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SELECTION THEOREMS AND THE REDUCTION PRINCIPLE

BY

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ABSTRACT. In this paper we attempt a unification of several selection theorems in the literature. It is proved that the existence of "nice" selectors in a certain class of selection problems is essentially equivalent to the fact that certain families of sets satisfy a weak version of Kuratowski's reduction principle. Various special cases are discussed.

1. **Introduction.** Several authors, notably Kuratowski and Ryll-Nardzewski [6] and Michael [8], have proved selection theorems which may loosely be described as follows: X is a set, Φ a family of subsets of X, Y is a complete metric space, $F: X \to 2^Y$, the space of nonempty closed subsets of Y, satisfies the condition:

$$\{x \in X: F(x) \cap V \neq \varnothing\} \in \Phi$$

for each open set V in Y; then, under suitable conditions on Φ and Y, the writers mentioned above establish the existence of "nice" selectors for F. The condition imposed on Φ by Kuratowski and Ryll-Nardzewski is of a set-theoretic nature, while Michael's conditions are topological.

The aim of this article is to unify these theorems. Indeed, it is shown that the existence of "nice" selectors for the class of selection problems described above is essentially equivalent to the fact that Φ satisfies a weak version of the reduction principle of Kuratowski [5]. It then becomes possible to derive various well-known selection theorems by simply noting that certain families satisfy the reduction principle.

In §2 we give the basic definitions and notation. §3 contains the statements and proofs of the main results. In §4 we deduce several known and some new results from our main theorems. §5 deals with the more general problem of finding selectors for functions $F: X \to P(Y) - \{\varnothing\}$, where Y is, as usual, a complete

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metric space and $P(Y)$ is the power set of Y. Here we formulate a theorem which is essentially an abstraction of the method used by Shchegolkov (cf. §15 in [1]) to prove that a planar Borel set, all of whose vertical sections are F_σ sets, can be uniformized by a Borel set.

2. **Definitions and notation.** Throughout X is a set, $\mathcal{F}$ a family of subsets of X. We identify cardinals with initial ordinals. λ, μ stand for infinite cardinals, α, β, γ for ordinals. λ^+ is the successor cardinal to λ.

Say that $\mathcal{F}$ is *λ-additive* (*λ-multiplicative*) if whenever $\{A_\alpha, \alpha < \beta\} \subseteq \mathcal{F}$ and $\beta < \lambda$, $\bigcup_{\alpha<\beta} A_\alpha \in \mathcal{F}$ ($\bigcap_{\alpha<\beta} A_\alpha \in \mathcal{F}$). Denote by $\mathcal{F}_\lambda$ the smallest λ-additive family containing $\mathcal{F}$. We shall sometimes write $\mathcal{F}_\sigma$ for $\mathcal{F}_{\aleph_1}$. $\mathcal{F}$ is said to be *subtractive* if $A, B \in \mathcal{F}$ implies that $A - B \in \mathcal{F}$. $\mathcal{F}^c$ stands for the family of subsets of X whose complements belong to $\mathcal{F}$.

According to Kuratowski [5], $\mathcal{F}$ satisfies the *reduction principle* (and we write RP($\mathcal{F}$)) just in case whenever $A_1, A_2 \in \mathcal{F}$, there exist sets B_1, B_2 such that (i) $B_i \in \mathcal{F}$, $i = 1, 2$, (ii) $B_i \subseteq A_i$, $i = 1, 2$, (iii) $B_1 \cap B_2 = \emptyset$ and (iv) $B_1 \cup B_2 = A_1 \cup A_2$. Analogously, say that $\mathcal{F}$ satisfies the *λ-reduction principle* (and write λ-RP($\mathcal{F}$)) if whenever $\{A_\alpha, \alpha < \beta\} \subseteq \mathcal{F}$ and $\beta < \lambda$, there exist sets B_α, $\alpha < \beta$, such that

(a) $(\forall \alpha < \beta)(B_\alpha \in \mathcal{F})$,

(b) $(\forall \alpha < \beta)(B_\alpha \subseteq A_\alpha)$,

(c) $(\forall \alpha, \alpha' < \beta)(\alpha \neq \alpha' \rightarrow B_\alpha \cap B_{\alpha'} = \emptyset)$ and

(d) $\bigcup_{\alpha<\beta} B_\alpha = \bigcup_{\alpha<\beta} A_\alpha$.

