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BLANCHFIELD DUALITY AND SIMPLE KNOTS

BY

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ABSTRACT. The method of presentation for n-knots is used to classify simple $(2q-1)$-knots, $q > 3$, in terms of the Blanchfield duality pairing. As a corollary, we characterize the homology modules and pairings which can arise from classical knots.

0. Introduction. In this paper we use the results and techniques of [4] to give a classification of simple knots in terms of the Blanchfield duality pairing.

We work in the piecewise linear category throughout, and it is to be understood that all embeddings, submanifolds, and isotopies are locally flat. An *n-knot* is an oriented pair (S^{n+2}, S^n), where S^r denotes the r-sphere; two n-knots are *equivalent* if there is an isomorphism of pairs between them which preserves orientations.

A knot of S^{2q-1} in S^{2q+1} is *simple* if its complement has the homotopy $(q-1)$-type of a circle; in the terminology of [4], it is a $(q-1)$-simple $(2q-1)$-knot. The term simple is due to Levine [8].

Let K denote the complement of a simple $(2q-1)$-knot; if R is the integral group ring of the infinite cyclic group, and R_0 its field of fractions, then Blanchfield [1] shows that there is a nonsingular pairing of R-modules

$$V: H_q(\widetilde{K}, \partial\widetilde{K}) \times H_q(\widetilde{K}) \longrightarrow R_0/R,$$

where $\widetilde{K}$ denotes the infinite cyclic cover of K.

By Corollary 10.1 of [4], if $q > 3$, a simple $(2q-1)$-knot has a presentation with one 0-handle, some $(q-1)$- and q-handles, and one $(2q-1)$-handle. Such a presentation is called *simple*, and we use it to obtain a matrix presentation of $H_q(\widetilde{K})$ and V; we remark that $H_q(\widetilde{K}, \partial\widetilde{K}) \cong H_q(\widetilde{K})$ by the long exact sequence of homology.

By means of the matrix presentation, we show that for $q > 3$ a simple $(2q-1)$-knot is determined by its homology module $H_q(\widetilde{K})$ and the duality

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pairing V. Furthermore, we characterize those modules and pairings which can arise from such knots.

These results may profitably be compared to some theorems of Kervaire [5] and Levine [8]. Any simple $(2q-1)$-knot has associated with it an integer matrix A, called the Seifert matrix, such that

$$(*) \qquad A + (-1)^q A'$$

is unimodular. A is not unique, but is determined up to S-equivalence (see [8], [10]). Kervaire and Levine show that the S-equivalence classes of integer matrices satisfying (*) are in 1-1 correspondence with simple $(2q-1)$-knots for $q \geqslant 3$. They work in the differential category, but their proofs use handlebody techniques and are applicable to the PL category in which we work.

It is not hard to see that the Blanchfield pairing may be described in terms of A, using Seifert's construction for the infinite cyclic cover. Thus, via the geometric results, for $q > 3$ the Blanchfield pairing determines, and is determined by, the S-equivalence class of the Seifert matrix. But the algebra depends only on the value of q mod 2, and so this is true for all q.

Trotter [12] has a purely algebraic proof of this result.

By means of the above observation, we may extend the range of values of q: thus for $q \geqslant 2$ a simple $(2q-1)$-knot is determined by its homology module and duality pairing, and for $q \neq 2$ we characterize those modules and pairings which can arise from simple $(2q-1)$-knots.

The latter result is particularly striking in the case $q = 1$. Any classical knot is simple, and so by high-dimensional techniques we are able to characterize the homology modules and Blanchfield pairings which can arise from classical knots.

1. **Presentation of the knot module.** We use the notation of [4] throughout the present paper.

Consider a simple presentation of a $(2q-1)$-knot, $q > 3$. Suppose that there are k $(q-1)$-handles, added at level σ, and that τ is a level between the $(q-1)$- and q-handles. By the proof of Theorem 5.1 of [4], the $(q-1)$-handles are unknotted in their level, i.e. we may move the 0-handle up to level σ to obtain $P = h^0 \cup \bigcup_{i=1}^k h_i^{q-1}$ unknotted in S^{2q}, as in [4, §6]. Let M be the boundary of the submanifold to which the q-handles are attached, so that $M \cong \partial P$ is unknotted in S^{2q}, and let Q be the complement in S^{2q} of the interior of a regular neighborhood $N \cong B^2 \times M$ of M. Note that $K_\tau \cong Q$.

From [4, §6] we may choose a handle decomposition of Q by means of P, and hence a basis $[x_i]_1^k \cup [y_i]_1^k$ of $H_q(\widetilde{Q}, \partial\widetilde{Q})$. We remark that y_i corresponds to the cocore of the ith $(q-1)$-handle of the knot, but that the $[x_i]_1^k$ are to some extent arbitrary: we make a choice at this stage, but we shall be ready later on to alter this choice.

By [4, §2] we may assume that the q-handles are in normal position.

By the Euler characteristic of S^{2q-1}, there are k q-handles of the knot. The core of each q-handle meets Q in a q-ball, to which we assign an orientation and lift in $\widetilde{Q}$, thereby determining an element of $H_q(\widetilde{Q}, \partial\widetilde{Q})$. Writing these elements in terms of the basis $[x_i]_1^k \cup [y_i]_1^k$, we obtain a $k \times 2k$ matrix over R, the integral group ring of the infinite cyclic group.

We use the notation of [4, §6].

