

NIL AND POWER-CENTRAL POLYNOMIALS IN RINGS⁽¹⁾

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ABSTRACT. A polynomial in noncommuting variables is *vanishing*, *nil* or *central* in a ring, R , if its value under every substitution from R is 0, nilpotent or a central element of R , respectively.

THEOREM. *If R has no nonvanishing multilinear nil polynomials then neither has the matrix ring R_n .* **THEOREM.** *Let R be a ring satisfying a polynomial identity modulo its nil radical N , and let f be a multilinear polynomial. If f is nil in R then f is vanishing in R/N . Applied to the polynomial $xy-yx$, this establishes the validity of a conjecture of Herstein's, in the presence of polynomial identity.* **THEOREM.** *Let m be a positive integer and let F be a field containing no m th roots of unity other than 1. If f is a multilinear polynomial such that for some $n > 2$ f^m is central in F_n , then f is central in F_n .*

This is related to the (non)existence of noncrossed products among p^2 -dimensional central division rings.

Introduction. Let C be a fixed domain of operators which we assume to be a commutative ring with unity. All rings are unital C -algebras and all polynomials have their coefficients in C . Let R be a ring, $f(x_1, \dots, x_k)$ a polynomial in noncommuting variables. f is said to be *vanishing*, *nil* or *central* in R if under any substitution from R its value is 0, nilpotent or a central element of R , respectively. f is *multilinear* if it is homogeneous and linear in every one of its variables. In §1 we prove that the $n \times n$ -matrix ring R_n has no nonvanishing multilinear nil polynomials if R has none. This is the case, for example, when R has no nonzero nilpotent elements. If R satisfies a polynomial identity modulo its nil radical and $f(x_1, \dots, x_k)$ is a multilinear polynomial which is nil in R , it is then shown (§2) that the ideal $f(R)$ generated by the elements $f(r_1, \dots, r_k)$, $r_i \in R$, is nil. Applied to the polynomial $x_1x_2 - x_2x_1$, this establishes the validity of a conjecture of Herstein's [4, p. 30], in the presence of polynomial identity. (This corollary has been known to some people for some years, but has apparently never been published.) A polynomial is called *power-*

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DEFINITION. Let u be a simple sequence. Then u is called *even* if for some σ , $|u^\sigma| = be_{ii} \neq 0$, and *odd* if for some σ , $|u^\sigma| = be_{ij} \neq 0$ where $i \neq j$.

These terms are well defined by Lemma 1, and are explained by the observation that if $i = j$ (respectively $i \neq j$) then i and j have an even (respectively odd) number of occurrences in the indices of the unit matrices of u . Note that if $|u^\sigma| = 0$ for all σ , then u is neither odd nor even.

LEMMA 2 (REGEV). Let R be a ring, $f(x_1, \dots, x_k)$ a multilinear polynomial. Let $u = (A_1, \dots, A_k)$ be a simple sequence of matrices from R_n .

(a) If u is even then the matrix $f(u)$, obtained by substituting u in f , is diagonal.

(b) If u is odd then $f(u) = ae_{ij}$ for some $a \in R$ and $i \neq j$.

PROOF. Since f is multilinear, it has the form $f(x_1, \dots, x_k) = \sum_{\sigma \in S_k} c_\sigma x_{\sigma(1)}, \dots, x_{\sigma(k)}$, where S_k is the symmetric group on $\{1, \dots, k\}$ and $c_\sigma \in C$. Thus $f(u) = \sum_{\sigma \in S_k} c_\sigma |u^\sigma|$. Let $T = \{\sigma \in S_k \mid |u^\sigma| \neq 0\}$, and note that in the sum for $f(u)$ it suffices to let σ range over T .

(a) If u is even then, for some $\sigma \in T$, $|u^\sigma|$ is of the form ae_{ii} , whence, by Lemma 1, $|u^\sigma| = a_\sigma e_{i_\sigma i_\sigma}$ for every $\sigma \in T$. Thus

$$f(u) = \sum_{\sigma \in T} c_\sigma |u^\sigma| = \sum_{\sigma \in T} c_\sigma a_\sigma e_{i_\sigma i_\sigma},$$

that is, a diagonal matrix.