For our purposes, however, a weaker version of the above principle is needed. Accordingly, we make the following definitions. $\mathcal{F}$ satisfies the *weak reduction principle* (and we write WRP($\mathcal{F}$)) if whenever $A_1, A_2 \in \mathcal{F}$ and $A_1 \cup A_2 = X$, there exist sets B_1, B_2 satisfying conditions (i)–(iv) of the previous paragraph. $\mathcal{F}$ satisfies the *λ-weak reduction principle* just in case whenever $\{A_\alpha, \alpha < \beta\} \subseteq \mathcal{F}$, $\beta < \lambda$ and $\bigcup_{\alpha<\beta} A_\alpha = X$, there exist sets B_α, $\alpha < \beta$, satisfying conditions (a)–(d) of the previous paragraph. We use λ-WRP($\mathcal{F}$) as an abbreviation of "$\mathcal{F}$ satisfies the λ-weak reduction principle."

Let Y be a complete metric space. A function $F: X \rightarrow 2^Y$ is said to be *$\mathcal{F}$-normal* if $\{x \in X: F(x) \cap V \neq \emptyset\} \in \mathcal{F}$ for each open set V in Y. In case X is a topological space and $\mathcal{F}$ is the family of open subsets of X, a $\mathcal{F}$-normal function is also called a *lower semicontinuous* (l.s.c.) function. A function $f: X \rightarrow Y$ is said to be a *selector* for $F: X \rightarrow 2^Y$ just in case $(\forall x \in X)(f(x) \in F(x))$. $f: X \rightarrow Y$ is said to be *$\mathcal{F}$-measurable* if $f^{-1}(V) \in \mathcal{F}$ for every open set V in Y.

Say that a set $A \subseteq [0, 1] \times [0, 1]$ is *elementary coanalytic* if there exists a sequence A_n, $n = 1, 2, \cdots$, of coanalytic subsets of $[0, 1] \times [0, 1]$ such that

(i) $A_n \supseteq A_{n+1}$, $n = 1, 2, \cdots$.

(ii) $A = \bigcap_{n=1}^{\infty} A_n$.

(iii) For each $x \in [0, 1]$ and each $n \geqslant 1$, A_n^x ($\stackrel{\text{def}}{=} \{y \in [0, 1]: (x, y) \in A_n\}$) is empty or a finite union of nondegenerate closed intervals.

The rest of our terminology and notation is quite standard and we refer the reader to [2] and [4].

3. Main results.

THEOREM 1. *Let X be a set, Φ a family of subsets of X such that $\varnothing$, $X \in \Phi$, Φ is λ^+-additive and $\aleph_0$-multiplicative. Then the following conditions are equivalent*:

(a) λ^+-WRP(Φ).

(b) *If Y is a complete metric space which has topological weight $\leqslant \lambda$, then any Φ-normal function F: $X \to 2^Y$ admits a $(\Phi \cap \Phi^c)_{\lambda^+}$-measurable selector.*

(c) *If Z is a discrete space of cardinality $\leqslant \lambda$, then any Φ-normal function F: $X \to 2^Z$ admits a $(\Phi \cap \Phi^c)$-measurable selector.*

PROOF. (a)$\to$(b). We follow the idea of the proof of the main theorem in [6].

