ON THE ANALYTIC CONTINUATION
OF THE MINAKSHISUNDARAM-PLEIJEL ZETA FUNCTION
FOR COMPACT RIEMANN SURFACES

BY

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ABSTRACT. A formula is derived for the Minakshisundaram-Pleijel zeta function in the half-plane \( \text{Re } s < 0 \).

Let \( S \) be a compact Riemann surface, which we will regard as the quotient of the upper half-plane \( H \) by a discontinuous group \( \Gamma \) of hyperbolic transformations. We will assume that \( H \) is endowed with the metric \( y^{-2}((dx)^2 + (dy)^2) \), and we will denote the area of \( S \) by \( A \). Let \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \) be the eigenvalues corresponding to the problem \( \Delta f + \lambda f = 0 \) on \( S \), where \( \Delta \) is the Laplace operator on \( S \), derived from the metric induced on \( S \) by that of \( H \). In the coordinates of \( H \), the Laplacian is \( y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) \). Finally, let \( Z(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} \). Since it is known [3] that \( A/\pi = \sum_{n=1}^{\infty} \lambda_n^{-1} \) is asymptotic to \( (A/4\pi)T \), it follows that the series for \( Z(s) \) converges absolutely in the half-plane \( \text{Re } s > 1 \).

In this note we will use the Selberg trace formula to derive an expression for the continuation of \( Z(s) \) in the half-plane \( \text{Re } s < 0 \). Accounts of the trace formula can be found in [1], [2], and [4]. The formula, adjusted to the present situation, goes as follows.

Suppose \( h(z) \) is an even function, holomorphic in a strip of the form \( \text{Im } z < \frac{1}{2} + \epsilon \) \( (\epsilon > 0) \), and satisfying a growth condition of the form \( |h(z)| = O((1 + |z|^2)^{-1 + \epsilon}) \) uniformly in the strip. Associate with the sequence \( \lambda_0, \lambda_1, \lambda_2, \cdots \) of eigenvalues the set \( \mathcal{R} \) consisting of those numbers which satisfy an equation of the form \( \lambda_n^2 = \lambda^2 + r^2 \) \( (n = 0, 1, 2, \cdots) \). Apart from the possibility \( r = 0 \), the elements of \( \mathcal{R} \) will then occur in pairs, of which each member is the negative of the other, and it is always the case that every element of \( \mathcal{R} \) is either real or pure imaginary, with imaginary part \( \leq \frac{1}{2} \). If one of the \( \lambda_n \)'s happens to be \( \frac{1}{2} \), the corresponding \( r = 0 \) will be counted with double multiplicity in its occurrence on the left side of the trace formula.

Now all the elements of \( \Gamma \) except the identity are hyperbolic. I.e., each \( \gamma \in \Gamma \) is conjugate in \( \text{PSL}(2, R) \) to a unique transformation of the form

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$z \rightarrow e^{t\gamma z}$, where $l_{\gamma}$ is real and positive. For geometric reasons, we will call the number $l_{\gamma}$ the length of the transformation $\gamma$ (cf. [2]). Clearly $l_{\gamma}$ is the same within a conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to $\gamma$ within $\Gamma$ itself. Also, we will call $\gamma \in \Gamma$ primitive, if it is not a positive integral power of any other element of $\Gamma$. Clearly we can speak of a conjugacy class in $\Gamma$ as being primitive. The trace formula then reads

$$
\sum_{\gamma \in \Gamma} h(\gamma) = \frac{A}{2\pi} \int_{-\infty}^{\infty} h(t) \ t \ 	ext{tanh} \ t \ r \ dr + \sum_{\{\gamma\}} \sum_{n=1}^{\infty} (l_{\gamma} \text{csch} \ \frac{1}{2} nl_{\gamma}) \hat{h}(nl_{\gamma}),
$$

where

$$
\hat{h}(x) = (2\pi)^{-1} \int_{0}^{\infty} e^{-ixr} h(r) \ dr,
$$

and the outer sum is taken over all primitive conjugacy classes in $\Gamma$. Moreover, all series in the formula converge absolutely.

