considered a \( Z_{2k-1} \) action. If \( k > 1 \) it is necessary to know the fixed set of \( Z_{2j} < Z_{2k-1} \) acting on \( F_{Z_2}^{-}(BO_s, Z_{2k}) \). \( F_{Z_2}^{-}(BO_s, Z_{2k}) \) is the classifying space for \( Z_{2k} \) bundles \( E \to X \) which have the properties

(a) \( Z_{2j+1} < Z_{2k} \) fixes \( X \).

(b) \( Z_2 < Z_{2j+1} < Z_{2k} \) acts on the fibers of \( E \) as multiplication by \(-1\).

For such a bundle \( E \) splits into subbundles with respect to the irreducible representations of \( Z_{2j+1} \). Since each irreducible representation of \( Z_{2j+1} \) which satisfies (b) is the realification of an irreducible complex representation, each of the subbundles of \( E \) has a complex structure. Thus if there are \( r \) irreducible real representations of \( Z_{2j+1} \) satisfying (b) one has

Proposition 3.3. \( F_{Z_2}^{-}(F_{Z_2}(BO_s, Z_{2k})) \) is \( Z_{2k-1} \) homotopy equivalent to \( \bigcup BU_{t_1} \times \cdots \times BU_{t_r} \) with \( \sum t_i = s \). □

4. A special case of equivariant transverse regularity. Let \( \gamma_{2s} \) represent the canonical \( 2s \) plane bundle over \( F_{Z_2}^{-}(BO_{2s}, Z_{2k}) \). (Note that \( (BO_{2s}, Z_{2k-1}) \) is \( Z_{2k-1} \) homotopy equivalent to \( F_{Z_2}^{-}(BO_{2s}, Z_{2k}) \).) Since \( Z_2 < Z_{2k} \) acts by \(-1\) in the fibers of \( \gamma_{2s} \) and since the determinant of \(-1\) acting on an even dimensional vector space is \(+1\), the \( Z_2 \) action dies when one takes the determinant bundle of \( \gamma_{2s} \) together with its induced action. In other words, \( \det \gamma_{2s} \to F_{Z_2}^{-}(BO_{2s}, Z_{2k}) \) is a \( Z_{2k-1} \) bundle.

Proposition 4.1. If

\[ f: (M, \partial M, Z_{2k-1} \text{ action}) \to (D(\det \gamma_{2s}), S(\det \gamma_{2s}), \det(Z_{2k} \text{ action})) \]

is an equivariant map, then \( f \) may be equivariantly homotoped to be transverse regular on the zero section of \( \det \gamma_{2s} \). Further, if \( A \subset M \) is a closed subspace and if \( f/A \) is already transverse regular, the homotopy can be chosen to fix \( A \).

Proof. One needs only to check that the hypotheses for Lemma 4.2 in [10] are satisfied. Therefore one looks at the fixed set of \( Z_{2j} < Z_{2k-1} \) acting on \( F_{Z_2}^{-}(BO_{2s}, Z_{2k}) \) for all \( 1 \leq j \leq k-1 \), and one checks that if \( x \in BO_{2s} \) is fixed by \( Z_{2j} \), then \( v \) is fixed by \( Z_{2j} \) for all \( v \in \det \gamma_{2s}/x \).

If \( T \) is the generator of \( Z_{2j+1} \) acting on \( \gamma_{2s} \), \( T \) acts as a real linear transformation on \( \gamma_{2s}/x \) such that \( T^{2j} \) acts as multiplication by \(-1\). Further, the minimum polynomial of \( T, m_T, \) must divide \( y^{2j+1} - 1 = (y - 1) \cdot (y + 1) \cdot q_1(y) \cdot \cdots \cdot q_{2j-1}(y) \) where \( q_i(y) \) is an irreducible quadratic of the form
\[ y^2 + ay + 1. y - 1 \] does not divide \( m_T \) since this would imply that \( T \) is multiplication by 1 on some one-dimensional subspace of \( \gamma_{2s}/x \). Elementary linear algebra then yields that \( \det T = +1 \) which implies that \( \det T \) fixes pointwise the fiber \( \det \gamma_{2s}/x \). \[ \square \]

5. The oriented bordism of \( \mathbb{Z}_k \). For a group \( G \), denote by \( \Omega_*^G \) the equivariant bordism module \( \Omega_*^G(F\{pt\}) \). In this section \( \Omega_*^{\mathbb{Z}_k}(\mathbb{R}^2) \) is computed. Let \((X, A)\) be a c.w. pair with \( \mathbb{Z}_k \) action having the property that \( F_{\mathbb{Z}_k}(X, A) \) is a c.w. pair for \( 0 \leq j \leq k \) where \( F_{\mathbb{Z}_k}(X, A) \) is the fixed set of \( \mathbb{Z}_k \) acting on \((X, A)\). For a bundle \( E \) with unit disc, \( D(E) \), and unit sphere, \( S(E) \), one denotes by \( T(E) \) the space \( D(E)/S(E) \), the Thom space of \( E \). The primary tool of this paper is the following theorem.