LEMMA 1.1. $H_q(\widetilde{K}_{(\tau)}) = [u_1, \cdots, u_k, v_1, \cdots, v_k : v_1 = \cdots = v_k = 0]$.

PROOF. P collapses to a wedge of k $(q-1)$-spheres contained in its interior; let P' be a regular neighborhood of this wedge in P. A regular neighborhood of P in S^{2q} has the form $B^1 \times P''$, where $P'' = P \cup$ collar. Let X denote the closed complement of $B^1 \times P''$ in S^{2q}, and let $K_{(r,s)}$ denote $K \cap (S^{2q} \times [r, s])$; recall that $K_{(s)} = K_{(0,s)}$.

By deformation retraction along the I factor, we have the following homotopy equivalences:

$$K_{(\sigma,\tau)} \simeq X \cup (B^1 \times P'), \qquad K_{(\sigma)} \simeq X \cup (B^1 \times P'') = S^{2q}.$$

By the work of [4, §6] we see that

$$H_q(\widetilde{X}) = [v_1, \cdots, v_k: \ ],$$

and that

$$H_q(\widetilde{K}_{(\sigma,\tau)}) = H_q(\widetilde{X}) \oplus [u_1, \cdots, u_k: \ ],$$

using the first homotopy equivalence.

An application of the Mayer-Vietoris sequence to the couple $(\widetilde{K}_{(\sigma,\tau)}, \widetilde{K}_{(\sigma)})$ shows that

$$H_q(\widetilde{K}_{(\tau)}) = [u_1, \cdots, u_k, v_1, \cdots, v_k: v_1 = \cdots = v_k = 0]. \quad \square$$

Consider the dual presentation of the knot, and let $[\chi_i]_1^k \cup [\xi_i]_1^k$ be a basis of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ similar to $[x_i]_1^k \cup [y_i]_1^k$, with a basis $[\alpha_i]_1^k \cup [\beta_i]_1^k$ of $H_q(\widetilde{Q})$ similar to $[u_i]_1^k \cup [v_i]_1^k$. By taking the dual presentation we reverse the orientation of I in $S^{2q} \times I$, and so must reverse the orientation of S^{2q} to preserve that of $S^{2q} \times I$. This means that the sense of the normal to a codimension one submanifold of S^{2q} is reversed, and so the construction of [4, Lemma 6.4] leads to the relations

$$i_*\alpha_i = (1 - t^{-1})\chi_i, \qquad i_*\beta_i = (1 - t)\xi_i.$$

Note that ξ_i corresponds to the cocore of the ith dual $(q-1)$-handle of the knot, i.e. to the core of the ith q-handle. If $\xi_i = a_{ij}x_j + d_{ij}y_j$, we shall write this in matrix notations as $\xi = Ax + Dy$.

LEMMA 1.2. *If* $q > 3$, *and* $\xi = Ax + Dy$, *then*

$$H_q(\widetilde{K}) = [u, v\colon v = 0, Au - tDv = 0] = [u\colon Au = 0].$$

PROOF. We note that $H_q(\widetilde{K}) = H_q(\widetilde{K}_{(1)})$ as $q > 3$, and K is obtained from $K_{(1)}$ by adding two $(2q+1)$-handles.

Lemma 1.1, applied to the presentation and its dual, yields

$$H_q(\widetilde{K}_{(\tau)}) = [u, v\colon v = 0], \qquad H_q(\widetilde{K}_{(\tau,1)}) = [\alpha, \beta\colon \beta = 0].$$

Note that, again in matrix notation,

$$\begin{aligned} i_*\beta = (1-t)\xi &= (1-t)Ax + (1-t)Dy \\ &= A(1-t)x + D(1-t)y = i_*(Au - tDv). \end{aligned}$$

Applying the Mayer-Vietoris sequence to $(\widetilde{K}_{(\tau)}, \widetilde{K}_{(\tau,1)})$ we obtain

$$\begin{aligned} H_q(\widetilde{K}_{(1)}) &= [u, v\colon v = 0, \beta = 0] \\ &= [u, v\colon v = 0, Au - tDv = 0] = [u\colon Au = 0]. \quad \square \end{aligned}$$

LEMMA 1.3. *If* $q > 3$, *and* $\xi = Ax + Dy$, *then*

$$H_q(\widetilde{K}, \partial\widetilde{K}) = [x, y\colon y = 0, Ax + Dy = 0] = [x\colon Ax = 0].$$

The proof is similar to the argument above.

We remark that $\det A \neq 0$, being the Alexander polynomial of the knot in dimension q.

2. **An equivalence lemma.** Consider simple presentations of two $(2q-1)$-knots, $q > 3$, each with k $(q-1)$-handles. Assume that the $(q-1)$-handles of each knot are added at level $\alpha < ½$, the q-handles at level $\beta > ½$. The part of the knot below level ½ we shall call the lower half, the part above level ½ the upper half. As a $(2q-1)$-ball unknots in S^{2q}, we may isotop the 0-handle of the first knot onto the 0-handle of the second, and by the proof of [4, Theorem 5.1] we may isotop the $(q-1)$-handles of the first knot to coincide with those of the second. Thus we may isotop the lower half of the first knot to coincide with the lower half of the second.