Let d be a complete metric on Y such that the diameter of Y is < 1. It suffices to define a sequence of functions f_n: $X \to Y$, $n = 0, 1, 2, \cdots$, such that

(i) $(\forall x \in X)(d(f_n(x), F(x)) < 2^{-n})$, $n \geqslant 0$,

(ii) $(\forall x \in X)(d(f_n(x), f_{n-1}(x)) < 2^{-(n-2)})$, $n > 0$, and

(iii) f_n is $(\Phi \cap \Phi^c)_{\lambda^+}$-measurable, $n \geqslant 0$.

For then setting $f(x) = \lim_n f_n(x)$, $x \in X$, we get a selector f for F; since $\lambda \geqslant \aleph_0$, it follows from the lemma of §1 in [6] that f is $(\Phi \cap \Phi^c)_{\lambda^+}$-measurable.

To construct the functions f_n, choose a dense set D in Y such that $\mathrm{Card}(D) \leqslant \lambda$. Enumerate the elements of D: $r_0, r_1, \cdots, r_\alpha, \cdots, \alpha < \mathrm{Card}(D)$. Define $f_0 \equiv r_0$. Let $n > 0$ and suppose f_{n-1} has been defined to satisfy (i)–(iii) above. Let

$$C_\alpha^n = \{x: d(r_\alpha, F(x)) < 2^{-n}\}, \qquad \alpha < \mathrm{Card}(D),$$

$$D_\alpha^n = \{x: d(r_\alpha, f_{n-1}(x)) < 2^{-(n-2)}\}, \qquad \alpha < \mathrm{Card}(D),$$

$$A_\alpha^n = C_\alpha^n \cap D_\alpha^n, \qquad \alpha < \mathrm{Card}(D).$$

Plainly, the sets $A_\alpha^n \in \Phi$. Claim: $X = \bigcup_{\alpha < \mathrm{Card}(D)} A_\alpha^n$. To see this, let $x \in X$. Choose $y \in F(x)$ such that $d(y, f_{n-1}(x)) < 2^{-(n-1)}$. Pick $r_\alpha \in D$ so that

$d(y, r_\alpha) < 2^{-n}$. It follows that $d(r_\alpha, F(x)) < 2^{-n}$ and $d(r_\alpha, f_{n-1}(x)) < 2^{-(n-2)}$, so $x \in A_\alpha^n$.

Since λ^+-WRP(Φ), there exist sets B_α^n, $\alpha < \mathrm{Card}(D)$, such that $B_\alpha^n \in \Phi$, $B_\alpha^n \subseteq A_\alpha^n$, $B_\alpha^n \cap B_\beta^n = \emptyset$, if $\alpha \neq \beta$, and $\bigcup_{\alpha<\mathrm{Card}(D)} B_\alpha^n = X$. In fact, since Φ is λ^+-additive, the sets $B_\alpha^n \in (\Phi \cap \Phi^c)$. Now define $f_n\colon X \to Y$ as follows: set $f_n(x) = r_\alpha$, if $x \in B_\alpha^n$. A routine verification shows that f_n satisfies conditions (i)–(iii) and the proof of (a) $\to$ (b) is complete.

(b) $\to$ (c) is entirely trivial.

(c) $\to$ (a). Let $\{A_\alpha, \alpha < \beta\} \subseteq \Phi$, let $\beta < \lambda^+$ and suppose that $\bigcup_{\alpha<\beta} A_\alpha = X$. Take $Z = \beta$, i.e., the set of ordinals less than β. Equip Z with the discrete topology. Note that $\mathrm{Card}(Z) \leqslant \lambda$. Define $F\colon X \to 2^Z$ as follows:

$$F(x) = \{\alpha \in Z\colon x \in A_\alpha\}.$$

Since Φ is λ^+-additive, it follows that F is Φ-normal. Consequently, there is a $(\Phi \cap \Phi^c)$-measurable selector f for F. For $\alpha \in Z$, let $B_\alpha = f^{-1}(\{\alpha\})$. Verify that $B_\alpha \in \Phi$, $B_\alpha \cap B_{\alpha'} = \emptyset$, if $\alpha \neq \alpha'$, $B_\alpha \subseteq A_\alpha$ and $\bigcup_{\alpha<\beta} B_\alpha = X$. Hence λ^+-WRP(Φ). This completes the proof of Theorem 1.