In order to study $Z(s)$, it is convenient to begin by studying a more general Dirichlet series. Namely, suppose $e > 0$, and define $Z_{e}(s) = \sum_{n=0}^{\infty} (\lambda_{n} + e)^{-s}$. As before, the series converges absolutely in the half-plane $\Re s > 1$. Next define $\alpha \geq \frac{1}{2}$ by requiring that $\alpha^{2} - \frac{1}{4} = e$. Letting $t$ be a positive number and setting $h(r) = e^{-(\alpha^{2} + r^{2})t}$ in the trace formula, we obtain

$$
\sum_{n=0}^{\infty} e^{-(\lambda_{n} + e)t} = \frac{A}{4\pi} \int_{-\infty}^{\infty} e^{-(\alpha^{2} + r^{2})t} \ t \ 	ext{tanh} \ t \ r \ dr
$$

$$
+ \frac{1}{2} (4\pi t)^{-1/2} \ \sum_{\{\gamma\}} \sum_{n=1}^{\infty} (l_{\gamma} \text{csch} \ \frac{1}{2} nl_{\gamma}) e^{-\frac{1}{4}(\alpha^{2} + r^{2})/t} / 4t
$$

with all series convergent.

Denote by $\theta_{1}(t)$ and $\theta_{2}(t)$, respectively, the first and second terms on the right side of (1). Then $\sum_{n=1}^{\infty} e^{-(\lambda_{n} + e)t} = \theta_{1}(t) + \theta_{2}(t) - e^{-et}$.

Throughout what follows, when we say that a result holds uniformly in $e$, we will mean that it holds uniformly for $e \in [0, 1]$, or what is the same thing, for $\alpha \in [1/2, \sqrt{5}/2]$.

The following two lemmas are obvious.

**Lemma 1.** $\sum_{n=1}^{\infty} e^{-(\lambda_{n} + e)t} = O(e^{-\lambda_{1}t})$ uniformly in $e$, as $t \rightarrow \infty$.

**Lemma 2.** $\theta_{1}(t) = O(e^{-t/4})$ uniformly in $e$, as $t \rightarrow \infty$.

Combining Lemmas 1 and 2, we obtain the following lemma.

**Lemma 3.** $\theta_{2}(t) - e^{-et} = O(e^{-\min(\lambda_{1}, 1/4))t})$ uniformly in $e$, as $t \rightarrow \infty$.

**Lemma 4.** $\theta_{2}(t)$ is of rapid decrease as $t \rightarrow 0$, uniformly in $e$. I.e., for any negative integer $k$, $t^{k} \theta_{2}(t) \rightarrow 0$ as $t \rightarrow 0$, uniformly in $e$. 

Proof. For any $k$, $t^k \theta_2^k(t)$ is equal to a convergent series of positive terms. Moreover, for $t$ less than some $\eta$, which depends on $k$ but can be taken independent of $\epsilon$, the terms all tend monotonically to zero as $t \downarrow 0$. The result then follows from Beppo Levi's theorem.

Lemma 5. Define $\theta_2^\epsilon(0) = 0$. Then for each $\epsilon \geq 0$, $\theta_2^\epsilon(t)$ is continuous, and the series for $\theta_2^\epsilon(t)$ converges uniformly to $\theta_2^0(t)$ on compact subsets of $[0, \infty)$.

Proof. For any $\epsilon \geq 0$, $\theta_2^\epsilon(t)$ is continuous at $t = 0$ by Lemma 4. For $t > 0$, $\theta_2^\epsilon(t)$ is clearly continuous, being a linear combination of three continuous functions. Moreover, since all the terms of the series that defines $\theta_2^\epsilon(t)$ are positive, it follows from Dini's theorem that the series converges uniformly on compact subsets of $[0, \infty)$.

Lemma 6. On any compact subset of $[0, \infty)$, $\theta_2^\epsilon(t) \rightarrow \theta_2^0(t)$ uniformly, as $\epsilon \rightarrow 0$.

Proof. $\theta_2^\epsilon(t)$ increases monotonically to $\theta_2^0(t)$ as $\epsilon \downarrow 0$. The result thus follows from Dini's theorem.

