ABSTRACT. The object of this paper is to show that there exists a polynomial $P_n(x)$ of degree $\leq 2n - 1$ which interpolates a given function exactly at the zeros of $n$th Tchebycheff polynomial and for which $\|f - P_n\| \leq C_k w_k(1/n, f)$ where $w_k(1/n, f)$ is the modulus of continuity of $f$ of $k$th order.

1. Introduction. The classical theorems of D. Jackson extend the Weierstrass approximation theorem by giving quantitative information on the degree of approximation to a continuous function in terms of its smoothness. Specifically, Jackson proved

**Theorem 1.** Let $f(x)$ be continuous for $|x| < 1$ and have modulus of continuity $w(t)$. Then there exists a polynomial $P(x)$ of degree $n$ at most such that $|f(x) - P(x)| < Aw(1/n)$ for $|x| < 1$, where $A$ is a positive numerical constant.

In 1951 S. B. Stečkin [3] made an important generalization of the Jackson theorem.

**Theorem 2 (S. B. Stečkin).** Let $k$ be a positive integer; then there exists a positive constant $C_k$ such that for every $f \in C[-1, +1]$ we can find an algebraic polynomial $P_n(x)$ of degree $n$ so that

$$\|f - P_n\| \leq C_k w_k(1/n, f),$$

where $w_k(1/n, f)$ is the modulus of continuity of $f(x)$ of $k$th order.

During personal conversation in Poznań, S. B. Stečkin raised the following
question concerning Theorem 2: Does there exist an algebraic polynomial \( P_n(x) \) of degree \( Cn \ (C > 1) \) interpolating at \( n \) points and which satisfies (1.1)? The case \( k = 1 \) was previously raised by P. L. Butzer [1] and solved consequently by G. Freud [2]. The object of this paper is to prove that an algebraic interpolatory polynomial satisfying (1.1) does exist. The approach we have adapted is to modify the classical Hermite-Fejér interpolation polynomials on the Tchebycheff nodes. Moreover the degree of the new interpolation process is still \( 2n - 1 \). It may be interesting to point out that recently the author [5], [6] has solved the problem of obtaining a trigonometric polynomial which interpolates a given \( 2\pi \) periodic continuous function at \( x_k = 2k\pi/n, \ k = 0, 1, \cdots, n - 1, \) and for which (1.1) is also true.

2. It is well known that the Hermite-Fejér interpolation polynomial of degree \( \leq 2n - 1 \) is defined by

\[
H_n[f, x] = \sum_{k=1}^{n} f(x_k^n)(1 - xx_k^n)
\]

where

\[
x_k^n = \cos((2k - 1)\pi/2n), \quad k = 1, 2, \cdots, n,
\]

are the zeros of Tchebycheff polynomial \( T_n(x) = \cos(n \arccos x) \). Let us express \( H_n[f, x] \) as a linear combination of \( T_0(x), T_1(x), \cdots, T_{2n-1}(x) \). For this purpose we define

\[
C_0(f) = \frac{1}{n} \sum_{k=1}^{n} f(x_k^n), \quad C_j(f) = \frac{2}{n} \sum_{k=1}^{n} f(x_k^n)T_j(x_k^n)
\]

for \( j = 1, 2, \cdots, 2n - 1 \). A simple computation shows that

\[
H_n[f, x] = \sum_{j=0}^{2n-1} C_j(f)\left(\frac{2n-j}{2n}\right)T_j(x).
\]

This representation of \( H_n[f, x] \) suggests the definition

\[
R_n(f) = R_n[f, x] = \sum_{j=0}^{2n-1} C_j(f)\alpha_{j,M}T_j(x),
\]

where \( M \) is an arbitrary fixed positive integer and

\[
\alpha_{0,M} = 1, \quad \alpha_{j,M} + \alpha_{2n-j,M} = 1, \quad j = 1, 2, \cdots, n, \quad \alpha_{j,M} = 0, \quad j > 2n.
\]

A simple example of \( \alpha_{j,M} \) satisfying (2.5) is given by

\[
\alpha_{j,M} = (2n-j)^M/((2n-j)^M + j^M), \quad j = 0, 1, \cdots, 2n - 1,
\]

\[
= 0, \quad j \geq 2n.
\]
For our purpose we make further restrictions on $\alpha_{j,M}$. We denote

$$
\mu_{j,M} = (1 - \alpha_{j,M})^j M, \quad j = 1, 2, \ldots, 2n,
$$

(2.7)

$$
= 0, \quad j = 0.
$$

Let us suppose that

$$
|\mu_{j+1,M} - \mu_{j,M}| = O(1/n^{M+1}), \quad j = 1, \ldots, 2n - 1,
$$

(2.8)

and

$$
|\mu_{j+1,M} - 2\mu_{j,M} + \mu_{j-1,M}| = O(1/n^{M+2}), \quad j = 1, \ldots, 2n - 1.
$$