The preceding result can be stated more elegantly for compact metric spaces.

THEOREM 2. *Let X be a set, Φ a family of subsets of X such that $\emptyset$, $X \in \Phi$, Φ is $\aleph_1$-additive and $\aleph_0$-multiplicative. Then the following conditions are equivalent:*

(a) *Φ^c satisfies the first principle of separation, i.e., whenever $E, F \in \Phi^c$ and $E \cap F = \emptyset$, there is a set $H \in (\Phi \cap \Phi^c)$ such that $E \subseteq H$ and $H \cap F = \emptyset$.*

(b) WRP(Φ).

(c) $\aleph_0$-WRP(Φ).

(d) *If Y is a compact metric space, then any Φ-normal function $F\colon X \to 2^Y$ admits a $(\Phi \cap \Phi^c)_\sigma$-measurable selector.*

(e) *If Z is a finite discrete space, then any Φ-normal function $F\colon X \to 2^Z$ admits a $(\Phi \cap \Phi^c)$-measurable selector.*

PROOF. The implications (a) $\leftrightarrow$ (b), (d) $\to$ (e) $\to$ (*b*) are quite obvious.

(b)$\to$(c). We shall establish (c) by induction. Suppose the proposition is true for m sets, $m \geqslant 2$. Let $A_1, A_2, \cdots, A_{m+1} \in \Phi$ and suppose that $\bigcup_{i=1}^{m+1} A_i = X$. By induction hypothesis, we can "reduce" the sets $A_1, A_2, \cdots, A_{m-1}, A_m \cup A_{m+1} \in \Phi$. We get sets $B_1, B_2, \cdots, B_m \in \Phi$ such that $B_i \cap B_j = \emptyset$ for $1 \leqslant i \neq j \leqslant m$, $B_i \subseteq A_i$, $1 \leqslant i \leqslant m-1$, $B_m \subseteq A_m \cup A_{m+1}$ and $\bigcup_{i=1}^m B_i = X$. Since Φ is $\aleph_0$-additive, $B_i \in (\Phi \cap \Phi^c)$, $1 \leqslant i \leqslant m$. Consequently, $B_m - A_m$, $B_m - A_{m+1} \in \Phi^c$. Moreover, $(B_m - A_m) \cap (B_m - A_{m+1}) = \emptyset$. Since (b), or equivalently (a), holds, there exists

$V \in (\Phi \cap \Phi^c)$ such that $B_m - A_m \subseteq V$ and $V \cap (B_m - A_{m+1}) = \varnothing$. Hence $B_m - V \subseteq A_m$ and $B_m \cap V \subseteq A_{m+1}$. Also $B_m - V, B_m \cap V \in \Phi$. Now set $C_i = B_i$, $1 \leqslant i \leqslant m-1$, $C_m = B_m - V$ and $C_{m+1} = B_m \cap V$. Then $C_i \in \Phi$, $1 \leqslant i \leqslant m+1$, $C_i \subseteq A_i$, $1 \leqslant i \leqslant m+1$, $C_i \cap C_j = \varnothing$ for $1 \leqslant i \neq j \leqslant m+1$, and $\bigcup_{i=1}^{m+1} C_i = \bigcup_{i=1}^{m-1} B_i \cup C_m \cup C_{m+1} = \bigcup_{i=1}^{m} B_i = X$. This shows (c) is true for $(m+1)$ sets, and the proof of (b) $\rightarrow$ (c) is complete.