LEMMA 2.1. *Suppose that the presentations above yield the same matrices* A, D *of Lemma* 1.2, *with respect to the same bases. Then the two knots are isotopic.*

PROOF. Let K denote the complement of the first knot, and Q the intersection of K with level ½. Let $\hat{K}$, $\hat{Q}$ be similarly defined for the second knot, and note that $Q = \hat{Q}$.

Let h_i^r denote the ith r-handle of the first knot, $\hat{h}_i^r$ that of the second. Let E_i denote the shadow in level ½ of h_i^q, and let C_i denote the shadow of its core, with $\hat{E}_i$, $\hat{C}_i$ similarly defined for the second knot.

Then $Q \cap C_1$ and $Q \cap \hat{C}_1$ represent the same element of $H_q(\widetilde{Q}, \partial\widetilde{Q})$, and so by the Hurewicz and covering space theorems they represent the same element of $\pi_q(Q, \partial Q)$. By the proof of [4, Lemma 7.3], quoting Irwin's theorem in place of general position, we may assume that they are homotopic keeping the boundary fixed. Comparison with the work of Hacon [13] shows that the obstruction σ to isotoping $Q \cap C_1$ onto $Q \cap \hat{C}_1$, keeping the boundary fixed, lies in $\bigoplus_{-\infty}^{\infty} \pi^1$, where π^1 is $\pi_q(S^{q-1}) = Z_2$. Let us add a trivially cancelling handle pair (k_1^{q-1}, k_1^q) to the first knot, and a similar pair $(k_1^{q-1}, \hat{k}_1^q)$ to the second, such that k_1^{q-1} coincides with $\hat{k}_1^{q-1}$, but the obstruction to isotoping the core of k_1^q onto that of $\hat{k}_1^q$ is σ. Move h_1^q over k_1^q; the obstruction for h_1^q becomes $\sigma + \sigma = 0$, and so we can isotop C_1 onto $\hat{C}_1$.

By the uniqueness of regular neighborhoods, we may assume that the attaching tube of E_1 coincides with that of $\hat{E}_1$. If we try to apply the argument of [4, Lemma 7.1] to isotop E_1 onto $\hat{E}_1$, we find an obstruction in $\pi_q(S^{q-1}) = Z_2$. We add another trivially cancelling handle pair (h^{q-1}, h^q) and $(\hat{h}^{q-1}, \hat{h}^q)$ in each presentation, such that h^{q-1} coincides with $\hat{h}^{q-1}$, and the core of h^q coincides with that of $\hat{h}^q$, but such that the obstruction in Z_2 is nonzero.

Suppose that the obstruction to isotoping E_1 onto $\hat{E}_1$ is nonzero: then we move h_1^q over h^q and $\hat{h}_1^q$ over $\hat{h}^q$. Now moving one q-handle over another may be achieved by adding together two q-balls (representing the cores of the handles) which meet in a common face. Thus the obstruction in $\pi_q(S^{q-1})$ is additive over handle addition, and the obstruction in Z_2 vanishes for the new h_1^q, $\hat{h}_1^q$.

This enables us to isotop E_1 onto $\hat{E}_1$, i.e. we may isotop h_1^q to coincide with $\hat{h}_1^q$. By considering the dual presentations of the knots, we see that the C_i form part of a basis of $H_q(\widetilde{Q}, \partial\widetilde{Q})$; if we slide h_1^q $(= \hat{h}_1^q)$ down below level ½, then $C_2, \cdots, C_k$ still form part of a basis for $H_q(\widetilde{Q}, \partial\widetilde{Q})$, now a free R-module of lower rank, which may be identified with a submodule of the original $H_q(\widetilde{Q}, \partial\widetilde{Q})$. Therefore C_2 represents the same element as $\hat{C}_2$ in this module. Thus we may repeat the argument above, to isotop E_2 onto $\hat{E}_2$.

Reiterating this procedure, we may isotop the first knot until h_i^q coincides with $\hat{h}_i^q$, for $1 \leqslant i \leqslant k$. We still have to consider the handle pairs that we have introduced. In the dual presentation, the dual of h^{q-1} cancels the dual of h^q, and the shadow of the dual of h_i^{q-1} does not meet the dual of h^q for any i; similar remarks hold for the other pairs. Thus in the dual presentation we may isotop the (dual) 0-handles to coincide, the $(q-1)$-handles to coincide, and the duals of the h_i^{q-1} to coincide with the duals of the $\hat{h}_i^{q-1}$; sliding the dual of h^q up to the level of the dual of h^{q-1}, we may cancel, and similarly for the other handle pairs. Isotoping the $(2q-1)$-handles to coincide completes the task. □

COROLLARY 2.2. *Suppose that two simple $(2q-1)$-knots, $q > 3$, each have simple presentations yielding the same matrices A, D of Lemma 1.2. Then the two knots are equivalent.*

PROOF. As the $(q-1)$-handles unknot, we may construct an isomorphism $h\colon B^{2q+1} \longrightarrow B^{2q+1}$ which takes the lower half of the first knot onto the lower half of the second, and which preserves the chosen bases of homology. The proof of the lemma then applies to complete the argument. □

3. **Change of base.** Recall the bases of homology that we have used so far. From [4, §6] we have bases $[x_i]_1^k \cup [y_i]_1^k$ of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ and $[u_i]_1^k \cup [v_i]_1^k$ of $H_q(\widetilde{Q})$ with the following properties:

$$i_*u_i = (1-t)x_i, \qquad i_*v_i = (1-t^{-1})y_i,$$

$$S(x_i, u_j) = 0, \qquad S(y_i, u_j) = (-1)^q\delta_{ij},$$

$$S(x_i, u_j) = \delta_{ij}, \qquad S(y_i, v_j) = 0.$$

If Q arises from a simple presentation of a knot, then we are given an unknotted embedding of $P = h^0 \cup \bigcup_1^k h_i^{q-1}$ in S^{2q}, so that the y_i are determined by the cocores of the $(q-1)$-handles of the knot. There is a certain amount of choice for the x_i, which we now investigate.