(2.9)

We also require that

$$
1 - \alpha_{1,M} = O(1/n^M),
$$

(2.10)

$$
|\alpha_{j+1,M} - 2\alpha_{j,M} + \alpha_{j-1,M}| = O(1/n^2), \quad j = 1, \ldots, 2n - 1.
$$

(2.11)

We now state our main theorem,

**Theorem 3.** Let $f(x)$ be continuous for $|x| < 1$. Then $R_n(f)$ as defined by (2.4) and (2.5) satisfy

$$
R_n[f, x_{in}] = f(x_{in}), \quad i = 1, 2, \ldots, n,
$$

(2.12)

and

$$
R_n[1, x] = 1.
$$

(2.13)

Moreover under the assumptions (2.7)-(2.11) we have ($f$ a polynomial of degree $\leq m - 1$)

$$
\|R_n(f) - f\| \leq C_M w_{M-1}(1/n, f).
$$

(2.14)

It is easy to verify that the choice of $\alpha_{j,M}$ given by (2.6) satisfies all the requirements needed in Theorem 3.

**Proof of Theorem 3.** First we will prove that $R_n(f)$ as defined by (2.4) and (2.5) is an interpolation polynomial in $x$ of degree $\leq 2n - 1$ satisfying (2.12). For this purpose we express

$$
R_n[f, x] = \sum_{j=0}^{n-1} C_j(f)\alpha_{j,M} T_j(x) + \sum_{j=n+1}^{2n-1} C_j(f)\alpha_{j,M} T_j(x).
$$

(3.1)

In view of the fact that

$$
T_{2n-j}(x_{in}) = -T_j(x_{in}), \quad C_{2n-j}(f) = -C_j(f), \quad j = 1, \ldots, n,
$$

(3.2)

we obtain, on using (2.5) and (3.1),

$$
R_n[f, x_{in}] = C_0(f) - \sum_{j=1}^{n-1} C_j(f) T_j(x_{in}).
$$
From the definition of $C_j(f)$ as given in (2.3) it follows that

$$C_0(f) + \sum_{j=1}^{n-1} C_j(f) T_j(x_{in}) = f(x_{in}), \quad i = 1, \ldots, n.$$ 

Therefore we obtain $R_n[f, x_{in}] = f(x_{in}), \quad i = 1, \ldots, n$. This proves (2.12).

(2.13) is an immediate consequence of (2.3). For if $f(x) \equiv 1$ then $C_0(f) = 1$, $C_j(f) = 0, \quad j = 1, \ldots, 2n - 1$.

Next we hope to prove the existence of a positive constant $L$ independent of $n$ and $x$ such that

$$(3.3) \quad \|R_n[f]\| \leq L\|f\|.$$ 

To prove this we need some preliminary notation and estimates. We denote the Fejér kernel by

$$(3.4a) \quad t_j(\theta) = 1 + \frac{2}{j} \sum_{i=1}^{j} (j - i) \cos i\theta, \quad j = 2, 3, \ldots,$$

$$(3.4b) \quad t_1(\theta) \equiv 1.$$ 

Associated with this kernel we introduce

$$(3.4) \quad \tau_{j,k}(\theta) = \frac{1}{2}(t_j(\theta + \theta_{kn}) + t_j(\theta - \theta_{kn})).$$

It is easy to verify that

$$(3.5) \quad (j + 1)\tau_{j+1,k}(\theta) - 2j\tau_{j,k}(\theta) + (j - 1)\tau_{j-1,k}(\theta) = 2 \cos j\theta \cos j\theta_{kn},$$

and

$$(3.6) \quad \sum_{k=1}^{n} |\tau_{j,k}(\theta)| = n, \quad j = 1, 2, \ldots.$$ 

From (2.3) and (2.4) it follows that

$$(3.7) \quad R_n[f, x] = \sum_{k=1}^{n} f(x_{kn})P_{kn}(x),$$

where

$$(3.8) \quad P_{kn}(x) = \frac{1}{n} \left[ 1 + 2 \sum_{j=1}^{2n-1} \alpha_{j,M} T_j(x_{kn})T_j(x) \right].$$

Now we prove that

$$(3.9) \quad \sum_{k=1}^{n} |P_{kn}(x)| \leq L,$$

from which (3.3) follows on using (3.7). For this purpose we express (3.8) in terms of $\tau_{j,k}(\theta)$ as defined in (3.9). On using (3.5) we obtain
On using (2.10), (2.11) and (3.6), (3.9) follows immediately. This proves (3.9) and in turn (3.3). By proving (3.3) it follows easily that \( R_n[f, x] \) converges uniformly to \( f(x) \) on \([-1, +1]\) for every continuous function on \([-1, +1]\). Here our aim is to obtain error estimates (2.14). The proof of (2.14) is based on following [4]