(c) $\rightarrow$ (d). The proof is exactly like the proof of the implication (a) $\rightarrow$ (b) in Theorem 1, except for the following modifications:

Define $f_0 \equiv y_0$, where y_0 is a fixed but arbitrary element of Y. Suppose now $n > 0$ and f_{n-1} has been defined to satisfy conditions (i)–(iii) with $\lambda = \aleph_0$. Since Y is compact, we can choose a 2^{-n} net $\{y_1^n, y_2^n, \cdots, y_{k_n}^n\}$ in Y. Let

$$C_i^n = \{x\colon d(y_i^n, F(x)) < 2^{-n}\}, \qquad 1 \leqslant i \leqslant k_n,$$

$$D_i^n = \{x\colon d(y_i^n, f_{n-1}(x)) < 2^{-(n-2)}\}, \qquad 1 \leqslant i \leqslant k_n,$$

$$A_i^n = C_i^n \cap D_i^n, \qquad 1 \leqslant i \leqslant k_n.$$

As before, the sets $A_i^n \in \Phi$ and $\bigcup_{i=1}^{k_n} A_i^n = X$. Since $\aleph_0$-WRP(Φ), we can "reduce" the sets A_i^n, $1 \leqslant i \leqslant k_n$, by disjoint sets $B_i^n \in (\Phi \cap \Phi^c)$, $1 \leqslant i \leqslant k_n$, such that $B_i^n \subseteq A_i^n$ and $\bigcup_{i=1}^{k_n} B_i^n = X$. Now set $f_n(x) = y_i^n$, if $x \in B_i^n$. Verify that f_n works. This completes the proof of Theorem 2.

REMARK 1. Note that in proving the implication (b) $\rightarrow$ (c) in Theorem 2, we did not use the full hypothesis regarding additivity of the family Φ; just $\aleph_0$-additivity of Φ suffices. It is also worth noting that, in general, $\aleph_0$-WRP(Φ) need not imply $\aleph_1$-WRP(Φ), even if Φ is closed under arbitrary unions and intersections. Indeed, let X = the real line and take $\Phi = \{\varnothing, X, [-n, n], n \geqslant 1\}$. Then $\aleph_0$-WRP(Φ), but $\neg(\aleph_1$-WRP(Φ)).

REMARK 2. The weak reduction principle is strictly weaker than the reduction principle. For example, take X to be the real line and Φ = the family of all open subsets of X which do not contain zero. Trivially, λ-WRP(Φ) for each cardinal λ. But $\neg$(RP(Φ)).

4. Special cases. As mentioned in the Introduction, our results, especially Theorem 1, are an attempt at a unification of a certain class of selection theorems. In this section, we shall deduce several known results from Theorems 1 and 2.

It is convenient at this point to quote a result of Kuratowski, which we shall find useful in the sequel. For a proof, the reader is referred to [5].

PROPOSITION. *If $\mathcal{F}$ is a λ-additive and subtractive family of subsets of X, then* λ^+-RP($\mathcal{F}_{\lambda^+}$).

We now deduce the main result of [6] (see also [9, p. 50]).

THEOREM 3. *Let L be a field of subsets of X. If Y is a Polish space, then any L_σ-normal function $F\colon X \to 2^Y$ admits a L_σ-measurable selector.*

PROOF. From the Proposition with $\lambda = \aleph_0$, it follows that $\aleph_1\text{-RP}(\mathsf{L}_\sigma)$. In Theorem 1 take $\lambda = \aleph_0$. The desired conclusion follows from the implication (a) $\to$ (b) of that theorem.

Several interesting applications of Theorem 3 are pointed out in [6] and in [9]. We give below some new applications. The first generalizes a result of Kuratowski [4, p. 434] on the extension of Borel functions.