Similarly, from the dual presentation of the knot we obtain bases $[\chi_i]_1^k \cup [\xi_i]_1^k$ of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ and $[\alpha_i]_1^k \cup [\beta_i]_1^k$ of $H_q(\widetilde{Q})$ with the following properties:

$$i_*\alpha_i = (1-t^{-1})\chi_i, \qquad i_*\beta_i = (1-t)\xi_i,$$

$$S(\chi_i, \alpha_j) = 0, \qquad S(\xi_i, \alpha_j) = (-1)^q\delta_{ij},$$

$$S(\chi_i, \beta_j) = \delta_{ij}, \qquad S(\xi_i, \beta_j) = 0.$$

LEMMA 3.1. *Suppose that a change of base is made in $H_q(\widetilde{Q}, \partial\widetilde{Q})$ and $H_q(\widetilde{Q})$, replacing $[x_i]_1^k$ by $[x_i + L_{ij}y_j]_1^k$ and $[u_i]_1^k$ by $[u_i - tL_{ij}v_j]_1^k$. Then a necessary and sufficient condition for the new bases to have the same intersection properties as old is that $L = (-1)^q t^{-1}L^*$, where $L^* = \bar{L}'$, the conjugate transpose of L.*

PROOF. Note that $i_*(u_i - tL_{ij}v_j) = (1-t)(x_i + L_{ij}y_j)$.

$$\begin{aligned} S(x_i + L_{ij}y_j, u_k - tL_{km}v_m) \\ &= S(x_i, u_k) + S(x_i, -tL_{km}v_m) + S(L_{ij}y_j, u_k) + S(L_{ij}y_j, -tL_{km}v_m) \\ &= L_{ij}S(y_j, u_k) - t^{-1}\bar{L}_{km}S(x_i, v_m) \\ &= (-1)^q L_{ik} - t^{-1}\bar{L}_{ki} = (-1)^q [L - (-1)^q t^{-1}L^*]_{ik}. \end{aligned}$$

Thus the new bases satisfy the analogue of $S(x_i, u_j) = 0$ if and only if $L - (-1)^q t^{-1}L^* = 0$. The other relations are proved similarly. □

LEMMA 3.2. *Suppose that a change of base is made in* $H_q(\widetilde{Q}, \partial\widetilde{Q})$ *and* $H_q(\widetilde{Q})$, *replacing* $[\chi_i]_1^k$ *by* $[\chi_i + L_{ij}\xi_j]_1^k$ *and* $[\alpha_i]_1^k$ *by* $[\alpha_i - t^{-1}L_{ij}\beta_j]_1^k$. *Then a necessary and sufficient condition for the new bases to have the intersection properties of the old is that* $L = (-1)^q tL^*$.

PROOF. Similar to the proof of Lemma 3.1. □

LEMMA 3.3. *Let* $[x_i]_1^k \cup [y_i]_1^k$, $[u_i]_1^k \cup [v_i]_1^k$ *be bases of* $H_q(\widetilde{Q}, \partial\widetilde{Q})$, $H_q(\widetilde{Q})$ *related as above, and let* $[\chi_i]_1^k \cup [\xi_i]_1^k$, $[\alpha_i]_1^k \cup [\beta_i]_1^k$ *be bases satisfying*

$$i_*\alpha = (1 - t^{-1})\chi_i, \qquad i_*\beta_i = (1 - t)\xi_i.$$

Suppose that the matrix relating them is given by

$$\begin{pmatrix} \chi \\ \xi \end{pmatrix} = \begin{pmatrix} B & C \\ A & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then a necessary and sufficient condition for the second pair of bases to have the intersection properties listed above is that the inverse of this matrix should be

$$\begin{pmatrix} -t^{-1}D^* & (-1)^q C^* \\ (-1)^q A^* & -tB^* \end{pmatrix}.$$

PROOF. First note that the matrix relating the bases of absolute homology is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -t^{-1}B & C \\ A & -tD \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

for $(1 - t)\xi = A(1 - t)x + D(1 - t)y$ and so $\beta = Au - tDv$ and similarly $\alpha = -t^{-1}Bu + Cv$.

