THE SPACE OF CONJUGACY CLASSES OF A TOPOLOGICAL GROUP

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ABSTRACT. The space \( G^\# \) of conjugacy classes of a topological group \( G \) is the orbit space of the action of \( G \) on itself by inner automorphisms. For a class of connected and locally connected groups which includes all analytic \([Z]\)-groups, the universal covering space of \( G^\# \) may be obtained as the space of conjugacy classes of a group which is locally isomorphic with \( G \), and the Poincaré group of \( G^\# \) is found to be isomorphic with that of \( G/G' \), the commutator quotient group. In particular, it is shown that the space \( G^\# \) of a compact analytic group \( G \) is simply connected if and only if \( G \) is semisimple. The proof of this fact has not appeared in the literature, even though more specialized methods are available for this case.

I. Definitions and elementary properties. Two elements \( x, y \) of a topological group \( G \) are called conjugate, and we write \( x \approx y \), if there is an element \( t \in G \) such that \( y = txt^{-1} \). The equivalence class of a point \( x \) under this relation is called the conjugacy class of \( x \), denoted \( I_x \). A subset of \( G \) which is a union of conjugacy classes is invariant under inner automorphisms and will be said to be invariant.

If \( G \) acts on itself by inner automorphisms, the inner automorphisms determined by the center \( Z(G) \) of \( G \) are trivial and \( G/Z(G) \) acts effectively on \( G \). The orbit space under the action of \( G \) or \( G/Z(G) \) is called the space of conjugacy classes of \( G \), denoted \( G^\# \). If \( G \) is the direct product of groups \( G_i \), then \( G^\# \) is homeomorphic with the Cartesian product of the spaces \( G_i^\# \) (see [5, p. 130]).

The space \( G^\# \) of a compact analytic group \( G \) is homeomorphic with the orbit space \( T/W \) of the action of the Weyl group \( W \) on a maximal toroid \( T \) of \( G \) [1, p. 95]. If \( G \) is semisimple, \( G^\# \) may be obtained by identifying certain boundary points of a compact convex polyhedron in the Lie algebra of \( T \) (see [2, Example 6]). Some elementary proofs and [11, p. 231] give the following:

LEMMA 1. If \( G \) is a compact analytic group, then \( G^\# \) is compact, Hausdorff, second countable, and locally arcwise simply connected.
The natural map $p: G \to G^\#$ may not be closed if $G$ is not discrete, and $G/Z(G)$ is not compact. From [4, p. 303], we have the following:

**Proposition 2.** If $G$ is a connected, locally compact group, then the following are equivalent:

(i) the natural map $p: G \to G^\#$ is closed;

(ii) each neighborhood of $e$ contains an invariant neighborhood of $e$ (the [$SIN$] property);

(iii) $G$ is the direct product of a compact group and a vector group.

**Example 1.** Let $H$ be the subgroup of $SL(3, \mathbb{R})$ consisting of matrices of the form

$$M(r, s, t) = \begin{pmatrix} 1 & r & . \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$ 

It is easily checked that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1} = M(r, s, t + as - br)$$

and that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1}(M(r, s, t))^{-1} = M(0, 0, as - br).$$

An element of the form $M(0, 0, t)$ is central, and $D = \{M(0, 0, n): n \in \mathbb{Z}\}$ is a discrete central subgroup. The conjugacy class of a noncentral element $M(r, s, t)$ is $\{M(r, s, w): w \in \mathbb{R}\}$. In particular, $\{M(1/n, 0, w): w \in \mathbb{R}\}$ is a conjugacy class for each $n \in \mathbb{Z}^+$. Hence, $H$ is not an [$SIN$] group. The quotient group $H/D$ has compact conjugacy classes and is not an [$SIN$] group.