**Theorem 4 [S. B. Stečkin]**. Let \( P \) be a natural number and \( U_n \) \((n = 1, 2, \cdots)\) be a linear method of approximation of functions having the following properties:

(i) for any function \( \phi(\theta) \in C_{2\pi} \), \( \|U_n(\phi)\| \leq L_o\|\phi\|; \)

(ii) for any function \( \phi(\theta) \in C_{2\pi} \) for which \( \phi^{(P)}(\theta) \in C_{2\pi} \), \( \|\phi - U_n(\phi)\| \leq L_p\|\phi^{(P)}\|/n^p \), \( n = 1, 2, \cdots \). Then for any function \( \phi(\theta) \in C_{2\pi} \) we have \( \|\phi - U_n(\phi)\| \leq B_p(L_o + L_p)w_p(1/n, \phi) \).

First let us choose \( U_n \) to be typical means of fourier series given by

\[
X_{2n,M}(\phi, \theta) = \frac{1}{2}a_0 + \sum_{j=1}^{2n-1} (a_j \cos j\theta + b_j \sin j\theta)(1 - j^M/(2n)^M),
\]

where \( a_j \)'s, \( b_j \)'s are fourier coefficients of \( \phi(\theta) \). From a theorem of A. Zygmund [7] it follows that conditions (i) and (ii) are satisfied for \( P = M - 1 \) \((M > 1)\) and, therefore, we conclude from Stečkin’s theorem that

\[
|X_{2n,M}(\phi, \theta) - \phi(\theta)| \leq C M w_M(1/n, \phi).
\]

Theorem 4 and (3.11) described above lead to a simple proof of (2.14). The representation of \( R_n[f, x] \) as given by (3.7) suggests that we consider a trigonometric polynomial

\[
A_n[\phi, \theta] = \sum_{k=1}^{\frac{n}{2}} \phi(\theta_k) \left\{ \frac{1}{n} + \frac{2}{n} \sum_{j=1}^{2n-1} a_{j,M} \cos j(\theta - \theta_k) \right\}
\]

where

\[
\phi(\theta) = f(\cos \theta) \equiv f(x).
\]

From (3.12) it follows that

\[
A_n[1, \theta] = 1,
\]

\[
A_n[\cos iu, \theta] = -\alpha_{2n-i}(\cos i\theta + \cos(2n-i)\theta);
\]

for \( i = 1, 2, \cdots, 2n - 1 \). From (3.10); (3.12)–(3.15) it follows that

\[
A_n[X_{2n,M}(\phi), \theta] - X_{2n,M}(\phi) = -(1 + \cos 2n\theta) \sum_{i=1}^{2n-1} a_i \cos i\theta \alpha_{2n-i}(1 - i^M/(2n)^M)
\]

\[
+ \sin 2n\theta \sum_{i=1}^{2n-1} a_i \sin i\theta \alpha_{2n-i}(1 - i^M/(2n)^M).
\]
Since $\phi(\theta)$ is an even function of $\theta$, its Fourier coefficients $b_i$ are all zero. Therefore we obtain

$$A_n [X_{2n, M}(t), \theta] - X_{2n, M}(\theta)$$

$$= -(1 + \cos 2n\theta) \sum_{i=1}^{2n-1} (a_i \cos i\theta - b_i \sin i\theta) \alpha_{2n-i} (1 - i^M/(2n)^M)$$

$$+ \sin 2n\theta \sum_{i=1}^{2n-1} (b_i \cos i\theta - a_i \sin i\theta) \alpha_{2n-i} (1 - i^M/(2n)^M).$$

On using an integral representation of Fourier coefficients we obtain

$$X_{2n, M}(\theta) - A_n [X_{2n, M}(t), \theta]$$

$$= \frac{(1 + \cos 2n\theta)}{\pi} \sum_{i=1}^{2n-1} i\delta_{l,M} \int_0^{2\pi} \phi(u) \cos i(u - \theta) du$$

$$+ \frac{\sin 2n\theta}{\pi} \sum_{i=1}^{2n-1} i\delta_{l,M} \int_0^{2\pi} \phi(u) \sin i(u - \theta) du,$$

where

$$i\delta_{l,M} = \alpha_{2n-l,M} (1 - i^M/(2n)^M);$$

we put

$$F(\theta) = \frac{1}{\pi} \sum_{i=1}^{2n-1} \delta_{l,M} \int_0^{2\pi} \phi(u + \theta) \sin i u du,$$

and rewrite

$$X_{2n, M}(\theta) - A_n [X_{2n, M}(t), \theta] = (1 + \cos 2n\theta) F'(\theta) - \sin 2n\theta \tilde{F}'(\theta).$$