Next consider $\chi_i = B_{ij}x_j + C_{ij}y_j$; taking the intersection with v_k we obtain

$$S(\chi_i, v_k) = S(B_{ij}x_j, v_k) + S(C_{ij}y_j, v_k) = B_{ij}S(x_j, v_k) = B_{ik}.$$

To prove the necessity, assume the inverse of $\begin{pmatrix} B & C \\ A & D \end{pmatrix}$ is $\begin{pmatrix} F & G \\ E & H \end{pmatrix}$, and note the following relations, obtained as above.

$$\begin{aligned} B_{ij} &= S(\chi_i, v_j), & C_{ij} &= (-1)^q S(\chi_i, u_j), \\ A_{ij} &= S(\xi_i, v_j), & D_{ij} &= (-1)^q S(\xi_i, u_j), \\ F_{ij} &= S(x_i, \beta_j), & G_{ij} &= (-1)^q S(x_i, \alpha_j), \\ E_{ij} &= S(y_i, \beta_j), & H_{ij} &= (-1)^q S(y_i, \alpha_j). \end{aligned}$$

Thus we see that

$$\begin{aligned} (1-t)F_{ij} &= S((1-t)x_i, \beta_j) = S(i_* u_i, \beta_j) \\ &= (-1)^q \overline{S(i_*\beta_j, u_i)} = (-1)^q \overline{S((1-t)\xi_j, u_i)} \\ &= (-1)^q (1-t^{-1}) \overline{S(\xi_j, u_i)} = -t^{-1}(1-t)\bar{D}_{ji} \end{aligned}$$

and therefore $F = -t^{-1}D^*$. The other entries are calculated in a similar way.

To prove sufficiency, we simply compute the intersection relations of the second pair of bases. For instance:

$$\begin{aligned} S(\chi_i, \beta_j) &= S(B_{ik}x_k + C_{ik}y_k, A_{jm}u_m - tD_{jm}v_m) \\ &= S(B_{ik}x_k, -tD_{jm}v_m) + S(C_{ik}y_k, A_{jm}u_m) \\ &= -B_{ik}t^{-1}\bar{D}_{jm}S(x_k, v_m) + C_{ik}\bar{A}_{jm}S(y_k, u_m) \\ &= -B_{ik}t^{-1}\bar{D}_{jk} + C_{ik}\bar{A}_{jk}(-1)^q \\ &= [-Bt^{-1}D^* + (-1)^q CA^*]_{ij} = \delta_{ij}. \quad \square \end{aligned}$$

LEMMA 3.4. *Given bases* $[x_i]_1^k \cup [y_i]_1^k$, $[u_i]_1^k \cup [v_i]_1^k$ *of* $H_q(\widetilde{Q}, \partial\widetilde{Q})$, $H_q(\widetilde{Q})$ *with the properties listed above, there is a corresponding handle decomposition of* Q, *if* $q > 3$.

PROOF. Regard Q as the complement of the interior of a regular neighborhood $N = B^1 \times B^1 \times M$ of $M = \partial P$ unknotted in S^{2q}. By piping together basis elements of $\pi_q(Q)$ and $\pi_q(Q, \partial Q)$, and using the map i_* of [4, Lemma 6.4], there are embeddings $f_i\colon B^q \times B^{q-1} \longrightarrow Q$ such that $f_i^{-1}(1 \times 0 \times M) = f_i^{-1}(\partial Q) = \partial B^q \times B^{q-1}$, and $f_i|_{B^q \times 0}$ represents the element $x_i \in H_q(\widetilde{Q}, \partial\widetilde{Q}) \cong \pi_q(Q, \partial Q)$. By the intersection relations, we may assume these embeddings to be disjoint.

Let B^{2q} be a regular neighborhood of $\operatorname{Im} f_1$ in Q meeting ∂Q regularly, and let Q_1 be the closed complement in S^{2q} of $N \cup B^{2q}$. Then $Q = Q_1 \cup B^{2q}$, and $Q_1 \cap B^{2q}$ is the closed complement in ∂B^{2q} of $\operatorname{Im} f_1$. Thus $Q_1 \cap B^{2q} \cong S^{q-1} \times B^q$.

We apply the Mayer-Vietoris sequence to the couple $(\widetilde{Q}_1, \widetilde{B}^{2q})$. In the middle dimensions we obtain

$$0 \longrightarrow H_q(\widetilde{Q}_1 \cap \widetilde{B}^{2q}) \longrightarrow H_q(\widetilde{Q}_1) \longrightarrow H_q(\widetilde{Q})$$
$$\longrightarrow H_{q-1}(\widetilde{Q}_1 \cap \widetilde{B}^{2q}) \longrightarrow H_{q-1}(\widetilde{Q}_1) \longrightarrow 0,$$

which gives

$$0 \longrightarrow H_q(\widetilde{Q}_1) \longrightarrow H_q(\widetilde{Q}) \xrightarrow{\partial_*} H_{q-1}(\widetilde{Q}_1 \cap \widetilde{B}^{2q}) \longrightarrow H_{q-1}(\widetilde{Q}_1) \longrightarrow 0.$$

Now $H_{q-1}(\widetilde{Q}_1 \cap \widetilde{B}^{2q}) = R$, and the intersection relations $S(x_1, u_i) = 0$, $S(x_1, v_i) = \delta_{1i}$ show that $\partial_* v_1$ is a generator of this module. It follows that $H_{q-1}(\widetilde{Q}) = 0$ and $H_q(\widetilde{Q}_1)$ is identified with the submodule of $H_q(\widetilde{Q})$ with basis $[u_i]_1^k \cup [v_i]_2^k$.