Now suppose $\Re s > 1$. Taking the Mellin transform of $\sum_{n=1}^\infty e^{-(\lambda_n + \epsilon)t}$, and adding $1/s$ to the result, we obtain

$$
\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{4\pi} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{-s} r \tanh \pi r \, dr
$$

or

$$
\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \sech^2 \pi r \, dr
$$

or

$$
\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \sech^2 \pi r \, dr
$$

or

$$
\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{8(s - 1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \sech^2 \pi r \, dr
$$

In view of Lemmas 3 and 4, the right side of the last equation gives, for any $\epsilon \geq 0$, a meromorphic continuation of $\Gamma(s)Z_\epsilon(s) + 1/s$, and hence of $\Gamma(s)Z_\epsilon(s)$, into $\Re s > -1$, and indeed, into the whole plane if $\epsilon = 0$ (since the first and third integrals are entire for any $\epsilon \geq 0$, and the second integral is holomorphic in $\Re s > -1$, and entire if $\epsilon = 0$). If $\epsilon > 0$, and we observe, using the power series for $e^{-\epsilon t}$, that $\int_0^1 (1 - e^{-\epsilon t}) t^s \, dt/t$ can be continued to the left of $\Re s > -1$, with simple poles at the negative integers, we obtain a meromorphic continuation of $\Gamma(s)Z_\epsilon(s)$ into the entire plane in this case as well. Since it is clear from this that the only possible poles of $\Gamma(s)Z_\epsilon(s)$ are simple poles at $1, 0, -1, -2, \cdots$, with the pole at $s = 1$ always present, we
conclude that for any $\epsilon > 0$, $Z_\epsilon(s)$ can be meromorphically continued into the whole plane, with a single simple pole at $s = 1$, having residue $A/4\pi$ (since $\int_{-\infty}^{\infty} \sech^2 \pi r \, dr = 2/\pi$).

Suppose now $\Re s > -1$, and $s \neq 0, 1$. Then in view of Lemma 6, it is evident, by inspecting the right side of (2), bearing in mind Lemmas 3 and 4, that $\Gamma(s)Z_\epsilon(s) + 1/s \to \Gamma(s)Z(s) + 1/s$, and hence $\Gamma(s)Z_\epsilon(s) \to \Gamma(s)Z(s)$, as $\epsilon \to 0$.

On the other hand, if $\Re s > 0$,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \sech^2 \pi r \, dr + \int_{0}^{\infty} (\theta_{\epsilon}^s(t) - e^{-\epsilon t}) t^s \frac{dt}{t},$$

and if $\epsilon > 0$, it is permissible to split the last integral into two integrals, since it follows from Lemma 3 that for positive $\epsilon$, $\theta_{\epsilon}^s(t)$ is of exponential decrease as $t \to \infty$. We thus obtain, at first for $\Re s > 0$, and then for the whole plane by analytic continuation,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\alpha^2 + r^2)^{1-s} \sech^2 \pi r \, dr + \int_{0}^{\infty} \theta_{\epsilon}^s(t) t^s \frac{dt}{t} - \Gamma(s)\epsilon^{-s}.$$

Now taking $-1 < \Re s < 0$, letting $\epsilon \to 0$, and bearing in mind that by Lemma 3, $\theta_{\epsilon}^s(t) = O(1)$ uniformly in $\epsilon$, as $t \to \infty$, we find that

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\lambda_4 + r^2)^{1-s} \sech^2 \pi r \, dr + \int_{0}^{\infty} \theta_{\epsilon}^s(t) t^s \frac{dt}{t}.$$

Since both integrals are holomorphic in $\Re s < 0$, we have obtained an expression for $\Gamma(s)Z(s)$ in the left half-plane.

Let us examine $\int_{0}^{\infty} \theta_{\epsilon}^0(t) t^s \frac{dt}{t}$, assuming $\Re s < 0$. Now

$$\int_{0}^{\infty} e^{-(t^2 + (n\lambda)^2)\lambda^4} t^s-1/2 \frac{dt}{t} = 2(n\lambda)^{s-1/2} K_{1/2-s}(\frac{1}{2}n\lambda) \quad [5, \text{p. 183}],$$

so we obtain the following result (the interchange of summation and integration being justified by Lemma 5):

**Theorem 1.** If $\Re s < 0$,

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)} \int_{-\infty}^{\infty} (\lambda_4 + r^2)^{1-s} \sech^2 \pi r \, dr + (4\pi)^{-1/2} \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma/n)^{1/2} \left(\operatorname{csch} \frac{1}{2}n\lambda_\gamma\right) (n\lambda)^s K_{1/2-s}(\frac{1}{2}n\lambda_\gamma).$$

Now if $\Re s < \frac{1}{2},$. 