COROLLARY 1. *Let L be a field of subsets of X and suppose that $A \subseteq X$ belongs to $(\mathsf{L}_\sigma)^c$. Let g be a $\mathsf{L}_\sigma \cap A$-measurable function on A into a Polish space Y, where $\mathsf{L}_\sigma \cap A$ is the trace of L_σ on A. Then there is a L_σ-measurable function f on X into Y such that f extends g.*

PROOF. Define $F\colon X \to 2^Y$ as follows:

$$F(x) = \begin{cases} \{g(x)\}, & \text{if } x \in A, \\ Y, & \text{otherwise.} \end{cases}$$

Check that F is L_σ-normal. By Theorem 3, there is a L_σ-measurable selector f for F. Plainly, f extends g.

Next, we give a new proof of a well-known result due to von Neumann [10] (see also [9, p. 65]).

COROLLARY 2. *Let A be an analytic set, and let h be a continuous function on A onto a separable metric space X. Let A be the σ-field on X generated by the analytic subsets of X. Then there is an A-measurable function $f\colon X \to A$ such that $h \circ f(x) = x$ for every $x \in X$.*

PROOF. Since A is analytic, there is a continuous function g on Σ, the space of irrationals, onto A. Define $F\colon X \to 2^\Sigma$ by $F(x) = (h \circ g)^{-1}(\{x\})$. For any open set V in Σ, $\{x\colon F(x) \cap V \neq \varnothing\} = h(g(V))$, which is analytic. So F is A-normal. By Theorem 3, there is a A-measurable selector $f'\colon X \to \Sigma$ for F. Take $f = g \circ f'$. Plainly, f does the trick!

As a final application of Theorem 3, we prove that a Borel set, all of whose (vertical) sections are compact, can be uniformized by a Borel set. This result is, of course, not new; however, our proof is.

COROLLARY 3. *Let X_1, X_2 be Polish spaces and let $B \subseteq X_1 \times X_2$ be Borel such that for every $x_1 \in X_1$, B^{x_1} $(\overset{\text{def}}{=} \{x_2 \in X_2\colon (x_1, x_2) \in B\})$ is compact. Then there exists a Borel function $f\colon \pi(B) \to X_2$ such that graph $f \subseteq B$. (Here π denotes projection of $X_1 \times X_2$ to X_1.)*

PROOF. By a result of Kunugui [3, Corollary 2, p. 100], $X \overset{\text{def}}{=} \pi(B)$ is Borel in X_1. Take L to be the Borel σ-field on X and define $F\colon X \to 2^{X_2}$ by $F(x) = B^x$. Now the above-mentioned result of Kunugui implies that F is L-normal, and the desired result falls out of Theorem 3.

In the remainder of this section, we deal with situations where Theorem 3 does not apply, either because the space Y is not separable or because the relevant family of subsets of X is not of the form L_σ, where L is a field.

The next result is proved in the theory ZFC + Martin's axiom (for a statement of Martin's axiom, see [7]).

COROLLARY 4. *Assume* ZFC + *Martin's axiom. Let R be the real line and let $\mathcal{A}$ be either the σ-field of Lebesgue measurable subsets of R or the σ-field of subsets of R having the Baire property. If Y is a complete metric space with topological weight $< 2^{\aleph_0}$ and $F\colon R \to 2^Y$ is $\mathcal{A}$-normal, then F admits an $\mathcal{A}$-measurable selector.*

PROOF. Let λ be a cardinal such that $\lambda < 2^{\aleph_0}$ and the topological weight of $Y \leqslant \lambda$. According to Corollary 1, p. 168 and Corollary 1, p. 171 in [7], $\mathcal{A}$ is $2^{\aleph_0}$-additive, and hence λ^+-additive. Consequently, by the Proposition, or directly, one sees that λ^+-RP($\mathcal{A}$). Now use the implication (a) $\to$ (b) of Theorem 1 to complete the proof.

Another consequence of our theorems is the following result of Shchegolkov [1].