By excision we note that $H_*(\widetilde{Q}_1, \partial\widetilde{Q}_1) = H_*(\widetilde{Q}, \partial\widetilde{Q} \cup \widetilde{B}^{2q})$. From the long exact sequence of the triad $(\widetilde{Q}, \partial\widetilde{Q} \cup \widetilde{B}^{2q}, \partial\widetilde{Q})$ we obtain

$$0 \longrightarrow H_{q+1}(\widetilde{Q}, \partial\widetilde{Q} \cup \widetilde{B}^{2q}) \longrightarrow H_q(\partial\widetilde{Q} \cup \widetilde{B}^{2q}, \partial\widetilde{Q})$$
$$\xrightarrow{j_*} H_q(\widetilde{Q}, \partial\widetilde{Q}) \longrightarrow H_q(\widetilde{Q}, \partial\widetilde{Q} \cup \widetilde{B}^{2q}) \longrightarrow 0.$$

$H_q(\partial\widetilde{Q} \cup \widetilde{B}^{2q}, \partial\widetilde{Q}) = R$, with a basis element that is mapped onto x_1 by j_*: it follows that $H_{q+1}(\widetilde{Q}_1, \partial\widetilde{Q}_1) = 0$ and $H_q(\widetilde{Q}_1, \partial\widetilde{Q}_1)$ may be identified with the submodule of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ generated by $[x_i]_2^k \cup [y_i]_1^k$. These bases of $H_q(\widetilde{Q}_1), H_q(\widetilde{Q}_1, \partial\widetilde{Q}_1)$ have properties similar to those for the homology of $\widetilde{Q}$.

Define M_1 to be $(B^1 \times 0 \times M) \cup f_1(B^q \times B^{q-1})$, and note that we have a proper embedding $f_2|_{B^q \times 0}\colon B^q \longrightarrow Q_1$, constructed above, which represents $x_2 \in H_q(\widetilde{Q}_1, \partial\widetilde{Q}_1)$. We repeat the argument above to obtain Q_2 and M_2, where $H_q(\widetilde{Q}_2)$ is the free R-module with basis $[u_i]_1^k \cup [v_i]_3^k$ and $H_q(\widetilde{Q}_2, \partial\widetilde{Q}_2)$ has basis $[x_i]_3^k \cup [y_i]_1^k$. Continuing in this way we arrive at Q_k, M_k.

Now we go back to the beginning, and remark that the elements y_i of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ may be represented by proper embeddings $g_i|_{B^q \times 0}\colon B^q \times 0 \longrightarrow Q$, with $g_i(\partial B^q \times B^{q-1}) \subset -1 \times 0 \times M$. Applying the procedure above, we construct $Q_{k+1}, M_{k+1}, \cdots, Q_{2k}, M_{2k}$.

At each stage we are killing off a pair of basis elements of $H_{q-1}(M)$ by a surgery, by [4, Lemma 6.5] so that as $q > 3, M_{2k} \cong S^{2q-2} \times B^1$. As $H_q(\widetilde{Q}_{2k}) = 0, Q_{2k}$ has the homotopy type of a circle, and so $S^{2q-2} \times 0$ in unknotted in S^{2q} [6]. As $\pi_{2q-2}(S^1) = 0$, the proof of [4, Lemma 7.1] shows that we may add two $(2q-1)$-handles to M_{2k} to form an unknotted sphere. This sphere induces a handle decomposition of Q as described in [4, §6]. □

4. **The Blanchfield pairing.** Recall that R is the integral group ring of the infinite cyclic group, and that R_0 is the field of fractions of R. *Conjugation* in R is defined as the linear extension of the map $t \longmapsto t^{-1}$, and the conjugate of an element α is denoted by $\bar{\alpha}$. R_0/R denotes the quotient of R_0 by R, regarded as R-modules.

Let K be the complement of an n-knot, and let x be an r-chain of $(\widetilde{K}, \partial\widetilde{K})$. If y is an $(n-r)$-chain of $\widetilde{K}$ in a dual triangulation, define

$$S(x, y) = \sum_{-\infty}^{\infty} T(x, t^k y)t^k \in R_0,$$

where T denotes the ordinary intersection of chains.

Now $H_*(\widetilde{K})$ and $H_*(\widetilde{K}, \partial\widetilde{K})$ are R-torsion-modules in each dimension (cf. [9]), and so if u is an $(n-r-1)$-cycle of $\widetilde{K}$, there exists an $(n-r)$-chain w and an $\alpha \in R$ such that $\bar{\alpha}u = \partial w$. Thus if x is an r-chain of $(\widetilde{K}, \partial\widetilde{K})$ we may define

$$V(x, u) = S(x, w)/\alpha \in R_0.$$

By taking values of V modulo R we obtain a pairing of homology classes to R_0/R which Blanchfield [1] shows to be *sesquilinear,* i.e. linear in the first variable and conjugate linear in the second.

We remark (cf. [1]) that if $\beta x = \partial y$ for $\beta \in R$ and an $(r+1)$-chain y of $(\widetilde{K}, \partial\widetilde{K})$, then

$$V(x, u) = S(x, w)/\alpha = S(y, u)/\beta.$$

We now set down a formula for V in terms of a simple presentation of a $(2q-1)$-knot; naturally we have only the case $V\colon H_q(\widetilde{K}, \partial\widetilde{K}) \times H_q(\widetilde{K}) \longrightarrow R_0/R$ to consider. Recall the bases of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ and $H_q(\widetilde{Q})$ considered in §1. According to Lemma 1.3, $H_q(\widetilde{K}, \partial\widetilde{K}) = [x, y\colon y = 0, Ax + Dy = 0]$. Thus, if φ is the quotient map, $\varphi y_i = 0$ for each i.

Similarly, by Lemma 1.2, there is a presentation of $H_q(\widetilde{K})$ as a quotient of $H_q(\widetilde{Q})$ with basis $[\alpha_i]_1^k \cup [\beta_i]_1^k$. If the quotient map is ψ, then $\psi\beta_i = 0$ for each i.