(b) If u is odd then for some $\sigma \in T$ $|u^\sigma|$ is of the form ae_{ij} with $i \neq j$; hence, by Lemma 1, $|u^\sigma| = a_\sigma e_{ij}$ for every $\sigma \in T$. Thus

$$\begin{aligned} f(u) &= \sum_{\sigma \in T} c_\sigma |u^\sigma| = \sum_{\sigma \in T} c_\sigma a_\sigma e_{ij} \\ &= \left(\sum_{\sigma \in T} c_\sigma a_\sigma \right) e_{ij} = ae_{ij}, \text{ where } a = \sum_{\sigma \in T} c_\sigma a_\sigma, \end{aligned}$$

and Lemma 2 is proved.

Lemmas 1 and 2 will now be applied to discuss polynomial identities and nil polynomials in matrix rings.

LEMMA 3. Let R be a ring and $f(x_1, \dots, x_k)$ a multilinear polynomial. If f vanishes under every even substitution from R_n then f vanishes in R_n .

PROOF. Since f is multilinear and the matrices of the form ae_{ij} , $a \in R$, generate R_n additively, it will suffice to show that f vanishes under every simple substitution $u = (A_1, \dots, A_k)$ from R_n . This vanishing is given for even substitutions, so assume u is odd. By Lemma 2, $f(u) = ae_{ij}$ for some $a \in R$ and $i \neq j$, and we wish to show $a = 0$. Consider the invertible matrix $A = 1 + e_{ji}$ (its inverse is $A^{-1} = 1 - e_{ji}$) and the inner automorphism $\varphi: x \mapsto Ax A^{-1}$ it induces in R_n . Writing u' for the image of the sequence u under φ , we have

$$f(u^\varphi) = f(u)^\varphi = ae_{ij}^\varphi = a(e_{jj} - e_{ii} + e_{ij} - e_{ji}),$$

since φ leaves a fixed. Now u^φ may not be a simple sequence, but we can write $f(u^\varphi) = \sum_{r=1}^m f(u^{(r)})$, where the $u^{(r)}$ are simple. We claim that the entries on the main diagonal of the matrix $f(u^{(r)})$ are all 0 for $r = 1, \dots, m$. Indeed, this is given for even $u^{(r)}$ and follows from Lemma 2 for odd $u^{(r)}$. Thus the main diagonal also vanishes in the matrix $f(u^\varphi) = a(e_{jj} - e_{ii} + e_{ij} - e_{ji})$ and, given $i \neq j$, this forces $a = 0$. Hence $f(u) = ae_{ij} = 0$, and the proof is completed.

REMARK. Lemma 3 asserts that a multilinear polynomial which vanishes under even substitutions from R_n must also vanish under odd ones. The converse of this statement is false for every n . For if R is, for example, a field of characteristic zero, then by [3] there exists a multilinear polynomial f , which is central and nonvanishing in R_n ; and, by Lemma 2, f vanishes under every odd substitution from R_n .

The main theorem of this section follows.

THEOREM 4. *If in a ring R every multilinear nil polynomial vanishes, then the same holds for R_n .*

PROOF. Suppose $f(x_1, \dots, x_k)$ is a multilinear polynomial which is nil in R_n . Then we shall show that f vanishes in R_n . By Lemma 3 we need only show that $f(u) = 0$ whenever

$$u = (a_1 e_{i_1 j_1}, \dots, a_k e_{i_k j_k})$$

is even. To this end, consider the matrices $x_\nu e_{i_\nu j_\nu}$, $\nu = 1, \dots, k$, with polynomial entries. Substitute these matrices in f and write

$$f(x_1 e_{i_1 j_1}, \dots, x_k e_{i_k j_k}) = \begin{pmatrix} f_1(x_1, \dots, x_k) & & 0 \\ & \ddots & \\ 0 & & f_n(x_1, \dots, x_k) \end{pmatrix},$$

where the f_i are multilinear polynomials over C in the variables x_1, \dots, x_k . The off-diagonal entries are guaranteed by Lemma 2 to be 0, since the original sequence u was even. If we now make any substitution $x_i = b_i \in R$ in the above relation, then the left-hand member becomes a nilpotent matrix since f is nil in R_n . Therefore, the diagonal matrix on the right also becomes nilpotent and it follows that each polynomial f_i is nil in R , hence vanishes in R . In particular, by substituting $x_i = a_i$, we obtain $f(u) = 0$. As mentioned above this completes the proof of the theorem.