**Example 2.** In the group $H$ of Example 1, consider the subgroup $G = \{M(m, n, t); m, n \in \mathbb{Z}; t \in \mathbb{R}\}$. The conjugacy class of a noncentral element $M(m, n, t)$ is $\{M(m, n, t + kd); k \in \mathbb{Z}; d$ the greatest common divisor of $m$ and $n\}$. The space $G^\#$ is normal, because each component of $G^\#$ is homeomorphic with $R/d\mathbb{Z}$ for some $d \in \mathbb{Z}$. The component of $e$ is exactly the center, so that $G$ is an [$SIN$] group which is not the direct product of a vector group and a compact group.

For connectedness, we have

**Proposition 3.** Suppose that $p$ is a closed map or that $G$ is locally connected or that each conjugacy class is connected. Then $G^\#$ is connected if and only if $G$ is connected.
Proof (of the nontrivial implication). Let $C$ be the (invariant) component subgroup of $G$. If $C \neq G$, there is an open and closed set $E$ which does not meet $C$. If $p$ is closed, then $p(E)$ is an open and closed set which does not meet $p(C)$.

In the other two cases, consider the space $(G/C)^#$ and the diagram:

$$
\begin{array}{c}
G \\
\downarrow \quad \downarrow \\
G/C \\
\downarrow \\
(G/C)^# \\
\downarrow \\
G^# \\
\end{array}
$$

If $G$ is locally connected, then $G/C$ and $(G/C)^#$ are discrete. If each conjugacy class is connected, then $G/C = (G/C)^#$ is totally disconnected [8, p. 60]. But in either case, $(G/C)^#$ is connected, hence, trivial. Thus, $G = C$.

Clearly, if $x, y \in G$ and $z \in Z(G)$, then $x \approx y$ if and only if $zx \approx zy$. This suggests that we define an action of $Z(G)$ on $G^#$ by

$$(*)
\begin{array}{c}
zI_x = I_{zx}.
\end{array}
$$

This action of $Z(G)$ on $G^#$ constitutes a transformation group, in the sense of [11], except that $G^#$ may not be a Hausdorff space.

Lemma 4. If $D$ is a closed subgroup of $Z(G)$, then the orbit space $G^#/D$ is homeomorphic with $(G/D)^#$. 

Proof. Consider the diagram:

$$
\begin{array}{c}
G/D \\
\downarrow \\
(G/D)^# \\
\downarrow \\
G^# \\
\downarrow \\
G^#/D
\end{array}
$$

II. Stability subgroups for the action of $Z(G)$ on $G^#$. The stability subgroups for the action $(*)$ are conveniently described in terms of the sets $I_x I_x^{-1}$. For each $x \in G$, the set $I_x I_x^{-1}$ is invariant and inversion-invariant and $e \in I_x I_x^{-1} \subset (G, G)$, the algebraic commutator subgroup. The following theorem, which was proved by Goto in [6], will be used to show that the main result of this paper (Theorem 16) holds for analytic $[Z]$-groups:

Theorem 5 (Goto). If $G$ is a compact semisimple analytic group, then there is an element $x \in G$ such that $I_x I_x^{-1} = G$.

In a more general situation, we have the following relationship between the algebraic and topological structure of the conjugacy classes:

Proposition 6. Suppose that the set $I_x I_x^{-1}$ is locally compact in its relative topology. Then $I_x \subset I_x I_x^{-1} I_x \subset x(G, G)$. Moreover, if the set $I_x I_x^{-1}$ is
closed under the group operation, then it is a closed invariant subgroup of \( G \) contained in \((G, G)\).