As a cycle in $(\widetilde{K}, \partial\widetilde{K})$, ξ_i bounds, and we may write it explicitly as $\xi_i = -\partial\theta_i \bmod \partial\widetilde{K}$, where $\theta_i \cong \xi_i \times I$ and $(\xi_i \times I) \cup (\partial\xi_i \times I) \subset \partial\widetilde{K}$. This uses the remark in [4, §2] about the form of $N(S^{2q-1}; S^{2q+1})$.

Lemma 4.1. $V(\varphi x_i, \psi\alpha_k) = (-1)^q(A^{-1})_{ik}$.

Proof. First note that $S(\theta_i, \alpha_k) = (-1)^{q+1}\delta_{ik}$ in $\widetilde{K}$, for $S(\xi_i, \alpha_k) = (-1)^q\delta_{ik}$ in $\widetilde{Q}$. Thus

$$\begin{aligned}(-1)^q\delta_{ik} &= -S(\theta_i, \alpha_k) = -V(\varphi\partial\theta_i, \psi\alpha_k)\\ &= V(\varphi\xi_i, \psi\alpha_k) = V(\varphi(A_{ij}x_j + D_{ij}y_j), \psi\alpha_k)\\ &= A_{ij}V(\varphi x_j, \psi\alpha_k),\end{aligned}$$

from which the result follows at once, as $\det A \neq 0$. □

5. **Equivalence of simple knots.** Now we are ready to prove an equivalence theorem for simple knots. Let K, $\underline{K}$ be the complements of two simple $(2q-1)$-knots. We shall say that these knots have *isomorphic Blanchfield pairings* if there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} H_q(\widetilde{K}) & \xrightarrow{\quad h \quad} & H_q(\underline{\widetilde{K}}) \\ \Big\downarrow j_* & & \Big\downarrow \underline{j}_* \\ H_q(\widetilde{K}, \partial\widetilde{K}) & \xrightarrow{\quad h \quad} & H_q(\underline{\widetilde{K}}, \partial\underline{\widetilde{K}}) \end{array}$$

where $j_*, \underline{j}_*$ come from the homology sequences of the pairs $(\widetilde{K}, \partial\widetilde{K})$, $(\underline{\widetilde{K}}, \partial\underline{\widetilde{K}})$, such that $\underline{V}(hx, hy) = V(x, y)$ for each $x \in H_q(\widetilde{K}, \partial\widetilde{K}), y \in H_q(\widetilde{K})$. Here $\underline{V}$ denotes the Blanchfield pairing of the second knot.

THEOREM 5.1. *If two simple $(2q-1)$-knots, $q > 3$, have isomorphic Blanchfield pairings, then they are equivalent knots.*

PROOF. Consider a simple presentation of the first knot. Take a level between the $(q-1)$- and q-handles of the knot, and let Q be the complement of the interior of a regular neighborhood of the intersection of the knot with this level. Let $[x_i]_1^n \cup [y_i]_1^n$ be the basis of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ as in §1, with y_i being represented by the cocore of the ith $(q-1)$-handle of the knot. Let $[\chi_i]_1^n \cup [\xi_i]_1^n$ be similarly defined from the dual presentation of the knot, and let $[u_i]_1^n \cup [v_i]_1^n$ and $[\alpha_i]_1^n \cup [\beta_i]_1^n$ be bases of $H_q(\widetilde{Q})$, defined as in §1. Then by Lemma 3.3 these bases are related in the following manner:

$$\begin{pmatrix}\chi\\ \xi\end{pmatrix} = \begin{pmatrix}B & C\\ A & D\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \quad \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-t^{-1}D^* & (-1)^q C^*\\ (-1)^q A^* & -tB^*\end{pmatrix}\begin{pmatrix}\chi\\ \xi\end{pmatrix},$$

where A, B, C, D are $n \times n$ matrices over R.

Let $\varphi: H_q(\widetilde{Q}, \partial\widetilde{Q}) \longrightarrow H_q(\widetilde{K}, \partial\widetilde{K})$, $\psi: H_q(\widetilde{Q}) \longrightarrow H_q(\widetilde{K})$ be the quotient maps; recall that $\varphi y_i = 0$, $\psi\beta_i = 0$ for each i, as in the previous section.

Let $i_*: H_q(\widetilde{Q}) \longrightarrow H_q(\widetilde{Q}, \partial\widetilde{Q})$ be defined as in [4, §6], and note that $j_*\psi = \varphi i_*$.

We use a similar machinery for the second knot, the notation in this case being underlined. Thus for instance we have a basis $[\underline{x}_i]_1^m \cup [\underline{y}_i]_1^m$ of $H_q(\underline{\widetilde{Q}}, \partial\underline{\widetilde{Q}})$.

Add m trivial pairs of $(q-1)$-, q-handles to the first knot, and n pairs to the second. Relabel the bases so that the new handles correspond to the last m basis elements of $H_q(\widetilde{Q}, \partial\widetilde{Q})$ and the first n basis elements of $H_q(\underline{\widetilde{Q}}, \partial\underline{\widetilde{Q}})$, respectively.