The following special case of Theorem 4 is of particular interest.

COROLLARY 5. *Let R be a ring with no nonzero nilpotent elements and let f be a multilinear polynomial. If f is nil in R_n then f vanishes in R_n .*

Examples of rings R as in Corollary 5 are fields, division rings and direct products thereof.

REMARK. If R is a ring with no nonzero nilpotent elements, it is an open question whether nonvanishing nil polynomials (necessarily nonmultilinear) for R_n exist. In case R is an infinite field the answer is negative, as follows from [1, Theorem 4], by noting that if f is nil in R_n then f^n must vanish in R_n .

2. **A generalized Herstein's conjecture.** One of the principal problems concerning nil polynomials is the following open problem, which was formulated by Herstein [4, p. 30] for the case of the polynomial $x_1x_2 - x_2x_1$: If $f(x_1, \dots, x_k)$ is nil in R , is the ideal generated in R by the elements $f(r_1, \dots, r_k)$ nil? The answer is known to be "yes" in case $f(x_1, x_2) = x_1x_2 - x_2x_1$ and the indices of nilpotency are bounded [ibid.]. In this case the ring satisfies the identity $(x_1x_2 - x_2x_1)^m$ for some m . We shall now show that the answer is still positive if R satisfies any polynomial identity modulo its nil radical and f is any multilinear polynomial. We start with a generalization to nil polynomials, of Kaplansky's classical theorem on primitive rings satisfying a polynomial identity.

THEOREM 6. *Let R be a (Jacobson) semisimple ring and let $f(x_1, \dots, x_k)$ be a multilinear polynomial of degree k which is nil in R . Then f vanishes in R . If, in particular, R is primitive, then it is a central simple algebra of dimension $\leq [k/2]^2$ over its center.*

PROOF. If $R = D_n$ is a matrix ring over a division ring, then f vanishes in R by Corollary 5. Suppose next that R is primitive. Then there exists a division ring D such that either $R \cong D_n$ for some n , or D_n is a homomorphic image of a subring of R for every n . But the latter is impossible for it leads to the absurd conclusion that f is nil (hence vanishes) in D_n for every n . Thus $R \cong D_n$ for some n and f vanishes in R . Finally, if R is semisimple then it is a subdirect product of primitive rings R_i and f is nil in each R_i . Thus f vanishes in each R_i , hence also in R .

THEOREM 7. *Let R be a ring, N its maximal nil ideal, and suppose R/N satisfies a polynomial identity. Let $f(x_1, \dots, x_k)$ be a multilinear polynomial which is nil in R . Then the ideal $f(R)$, generated in R by all the elements $f(r_1, \dots, r_k)$, $r_i \in R$, is nil.*

PROOF. Suppose first that R is an algebra over a field. By considering R/N instead of R , we may suppose that R satisfies a polynomial identity and has no nonzero nil ideals. We then wish to show that $f(R) = \{0\}$, i.e. f vanishes in R . Consider any element a of $f(R)$. a can be written as a finite sum of terms of the form $xf(r_1, \dots, r_k)y$, where $x, y, r_1, \dots, r_k \in R$. If R_0 is the subalgebra of

R generated by all the elements appearing in such an expression for a , then clearly $a \in f(R_0)$. Let J_0 be the (Jacobson) radical of R_0 . Then f is nil in the semisimple algebra R_0/J_0 and, by Theorem 6, f vanishes in R_0/J_0 . Thus $f(R_0) \subseteq J_0$ and in particular $a \in J_0$. But J_0 , being the radical of a finitely generated polynomial identity algebra over a field, is nil [2] and so a is nilpotent. Since a was an arbitrary element of $f(R)$, $f(R)$ is seen to be a nil ideal, hence $f(R) = \{0\}$. This establishes the theorem for algebras over fields. The transition to general rings is standard and will be omitted (see e.g. [5, p. 416]).

Applying Theorem 7 to the polynomial $f(x_1, x_2) = x_1x_2 - x_2x_1$, one obtains a proof of Herstein's conjecture, in the presence of polynomial identity.