**Proof.** The second part follows from [8, p. 35], for then \( I_xI_x^{-1} \) is a locally compact subgroup.

For the first part, let \( U, V \) be neighborhoods of \( e \) with \( \bar{U} \cap I_xI_x^{-1} \) compact and \( V^2 \subset U. \) Let \( w \in \bar{I}_x, z \in I_x^{-1} \cap w^{-1}V, \) and let \( \{w_i\} \) be a net in \( I_x \) converging to \( w. \) Then, eventually,

\[ w_i z \in Vw_i w^{-1}V \cap I_xI_x^{-1} = V^2 \cap I_xI_x^{-1} \subset U \cap I_xI_x^{-1}. \]

Thus, \( wz \) is in the closed set \( \bar{U} \cap I_xI_x^{-1} \) and \( w = wzz^{-1} \in I_xI_x^{-1}. \)

**Example 3.** If \( G \) is the affine group, \( \{(r, s) : r \in R^+, s \in R\} \), and \( x = (1, 0) \), then \( e \in \bar{I}_x, \) and \( I_xI_x^{-1} = I_x \cup I_x^{-1} \cup \{e\} = (G, G) \) (see [8, p. 350]).

We now identify the stability subgroups for the action \((*)\).

**Lemma 7.** If \( D \) is a closed subgroup of \( Z(G) \), then the set \( D_x = D \cap I_xI_x^{-1} = D \cap xI_x^{-1} \) is the stability subgroup in \( D \) of \( I_x \in G^\# \).

**Proof.** An element \( d \in D \) is in the stability subgroup if and only if it translates some (and hence, every) conjugate of \( x \) to another conjugate of \( x. \) Then, for some \( s, t \in G, d = sx^{-1}tx^{-1}t^{-1} = s^{-1}ds = xs^{-1}tx^{-1}t^{-1}s. \)

These stability subgroups are related to the zeros of characters of finite-dimensional irreducible representations:

**Corollary 8.** Let \( \pi \) be a finite-dimensional irreducible representation of \( G \) and let \( x \in G. \) If \( \text{trace}(\pi(x)) \neq 0, \) then \( Z(G) \cap I_xI_x^{-1} \subset \text{kernel}(\pi). \) If moreover, \( \pi \) is faithful, then the stability subgroup of \( I_x \in G^\# \) under \((*)\) is trivial.

**Proof.** Let \( z \in Z(G) \cap I_xI_x^{-1}, \) then Schur’s lemma shows that \( \text{trace}(\pi(x)) = \text{trace}(\pi(xz)) = \text{trace}(\pi(x))\text{trace}(\pi(z))/\text{trace}(\pi(e)). \)

**Corollary 9.** If \( G \) is a compact semisimple analytic group and \( x \in G \) is a regular point, that is, a point whose centralizer has minimum dimension, then \( D_x \) is isomorphic with a subgroup of the Weyl group \( W \) of \( G. \)

**Proof.** There is a maximal toroid \( T \) which contains \( x \) (and \( Z(G) \)) (see [1]) and for each \( d \in D_x \) there is exactly one \( nT \in W \) such that \( dx = nx^{-1}. \) This correspondence effects an isomorphism between \( D_x \) and an Abelian subgroup of \( W. \)

**Example 4.** If \( G = SU(2), \) there is only one conjugacy class with a non-trivial stability subgroup, that of \( (0, -1). \) The Weyl group is of order two.
III. The Poincaré group of $G^\#$. The relationship between the structures $G^#$ and $G^a = G/G'$, where $G'$ is the closed commutator subgroup of $G$, is a consequence of the fact that the natural map from $G$ to $G^a$ factors through $G^#$ (see [8, p. 358]).

**Lemma 10.** The map $q: G^\# \to G^a$ defined by $q(t_x) = xG'$ is continuous, open and surjective.

A connected and locally connected space $S$ will be said to be simply connected if, for each covering space $(U, f)$ of a space $T$ and continuous map $g: S \to T$, there is a unique continuous map $h: S \to U$ such that $f \circ h = g$ and $h(s) = u$, where $s$ and $u$ are prescribed points such that $f(u) = g(s)$.

This is the definition used by Hochschild [10], and is equivalent to that used by Chevalley [3], except that we do not require the Hausdorff property. The stability of this lifting property under two types of maps which appear leads to sufficient conditions for the spaces $G^#$ and $G^a$ to be simply connected.