**Theorem 5.1.** \( \Omega_*^{\mathbb{Z}_k}(F, E)(X, A) \) is isomorphic to

\[
\bigoplus_{s=0}^{\lfloor s/2 \rfloor} \Omega_*^{\mathbb{Z}_k-1}(F)(F_{\mathbb{Z}_k}(X)/F_{\mathbb{Z}_k}(A)) \land T(\det \gamma_{2s})
\]

where \( \gamma_{2s} \) is the canonical \( 2s \) plane bundle over \( F_{\mathbb{Z}_k}(BO_{2s}, \mathbb{Z}_k) \).

**Proof.** Let \( [M, M_0, M_1, T, f] \in \Omega_*^{\mathbb{Z}_k}(F, E)(X, A) \) where \( T \) generates the \( \mathbb{Z}_k \) action on \( M \). Let \( F_2 \) be the \((n-s)\)-dimensional component of the fixed set of \( \mathbb{Z}_2 < \mathbb{Z}_k \) acting on \( M \). Then \( F_2 \) is a submanifold of \( M \) with an induced action of \( \mathbb{Z}_k \) which is covered in the normal bundle to \( F_2 \) in \( M, v \), by an action of \( \mathbb{Z}_k \). Further, \( \partial F_2 = F_2 \cap M_1 \). Since one may identify the disc of the normal bundle equivariantly with a small tubular neighborhood of \( F_2 \), one knows that no elements of the disc of the normal bundle \(-\{\text{zero section}\}\) can be fixed by \( \mathbb{Z}_2 \) for \( 1 \leq j \leq k \). Since each fiber of \( v \) is a representation space for \( \mathbb{Z}_2 \), \( v \) is a \( \mathbb{Z}_k \) bundle over \( F_2 \) such that \( \mathbb{Z}_2 \) acts as \(-1\) in the fibers. One then knows that \( v \to F_2 \) is classified equivariantly into \( F_{\mathbb{Z}_2}(BO_{2s}, \mathbb{Z}_k) \) yielding a \( \mathbb{Z}_k \) bundle map

\[
\begin{array}{ccc}
\nu & \xrightarrow{g'} & \gamma_{2s} \\
\pi \downarrow & & \downarrow \\
F_2 & \xrightarrow{g} & BO_{2s}
\end{array}
\]

By taking the determinant bundles of \( \nu \) and \( \gamma_{2s} \) one gets a similar diagram of \( \mathbb{Z}_k \) bundle maps.

One may assume that \( \det g' \) maps the \((D, S)\) pair of \( \det \nu \) into the \((D, S)\) pair of \( \gamma_{2s} \). Letting \( \tilde{\pi} : \det \nu \to F_2 \) be the projection, and crossing \( \det g' \) with \( f \circ \tilde{\pi} \), one gets a map from the pair

\[
(D(\det \nu), D(\det \nu/\partial F_2) \cup S(\det \nu))
\]

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Since the first Stiefel-Whitney classes of $\nu$ and the tangent bundle of $F_2$, $\tau(F_2)$, are equal, $D(\det \nu)$ is an oriented manifold. Let $T'$ generate the $Z_{2k-1}$ action on $D(\det \nu)$. One notes that $T'$ is orientation preserving if $\det(dT') = T' \times 1$ on $\det \tau(D(\det \nu))$ [6, Lemma 3]. Since $\det dT' = \pi^*(\det d\tau)$ on $\pi^*(\det \tau(M)/F_2) \cong \det \tau(D(\det \nu))$ it follows that $T'$ is orientation preserving since $T$ is orientation preserving. Thus by summing over the discs of the determinant bundles of all possible components of the fixed set of $Z_2$, one may define a map

$$F: \Omega^*_{2k}(F, F_e)(X, A) \rightarrow \bigoplus_{s=0}^{Z_{2k-1}} \Omega^*_{2s+1}(F,F_Z_2(X, A) \times (\det \gamma_{2s}), S(\det \gamma_{2s})).$$

In order to define an inverse to $F$ consider

$$[N, \partial N, S, h] \in \Omega^*_{2k-1}(F,F_Z_2(X, A) \times (\det \gamma_{2s}), S(\det \gamma_{2s})).$$