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so if we define $\phi_s(A)$, for $\Re s > \frac{1}{2}$ and positive $A$, by setting $\phi_s(A) = (2\pi)^{-1} \int_0^\infty ((A/2)^2 + x^2)^{-s} \cos x \, dx$, we obtain the following reformulation of Theorem 1.

**Theorem 2.** If $\Re s < 0$,

$$
\Gamma(s) Z(s) = \frac{A \Gamma(s)}{8(s-1)} \int_{-\infty}^\infty (\frac{1}{4} + r^2)^{1-s} \sech^2 \pi r \, dr
$$

$$
+ \Gamma(1-s) \sum_{\gamma \neq 0} \sum_{n=1}^\infty \lambda(\csch \frac{1}{2} n \lambda) \phi_{1-s}(n \lambda).
$$

Suppose now $\Re s > 1$. Then

$$
\frac{A \Gamma(s)}{8(s-1)} \int_{-\infty}^\infty (\frac{1}{4} + r^2)^{1-s} \sech^2 \pi r \, dr = \frac{A \Gamma(s)}{2\pi} \int_0^\infty (\frac{1}{4} + r^2)^{-s} r \tanh \pi r \, dr.
$$

Now as we have pointed out, it is well known that $\Sigma_{\lambda_n < T} 1 \sim AT/4\pi$, so it follows that $\Sigma_{0 < r_n < T} 1 \sim AT^2/4\pi$. But $(A/2\pi) \int_0^T r \tanh \pi r \, dr \sim AT^2/4\pi$, and in view of the trace formula, is the correct principal term in the asymptotic analysis of $\Sigma_{0 < r_n < T} 1$. This suggests defining a remainder term

$$
R(T) = \sum_{0 < r_n < T} 1 - \frac{A}{2\pi} \int_0^T r \tanh \pi r \, dr.
$$

Then if we denote the eigenvalues in $(0, \frac{1}{4})$ by $\lambda_1, \cdots, \lambda_N$, and define $\lambda(r) = \frac{1}{4} + r^2$, Theorem 2 and the previous arguments tell us that $\Gamma(s) \{ \Sigma_{n=1}^N \lambda_n^{-s} + \int_0^\infty (\lambda(r))^{-s} dR(r) \}$ can be meromorphically continued from $\Re s > 1$ into the whole plane, and for $\Re s < 0$, equals $\Gamma(1-s)\Phi(1-s)$, where $\Phi(s)$ is defined in the half-plane $\Re s > 1$ by setting

$$
\Phi(s) = \sum_{\gamma \neq 0} \sum_{n=1}^\infty \lambda(\csch \frac{1}{2} n \lambda) \phi_s(n \lambda).
$$

If, now, we define $R^*(T) = R(\sqrt{T - \frac{1}{4}})$, integrate by parts, and make the change of variable $\lambda = \lambda(r)$, the previous statement becomes the statement that $\Gamma(s) \{ \Sigma_{n=1}^N \lambda_n^{-s} + \int_{1/4}^\infty \lambda^{-s-1} R^*(\lambda) \, d\lambda \}$ can be meromorphically continued into the whole plane, and for $\Re s < 0$, equals $\Gamma(1-s)\Phi(1-s)$.

Thus, setting

$$
\Psi(s) = s \int_{1/4}^\infty \lambda^{-s-1} R^*(\lambda) \, d\lambda = s \int_{\log 1/4}^\infty e^{-\lambda s} R^*(e^\lambda) \, d\lambda = \int_{\log 1/4}^\infty e^{-\lambda s} dR^*(e^\lambda),
$$
we find that $\Psi(s)$, the Laplace transform of the exponential form of the eigenvalue remainder measure, satisfies the following identity:

**Theorem 3.** If $\text{Re } s < 0$, $\Gamma(s) \{ \sum_{n=1}^{N} \lambda_n^{-s} + \Psi(s) \} = \Gamma(1 - s) \Phi(1 - s)$.

**Corollary.** If there are no eigenvalues in $(0, \frac{1}{2})$ and $\text{Re } s < 0$, we have $\Gamma(s)\Psi(s) = \Gamma(1 - s)\Phi(1 - s)$.

**REFERENCES**


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