For analytic $\mathbb{R}$-groups, we show that these spaces are arcwise simply connected if they are simply connected.

A space is said to be locally simply connected if each point has a simply connected neighborhood. A connected space has a simply connected covering space if and only if it is locally simply connected (see [10] and [3]).

**Lemma 11.** Let $G$ be a group which acts on a simply connected space $M$ with a fixed point $m$. Then the orbit space $M/G$ is simply connected.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{p} & M/G \\
\downarrow{h'} & & \downarrow{g'} \\
U & \xrightarrow{f} & T \\
\end{array}
$$

where $f$ is a covering, $g$ is continuous, $g' = g \circ p$ and $h'$ is a specified lift of $g'$. To show that there is a map $h$ as indicated, we show that $h'$ is constant on $G$-orbits. Since $h'$ is the unique map taking $m$ to $h'(m)$ and satisfying $f \circ h' = g'$, precomposition of $h'$ with an action of $G$ does not alter $h'$, that is $h'$ is constant on $G$-orbits.

We can now give some sufficient conditions for the spaces $G^#$ and $G^a$ to be simply connected:

**Proposition 12.** If $G$ is locally connected and $G^#$ is simply connected, then $G^a = G/G'$ is simply connected.
Proof. By Proposition 3, \( G \) is connected. Hence, the \( G' \)-cosets are connected [9, p. 142]. Use [10, p. 56], and Lemma 10.

Proposition 13. If \( G \) is simply connected, then \( G' \) is simply connected.

Proof. The stability subgroup of \( e \) for the action of \( G \) on itself by inner automorphisms is \( G \). Use Lemma 11.

Proposition 14. Let \( D \) be a discrete subgroup of \( Z(G) \) and let \( D_x \) be the stability subgroup of \( I_x \in G' \) under (*) . If \( G' \) is simply connected, then \( (G/D_x)' \) is simply connected.

Proof. The subgroup \( D_x \) is closed (we have not assumed any separation properties for \( G' \)). Use Lemmas 11 and 4.

Proposition 15. Let \( D \) be a discrete subgroup of \( Z(G) \) which is generated by the stability subgroups \( D_x \) under (*). If \( G' \) is simply connected, then \( (G/D)' \) is simply connected.

Proof. Partially order by inclusion the collection of subgroups \( D^* \) of \( D \) such that the orbit space \( G'/D^* \) is simply connected, and use Zorn's lemma. The uniqueness of lifts in the definition of "simply connected" implies that the union of the elements of a chain is an upper bound for the chain, and \( D \) is the only possible maximal element because of Proposition 14.

Example 5. In Example 1, the group \( H \) is the universal covering group of \( H/D \) and \( (H/D)' \) is simply connected.

We are now ready to prove the main result.

Theorem 16. Let \( G \) be a connected and locally simply connected group with universal covering group \( \tilde{G} \). If \( D \) is a discrete subgroup of \( Z(\tilde{G}) \) such that \( G \cong \tilde{G}/D \) and \( D \cap (\tilde{G})' \) is generated by the stability groups \( D_x \) under (*), and \( D(\tilde{G})'/(\tilde{G})' \) is discrete in \( (\tilde{G})^a = \tilde{G}/(\tilde{G})' \), then the Poincaré groups of \( G' \) and \( G^a \) are isomorphic with \( D/(D \cap (\tilde{G})') \).