One has $N \overset{p2 \circ h}{\Rightarrow} D(\det \gamma_{2s})$ and $p2 \circ h$ is an equivariant map which, by Proposition 4.1, may be considered to be transverse regular on the zero section, $BO_{2s}$, of $\det(\gamma_{2s})$. Let $N' = (p2 \circ h)^{-1}(BO_{2s})$. Since $\gamma_{2s}$ has a $Z_{2k}$ action covering the $Z_{2k-1}$ action on $BO_{2s}$, $(p2 \circ h)^*(\gamma_{2s}) \overset{Z_{2k}}{\Rightarrow} N'$ is a bundle with an induced $Z_{2k}$ action such that $Z_2 < Z_{2k}$ acts as $-1$ in the fibers. Let $S'$ generate the $Z_{2k}$ action on $(p2 \circ h)^*(\gamma_{2s})$. $D((p2 \circ h)^*(\gamma_{2s}))$ is oriented and one checks that $\det dS'$ acts as $S' \times 1$ on the determinant of the tangent bundle. Hence $S'$ is orientation preserving by [6, Lemma 3]. Hence by mapping $[N, \partial N, S, h]$ into

$$[D((p2 \circ h)^*(\gamma_{2s})), S(p2 \circ h)^*(\gamma_{2s}), D((p2 \circ h)^*(\gamma_{2s})/\partial N'), S', p1 \circ h \circ \pi']$$

one defines a map $K$ from

$$\Omega^*_{2k-1}(F,F_Z_2(X, A) \times (\det \gamma_{2s}), S(\det \gamma_{2s}))$$

into $\Omega^*_{2k}(F, F_e)(X, A)$.

To see that $F \circ K = \text{id}$ one notes that $D[\det(p2 \circ h)^*(\gamma_{2s})]$ may be regarded as a tubular neighborhood of $N'$ in $N$. By a deformation one may assume that $p2 \circ h$ maps

$$[N - [D(\det(p2 \circ h)^*(\gamma_{2s})) - S(\det(p2 \circ h)^*(\gamma_{2s}))]]$$

into $S(\det \gamma_{2s})$. Let $\pi''$ be the bundle projection, $\pi'': (p2 \circ h)^*(\det \gamma_{2s}) \rightarrow N'$. Since $N'$ is a strong equivariant homotopy retract of its tubular neighborhood there is an equivariant homotopy $J: N \times I \rightarrow F_Z_2(X)$ giving a homotopy
between $p_1 \circ h$ and $J_1$ where $J_1$ has the property that $J_1$ on $D((p_2 \circ h)^*(\det \gamma_2))$ is given by $(p_1 \circ h/N') \circ \pi''$. It follows that

$$
\{N \times I, \partial N \times I \cup N \times 1 - \text{int} D((p_2 \circ h)^*(\det \gamma_{2s})), S \times 1, (U \times (p_2 \circ h)) \times 1\}
$$

gives a bordism between $[N, \partial N, S, h]$ and $F \circ K([N, \partial N, S, h])$.

To see that $K \circ F = \text{id}$ it suffices to observe that $F_2$ is a strong equivariant retract of its tubular neighborhood, $D(\nu)$, and hence one may suppose $f$ is homotopic to a map $H$ such that $H/D(\nu) = f/F_2 \circ \pi$. Now $K \circ F$ is obtained by restricting to $D(\nu)$. Since $Z_{2k}$ acts freely in the complement of $F_2$, $[M, M_0, M_1, T, f] = K \circ F([M, M_0, M_1, T, f])$. □

Now suppose $(X, A)$ is a c.w. pair acted on by $G = Z_{2k}$. Let $q$ denote the quotient map onto the space pair $(X/G, A/G)$ obtained by identifying the orbits of the $G$ action. It is a well-known fact that $q^*: H^*(X/G, A/G; \mathbb{R}^2) \to H^*(X, A; \mathbb{R}^2)$ is a monomorphism onto the elements of $H^*(X, A; R^2)$ which are invariant under the $G$ action (see [5, Corollary 2.3]). This fact together with the appropriate universal coefficient theorem indicates that if $H^*(X, A; R^2)$ is a free $R_2$ module on even [odd] dimensional generators, then $H^*(X/G, A/G; R^2)$ is a free $R_2$ module on even [odd] dimensional generators.

In light of this fact one defines a space pair $(X, A)$ to be (2-even) [(2-odd)] if and only if $H^*(X, A; R^2)$ is a free $R_2$ module on even [odd] dimensional generators.

**Lemma 5.2.** Let $G = Z_{2k}$. If $(X, A)$ is a $G$ pair and if $(X, A)$ is (2-even) [(2-odd)], then $\Omega^G_*(F_e)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even [odd] dimensional generators.