Now, we need to obtain estimates of $F'(\theta)$ and $\tilde{F}'(\theta)$. For this purpose we assume that $\phi(\theta)$ is $(M-1)$ times continuously differentiable of function of $\theta$. Integrating by parts $(M-1)$ times we obtain after elementary calculation that

$$F(\theta) = \frac{(-1)^{M/2+1}}{\pi} \sum_{i=1}^{2n-1} \lambda_{l,M} \int_0^{2\pi} \phi^{(M-1)}(u + \theta) \cos i u du$$

for $M$ even integer, where

$$i^{M-1} \lambda_{l,M} = \delta_{l,M}. $$

From (3.4a) we obtain

$$2 \sum_{i=1}^{2n-1} \lambda_{l,M} \cos i u$$

$$= \sum_{i=1}^{2n-1} \left( \lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M} \right) i\gamma_i (u) + 2n \lambda_{2n-1,M} t_{2n}(u).$$
Therefore on using (3.20) and (3.21) we obtain
\[
F(\theta) = \frac{(-1)^{M/2+1}}{2\pi} \int_0^{2\pi} \phi^{(M-1)}(u + \theta) \left\{ \sum_{i=1}^{2n-1} (\lambda_{i,M} - 2\lambda_{i,M} + \lambda_{i+1,M}) t_i(u) + 2n\lambda_{2n-1,t_{2n}}(u) \right\} du.
\]
(3.22)

From (2.7), (3.17) and (3.21) it follows that
\[
|\lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M}| = O(1/n^{M+2}), \quad i = 1, \cdots, 2n-1,
\]
\[
\lambda_{2n-1,M} = O(1/n^{M+1}).
\]

From (3.4a) we obtain \( \int_0^{2\pi} \left| t_i(u) \right| du = 2\pi. \) On putting these estimates in (3.22) we obtain \( |F(\theta)| \leq c_M \|\phi^{(M-1)}\|/n^M. \) From (3.18) it follows that \( F(\theta) \) is a trigonometric polynomial of order \( \leq 2n. \) On using a well-known theorem of S. N. Bernstein (see Zygmund [8, volume 1, p. 118]) we obtain
\[
|F'(\theta)| \leq 2c_M \|\phi^{(M-1)}\|/n^{M-1}, \quad |\tilde{F}'(\theta)| \leq 2c_M \|\phi^{(M-1)}\|/n^{M-1}.
\]

Therefore, under the assumption that \( \phi^{(M-1)}(\theta) \in c_{2\pi} \) we obtain
\[
|X_{2n,M}(\theta) - A_n[X_{2n}(t), \theta]| \leq 2c_M \|\phi^{(M-1)}\|/n^{M-1}.
\]
(3.23)

Following a proof similar to that given for (3.3) it follows that for every \( \phi \in c_{2\pi} \) we have
\[
|A_n[\phi, \theta]| \leq B_M \|\phi\|.
\]
(3.24)

Now we claim that for every \( \phi^{(M-1)} \in c_{2\pi} \) we have
\[
|\phi(\theta) - A_n[\phi, \theta]| \leq B_M \|\phi^{(M-1)}\|/n^{M-1}.
\]
(3.25)

This follows from
\[
\phi(\theta) - A_n[\phi, \theta] = \phi(\theta) - X_{2n,M}(\theta) + X_{2n,M}(\theta) - A_n[X_{2n,M}(t), \theta]
\]
\[
= A_n[X_{2n,M}(t), \theta] - A_n[\phi, \theta],
\]
(3.23), (3.24) and (3.11). This proves (3.25).

(3.24) and (3.25) enable us to apply Stečkin Theorem 4 and we conclude that
\[
|\phi(\theta) - A_n[\phi, \theta]| \leq B_M w_{M-1}(1/n, \phi).
\]

But this inequality implies
\[
|\phi(\theta) - \frac{1}{2}(A_n[\phi, \theta] + A_n[\phi, -\theta])| \leq B_M w_{M-1}(1/n, \phi).
\]
Since \( \phi(\theta) = f(\cos \theta) \equiv f(x) \) and \( R_n[f, x] = \frac{1}{2}A_n[\phi, \theta] + A_n[\phi, -\theta], \) we obtain
\[
|f(x) - R_n[f, x]| \leq B_M w_{M-1}(1/n, \phi).
\]
(3.26)
It is well known that \( w_{M-1}(1/n, \phi) \leq c_M w_{M-1}(1/n, f) \). We obtain from (3.26) that

\[
|f(x) - R_n[f, x]| \leq e_M w_{M-1}(1/n, f).
\]

This proves (2.14) and thus completes the proof of Theorem 3 for \( M \) even integer. For \( M \)-odd positive integer \( > 1 \) a similar proof can be given.

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