Proof. Let \( D_1 = D \cap (\tilde{G})' \) and let \( f_1: G \to G/D_1 \) and \( f_2: G/D_1 \to G/D \) be the natural covering maps. Since \( D(\tilde{G})'/(\tilde{G})' \) is closed in \( (\tilde{G})^a \), \( f_1((\tilde{G})') = (\tilde{G}/D_1)' \) and \( f_2(f_1(D(\tilde{G})')) = (\tilde{G}/D)' \). Thus, we have the diagram

\[
\begin{array}{ccc}
\tilde{G} & \longrightarrow & (\tilde{G})' \\
\downarrow f_1 & & \downarrow f_1^a \\
\tilde{G}/D_1 & \longrightarrow & (G/D_1)' \\
\downarrow f_2 & & \downarrow f_2^a \\
G/D & \longrightarrow & (G/D)' \\
\end{array}
\]
where $f_1^a$, $f_2^#$ are the topological isomorphisms induced by $f_1$, $f_2^#$ and $f_2^a$ are induced by $f_2$ and other maps are as in Lemmas 4 and 10.

Propositions 13, 15, and 12 show that the spaces $(\tilde{G}/D_1)^#$ and $(\tilde{G}/D_1)^a$ are simply connected. It remains to show that $D/D_1$ is a properly discontinuous group of homeomorphisms of these spaces [12, p. 87]. In $(\tilde{G}/D_1)^a$, $D/D_1 \cong (D(\tilde{G})'/D_1)/(\tilde{G})'/D_1)$, a discrete subgroup. The action (*) of $D/D_1$ on $(\tilde{G}/D_1)^#$ is also properly discontinuous, because the elements of a $D/D_1$ orbit are conjugacy classes lying in distinct $(\tilde{G}/D_1)'$-cosets (in $\tilde{G}$, we have $(dI_x)^{-1}_x \subseteq (\tilde{G})'$ only if $d \in I_x^{-1}_x(\tilde{G})'$).

Corollary 17. If $G$ is an analytic $[Z]$-group, the spaces $G^#$ and $G^a$ are locally arcwise simply connected and have isomorphic fundamental groups.

Proof. First of all, the group $G$ is the direct product of a vector group and a compact group (Proposition 2), so we may assume that $G$ is compact. Then $\tilde{G}$ is the direct product of a vector group and a simply connected compact semisimple analytic group $H$ (see [10] and [13]). If $D$ is a discrete subgroup of $\mathbb{Z}(\tilde{G})$ such that $\tilde{G}/D = G$, then $DH/H = D(\tilde{G})'(\tilde{G})'$ is discrete [10, p. 6] and $D \cap H = D \cap (\tilde{G})' = D_x$ for some $x \in H$ (Theorem 5). The result follows from [12, p. 88], because $H^#$ is arcwise simply connected (see Proposition 13, Lemma 1, and [10]).

Corollary 18. A compact analytic group is semisimple if and only if $G^#$ is simply connected.

Proof. A compact analytic group $G$ is semisimple if and only if the toroid $G^a$ is trivial; use Corollary 17.

Example 6. One maximal toroid of $G = SO(4)$ consists of matrices

$$M(\theta, \varphi) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

and the nontrivial elements of the Weyl group are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
and
\[
C = AB = BA = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

One checks easily that
\[
A(M(\theta, \varphi))A^{-1} = M(2\pi - \theta, 2\pi - \varphi),
\]
\[
B(M(\theta, \varphi))B^{-1} = M(\varphi, \theta),
\]
\[
C(M(\theta, \varphi))C^{-1} = M(2\pi - \varphi, 2\pi - \theta),
\]
so that each conjugacy class is represented by a matrix \( M(\theta, \varphi) \) with \( 0 \leq \theta \leq \pi \) and \( \theta \leq \varphi \leq 2\pi - \theta \). The space \( G^\# \) may be realized as the small triangle on the left in the square
\[
\begin{array}{cc}
(0, 0) & (2\pi, 0) \\
(0, 2\pi) & (2\pi, 2\pi)
\end{array}
\]
where pairs \( M(0, \varphi), M(0, 2\pi - \varphi) \) on the left-hand boundary must be identified. The space \( G^\# \) is simply connected, as indicated by Corollary 18.

REFERENCES


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