**Proof.** From Proposition 2.3 in [10] one learns that $\Omega^G_*(F_e)(X, A) \cong \Omega_*(X \times_G EG, A \times_G EG)$ where $EG$ is the total space of the universal principal $G$ bundle. From the discussion preceding this lemma one learns that $(X \times_G EG, A \times_G EG)$ is (2-even) [(2-odd)]. As in [11, p. 145] one can show that if $(X, A)$ is a c.w. pair such that $H_*(X, A; R_2)$ is a torsion free $R_2$ module, then

$$
\Omega_*(X, A) \otimes R_2 \cong (\Omega_* \otimes R_2) \otimes_{R_2} H_*(X, A; R_2).
$$

This yields the desired result. □

Thus it is of interest to examine the homology of the spaces introduced in Theorem 5.1. From the homology exact sequence of the cofibration $S(\det \gamma_{2s}) \to D(\det \gamma_{2s}) \to T(\det \gamma_{2s})$ in which $BSO_{2s}$ is homotopy equivalent to $S(\det \gamma_{2s})$ and $BO_{2s}$ is homotopy equivalent to $D(\det \gamma_{2s})$ one learns that $T(\det \gamma_{2s})$ is (2-odd). From the proof of Proposition 4.1 one knows that for $Z_{2k}$ actions

This completes the proof. □
If $E \to X$ is an oriented bundle, $\det E$ is a trivial line bundle and thus $T(\det E) = \Sigma X^+$ where $\Sigma$ denotes reduced suspension. Now by Proposition 3.3, $F_{Z_2}(BO_{2s}, Z_{2k})$ is homotopic to $\bigcup BU(t)$ where $(t)$ is a $q$-tuple of nonnegative integers $(t_1, t_2, \cdots, t_q)$ and $BU(t) = BU_{t_1} \times BU_{t_2} \times \cdots \times BU_{t_q}$.

Since $\gamma_{2s}/BU(t)$ is complex, $\gamma_{2s}/BU(t)$ is trivial and $T(\gamma_{2s}/\bigcup BU(t)) = \sqrt{\Sigma BU(t)}$. It follows that $F_{Z_2}(T(\gamma_{2s}))$ is $(2$-odd$)$ for $0 < j < k - 1$.

Now from the appropriate cofibrations $X \vee Y \to X \times Y \to X \vee Y$ one reads off the result:

**Lemma 5.3.** If $F_{Z_2}(X, A)$ is $(2$-even$)$ $[(2$-odd$)]$ for $0 < j < k$ and if $Y = F_{Z_2}(X)/F_{Z_2}(A) \wedge T(\gamma_{2s})$, then $Y$ is a space with $Z_{2k-1}$ action such that $F_{Z_2}(Y)$ is $(2$-odd$)$ $[(2$-even$)]$ for $0 < j < k - 1$. □

This brings one finally to the computations.

**Theorem 5.4.** If $F_{Z_2}(X, A)$ is $(2$-even$)$ $[(2$-odd$)]$ for $0 < j < k$, then $\Omega^*_{2k}(F)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even $[odd]$ dimensional generators.

**Proof.** If $k = 0$ then both the even and the $[odd]$ case follow from Lemma 5.2. Assume that the theorem is true for $k' < k$. Let $(X, A)$ have a $Z_{2k}$ action satisfying the hypotheses. By Lemma 5.2, $\Omega^*_{2k}(F)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even $[odd]$ dimensional generators. Theorem 5.1 yields that

$$\Omega^*_{2k}(F, F_e)(X, A) \cong \bigoplus \Omega^*_{2k-1}(F)(F_{Z_2}/(X)/F_{Z_2}(A) \wedge T(\gamma_{2s})).$$

By Lemma 5.3 and induction hypothesis, this implies that $\Omega^*_{2k}(F, F_e)(X, A) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even $[odd]$ dimensional generators.

Now consider the exact triangle

$$\Omega^*_{2k}(F_e)(X, A) \otimes R_2 \to \Omega^*_{2k}(F)(X, A) \otimes R_2 \to \Omega^*_{2k}(F, F_e)(X, A) \otimes R_2.$$

Note that it is in fact a split short exact sequence. This gives the induction step. □

**Note.** If $(X, A) = (pt, \emptyset)$, Theorem 5.4 says that $\Omega^*_{2k} \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators.

**Note.** This is the best possible result in the following sense. In [3, p. 105] P. E. Conner computes the torsion of $\Omega^*_{2k}$. There is too much torsion for $\Omega^*_{2k}$ to be a free $\Omega_* \otimes R_2$ module.

**Note.** In the paper as originally submitted the author asserted that $\Omega^*_{2k} \otimes R_2$ is a free $\Omega_* \otimes R_2$ module for $G$ any finite cyclic group. However, the
referee kindly noted a logical error in the author's proof of this statement. Nonetheless it is still a very reasonable conjecture, and in fact seems to be true in certain special cases (e.g. $\mathbb{Z}_2 \times \mathbb{Z}_p$). Along this line it should also be noted that in the author's dissertation [13] he proves via a somewhat arduous and noninstructive argument that for $G$ a finite cyclic group the torison of $\Omega^*_G$ is all 2-torsion.

BIBLIOGRAPHY


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