COROLLARY 5. *If A is an elementary coanalytic subset of* $[0, 1] \times [0, 1]$ *such that $\pi(A)$ is Borel, then there is a Borel function $f\colon \pi(A) \to [0, 1]$ such that graph $f \subseteq A$. (Here π is the projection to the first coordinate.)*

PROOF. Let $X = \pi(A)$ and let Φ be the family of coanalytic subsets of X. Define $F\colon X \to 2^{[0,1]}$ by $F(x) = A^x$ $(= \{y \in [0, 1]\colon (x, y) \in A\})$. It follows from the definition of an elementary coanalytic set that F is Φ-normal. Moreover, according to a celebrated result of Luzin [4, p. 485], Φ^c satisfies the first principle of separation. The implication (a) $\to$ (d) of Theorem 2 and one more application of Luzin's theorem now complete the proof.

COROLLARY 6. *Let X_1, X_2 be Polish spaces, and let $E \subseteq X_1 \times X_2$ be a* PCA-*set such that for every $x_1 \in X_1$, E^{x_1} is a closed subset of X_2. Assume that $\pi(E)$, the projection of E to X_1, is simultaneously* PCA *and* CPCA. *Let $\mathcal{A}$ be the σ-field on $\pi(E)$ of sets which are simultaneously* PCA *and* CPCA. *Then there is an $\mathcal{A}$-measurable function $f\colon \pi(E) \to X_2$ such that graph $f \subseteq E$.*

PROOF. Let $X = \pi(E)$ and let Φ be the family of PCA subsets of X. Kuratowski has proved that $\aleph_1$-RP(Φ) [5, p. 187]. The rest of the proof is

exactly like the proof of the previous corollary, except that we use Theorem 1 (with $\lambda = \aleph_0$) instead of Theorem 2.

So far in this section we have considered problems which are essentially set-theoretic in nature. We now turn our attention to problems of a topological character.

THEOREM 4. *For a T_1-space X, the following conditions are equivalent.*

(i) *X is paracompact and strongly 0-dimensional.*

(ii) *If Y is a complete metric space, then every l.s.c. function $F: X \to 2^Y$ admits a continuous selector.*

PROOF. (i) $\to$ (ii). Take Φ to be the family of open subsets of X. In view of the implication (a) $\to$ (b) of Theorem 1 and the fact that Φ is closed under arbitrary unions, it suffices to prove λ^+-WRP(Φ) for each cardinal λ.

Let, then, $\{A_\alpha, \alpha < \lambda\}$ be an open cover of X. By paracompactness, there is a locally finite open refinement $\{B_i, i \in I\}$ of $\{A_\alpha, \alpha < \lambda\}$. Since X is paracompact, X is normal [2, Theorem 2, p. 207]. Consequently, as X is strongly 0-dimensional, $\{B_i, i \in I\}$ admits a disjoint open refinement $\{C_j, j \in J\}$ [8, Proposition 2]. It follows that $\{C_j, j \in J\}$ is a disjoint open refinement of $\{A_\alpha, \alpha < \lambda\}$. For $\alpha < \lambda$, set $D_\alpha = \bigcup\{C_j\colon C_j \subseteq A_\alpha$ and $(\forall \beta < \alpha)(C_j \nsubseteq A_\beta)\}$. Plainly, the sets D_α are disjoint, $D_\alpha \in \Phi$, $D_\alpha \subseteq A_\alpha$ and $\bigcup_{\alpha<\lambda} D_\alpha = \bigcup_{j\in J} C_j = X$. Hence, λ^+-WRP(Φ).

(ii) $\to$ (i). Condition (ii) implies, in particular, that condition (c) of Theorem 1 holds for each cardinal. Hence, from the implication (c) $\to$ (a) of Theorem 1, we get: λ^+-WRP(Φ) for every cardinal λ. From this, (i) follows immediately.

Implication (i) $\to$ (ii) of Theorem 4 is due to Michael [8]. The converse appears to be new. We conclude the section by stating one more topological result.