COROLLARY 8. *Let R be a ring, N its maximal nil ideal, and suppose R/N satisfies a polynomial identity. If all the commutators in R are nilpotent, then the ideal $C(R)$ generated by them is nil. (Equivalently, the collection of nilpotent elements of R is an ideal.)*

REMARKS. (a) Theorem 7 asserts that the condition that R/N satisfies a polynomial identity is sufficient for the validity of the "generalized Herstein's conjecture". It is interesting to note that this condition is also necessary. For if R is any ring and the ideal $f(R)$ is nil, then f vanishes in R/N .

(b) Theorem 7 and Corollary 8 still hold if, instead of assuming that R/N satisfies a polynomial identity, we assume that R is an algebra over an uncountable field.

3. Power-central polynomials.

LEMMA 9. *Let R be a ring, $f(x_1, \dots, x_k)$ a multilinear polynomial and $n > 2$ an integer. If $f(u) \in \text{Center}(R_n)$ for every even substitution u from R_n , then f is central in R_n . The conclusion still holds if $n = 2$ and $2a \neq 0$ for all $a \neq 0$ in R .*

The proof is similar to that of Lemma 3, but we repeat it for completeness.

PROOF. It is enough to show that f vanishes under every odd substitution from R_n . Let u be odd and let $f(u) = ae_{ij}$ for some $a \in R$ and $i \neq j$. Applying the same transformation φ as in the proof of Lemma 3, we have $f(u^\varphi) = a(e_{jj} - e_{ii} + e_{ij} - e_{ji})$. Write $f(u^\varphi) = \sum_{r=1}^m f(u^{(r)})$ where $u^{(r)}$ are all simple. Then, by Lemma 2, the sum over all the even $u^{(r)}$'s must equal $a(e_{jj} - e_{ii})$. But by the assumption of the lemma, this sum is in the center of R_n . Thus $a(e_{jj} - e_{ii}) \in \text{Center}(R_n)$. Since we are assuming $n > 2$ or $a \neq -a$, this forces $a = 0$. Hence $f(u) = ae_{ij} = 0$ and f is central.

Note that the restriction in Lemma 9 is essential, for if F is a field of characteristic 2, then the commutator $[x_1, x_2] = x_1x_2 - x_2x_1$ is a multilinear polynomial and is central under even substitutions from F_2 , but $[e_1, e_{12}] = e_{12} \notin \text{Center}(F_2)$.

THEOREM 10. *Let f be a multilinear polynomial, m and $n > 2$ positive integers, and F a field containing no m th roots of unity other than 1. If f^m is central in F_n , then so must be f . The conclusion holds also for $n = 2$, provided $\text{char}(F) \neq 2$.*

PROOF. Assume that f^m is central in F_n and let u be an even substitution from F_n . By Lemma 9, we need only show that $f(u) \in \text{Center}(F_n)$. Since u is even, $f(u)$ is a diagonal matrix, $f(u) = \sum_{i=1}^n a_i e_{ii}$, $a_i \in F$. We may assume $f(u) \neq 0$, say $a_1 \neq 0$. Since $f^m(u) = \sum_{i=1}^m a_i^m e_{ii}$ is in the center of F_n , we have $a_1^m = a_2^m = \cdots = a_n^m$. Thus, for any $1 < i \leq n$, $(a_i/a_1)^m = 1$; so $a_i/a_1 = 1$ and $a_i = a_1$, that is, $f(u) = \sum_{i=1}^n a_i e_{ii} \in \text{Center}(F_n)$.

Let Q denote the field of rational numbers, and for a prime q let $Z_q(t)$ denote the field of rational functions in t over the field of q elements. Let F denote either Q or $Z_q(t)$ and $q = \text{char} F$. In [6] Schacher and Small proved the equivalence of the following statements for an odd prime p :

(a) Every p^2 -dimensional division ring of characteristic q (0 or a prime) is a crossed product.

(b) There exists for F_p a noncentral polynomial f such that f^p is central.

Whether these statements are true or false is still an open problem (for $p \geq 5$).

However, we have the following.

COROLLARY 11. *Let q and $p > 2$ be primes such that $q \leq p$, and let F be Q or $Z_q(t)$. Then F_p has no noncentral multilinear polynomial f such that f^p is central.*

PROOF. Note that F has no p th roots of unity other than 1, then apply Theorem 10.

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