THEOREM 5. *For a T_1-space X, the following are equivalent:*

(a) *X is normal and strongly 0-dimensional.*

(b) *If Y is a compact metric space, then every l.s.c. function $F: X \to 2^Y$ admits a continuous selector.*

PROOF. Use Theorem 2.

5. A selection theorem for general set-valued mappings. In this section we discuss briefly the problem of finding "nice" selectors for set-valued mappings whose values are not necessarily closed sets.

THEOREM 6. *Let X be a set, Φ a family of subsets of X such that $\varnothing$, $X \in \Phi$, Φ is $\aleph_1$-additive and $\aleph_0$-multiplicative. Let $F: X \to P(Y) - \{\varnothing\}$,*

where Y is a Polish space. Suppose that there exists a sequence of sets A_n, $n = 1, 2, \cdots$, and functions $G_n: A_n \to 2^Y$, $n = 1, 2, \cdots$, such that $A_n \in \Phi$, G_n is $\Phi \cap A_n$-normal, $(\forall x \in A_n)(G_n(x) \subseteq F(x))$, $n = 1, 2, \cdots$, and $\bigcup_{n=1}^{\infty} A_n = X$. If $\aleph_1$-WRP(Φ), then there is a $(\Phi \cap \Phi^c)_\sigma$-measurable selector for F.

PROOF. Since Φ is $\aleph_1$-additive and $\aleph_1$-WRP(Φ), there exist sets B_n, $n = 1, 2, \cdots$, such that $B_n \in (\Phi \cap \Phi^c)$, $B_n \subseteq A_n$, $B_n \cap B_m = \varnothing$ for $n \neq m$, and $\bigcup_{n=1}^{\infty} B_n = X$. Let H_n be the restriction of G_n to B_n, i.e., $H_n = G_n \restriction B_n$, $n = 1, 2, \cdots$. Then H_n is $\Phi \cap B_n$-normal, $n = 1, 2, \cdots$. As $B_n \in (\Phi \cap \Phi^c)$, it follows that $\aleph_1$-WRP$(\Phi \cap B_n)$, $n = 1, 2, \cdots$. Hence, by Theorem 1, H_n admits a $((\Phi \cap \Phi^c) \cap B_n)_\sigma$-measurable selector $f_n: B_n \to Y$, $n = 1, 2, \cdots$. Define $f: X \to Y$ by setting $f = f_n$ on B_n, $n = 1, 2, \cdots$. Then f is a $(\Phi \cap \Phi^c)_\sigma$-measurable selector for F. This completes the proof.

We now use the above theorem to derive a known result.

COROLLARY 7. *Let X_1 be a 0-dimensional separable metric space and let X_2 be a 0-dimensional Polish space. If E is a nonempty open subset of $X_1 \times X_2$, then there is a continuous function f on $\pi(E)$, the projection of E to X_1, into Y such that graph $f \subseteq E$.*

PROOF. Let $X = \pi(E)$ and take Φ to be the family of open subsets of X. Since $X_1 \times X_2$ is a separable 0-dimensional metric space, there exist clopen subsets C_n of $X_1 \times X_2$ such that $E = \bigcup_{n=1}^{\infty} C_n$. Set $A_n = \pi(C_n)$, $n = 1, 2, \cdots$, so that $A_n \in \Phi$ and $\bigcup_{n=1}^{\infty} A_n = X$. Define $F: X \to P(X_2) - \{\varnothing\}$ by $F(x) = E^x$ and $G_n: A_n \to 2^{X_2}$ by $G_n(x) = C_n^x$, $n = 1, 2, \cdots$. Plainly, G_n is $(\Phi \cap A_n)$-normal and $(\forall x \in A_n)(G_n(x) \subseteq F(x))$. Finally, by a well-known theorem, $\aleph_1$-RP(Φ) in [4, Theorem 1, p. 279]. The desired conclusion now follows from Theorem 6.

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