As $\underline{\varphi}$ is onto, and $\underline{\varphi}\underline{x}_1 = 0$, there exist $k_{n+1}, \cdots, k_{n+m} \in R$ such that

$$h\varphi x_1 = \underline{\varphi x}_1 + k_{n+1}\underline{\varphi x}_{n+1} + \cdots + k_{n+m}\underline{\varphi x}_{n+m}.$$

Changing base in $H_q(\widetilde{Q}, \partial\widetilde{Q})$, $h\varphi x_1 = \underline{\varphi x}_1$.

Note that this may be effected by a series of handle additions, as in the proof of [4, Theorem 5.1]. Iteration yields

$$h\varphi x_i = \underline{\varphi x}_i, \qquad 1 \leqslant i \leqslant n.$$

So far we have only altered the $\underline{x}_i$, $1 \leqslant i \leqslant n$. Using the fact that φ is onto and $\varphi x_{n+1} = 0$, we have

$$h(k_1\varphi x_1 + \cdots + k_n\varphi x_n + \varphi x_{n+1}) = \underline{\varphi x}_{n+1},$$

for some $k_1, \cdots, k_n \in R$. Changing base in $H_q(\widetilde{Q}, \partial\widetilde{Q})$ yields $h\varphi x_{n+1} = \underline{\varphi x}_{n+1}$. As before, we obtain

$$h\varphi x_i = \underline{\varphi x}_i, \qquad n+1 \leqslant i \leqslant n+m.$$

This has been achieved by column operations on A, the matrix which presents $H_q(\widetilde{K}, \partial\widetilde{K})$ with generators $[x_i]_1^{n+m}$, i.e. by row operations on A^*, the matrix which presents $H_q(\widetilde{K}, \partial\widetilde{K})$ with generators $[\chi_i]_1^{n+m}$. Therefore, the values of $\varphi\chi_i$, $1 \leqslant i \leqslant n+m$, are unchanged. Thus we may repeat the procedure with the χ_i to obtain:

$$\left.\begin{aligned} h\varphi x_i &= \underline{\varphi x}_i \\ h\varphi\chi_i &= \underline{\varphi\chi}_i \end{aligned}\right\}, \qquad 1 \leqslant i \leqslant n+m.$$

Note that

$$\underline{j}_*h\psi\alpha_i = hj_*\psi\alpha_i = h\varphi i_*\alpha_i = (1 - t^{-1})h\varphi\chi_i,$$

$$\underline{j}_*\psi\underline{\alpha}_i = \underline{\psi i}_*\underline{\alpha}_i = (1 - t^{-1})\underline{\varphi\chi}_i = (1 - t^{-1})h\varphi\chi_i,$$

and so, $\underline{j}_*$ being an isomorphism,

$$h\psi\alpha_i = \underline{\psi\alpha}_i, \qquad 1 \leqslant i \leqslant n+m.$$

Using the hypothesis that the pairings are isomorphic, we have from Lemma 4.1:

$$A^{-1} \equiv \underline{A}^{-1} \mod R.$$

Thus $A^{-1} = \underline{A}^{-1} + E$, where E is an $(n+m) \times (n+m)$ matrix over R. We deduce

$$A^{-1} = \underline{A}^{-1} + E, \qquad I = \underline{A}^{-1}A + EA,$$

$$\underline{A} = (I + \underline{A}E)A, \qquad \underline{A} = TA,$$

where $T = I + \underline{A}E$.

As we remarked in §1, $\det A$ is the Alexander polynomial of the knot in dimension q. It follows that $\det A$ is nonzero and $\det A = \det \underline{A} \times$ (unit of R).

Hence $T \in GL(R)$, and as $\mathrm{Wh}((t:\ )) = 0$, $\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}$ may be written as a product of elementary matrices for some identity matrix I. Thus, after enlargement if necessary, $\underline{A}$ may be obtained from A by elementary row operations. These may be induced by handle moves, and so we may arrange for $\underline{A}$ to be equal to A: this we shall shortly do.

From the relationship between the bases,

$$h\varphi\chi_i = h\varphi(B_{ij}x_j + C_{ij}y_j) = B_{ij}h\varphi x_j$$

and

$$h\varphi\chi_i = \underline{\varphi\chi_i} = \underline{\varphi}(\underline{B}_{ij}\underline{x}_j + \underline{C}_{ij}\underline{y}_j) = \underline{B}_{ij}\underline{\varphi x}_j = \underline{B}_{ij}h\varphi x_j.$$

Therefore

$$(B_{ij} - \underline{B}_{ij})\varphi x_j = 0.$$

The relation matrix for $H_q(\widetilde{K}, \partial\widetilde{K})$ with generators $[x_i]_1^{n+m}$ is A, and so there exists an $(n+m) \times (n+m)$ matrix L over R such that $B - \underline{B} = LA$.

Now we make $\underline{A} = A$; as this involves only row operations on $\underline{A}$, $\underline{B}$ is not affected.

The form of the inverse matrix to $\begin{pmatrix} B & C \\ A & D \end{pmatrix}$ provides the equation $(-1)^q A^*B - tB^*A = 0$. Similarly, with $\underline{A} = A$, $(-1)^q A^*\underline{B} - t\underline{B}^*A = 0$. Therefore,

$$\begin{aligned} A^*[(-1)^q L - tL^*]A &= (-1)^q A^*(B - \underline{B}) - t(B^* - \underline{B}^*)A \\ &= (-1)^q A^*B - tB^*A - [(-1)^q A^*\underline{B} - t\underline{B}^*A] = 0. \end{aligned}$$

Since $\det A \neq 0$, $(-1)^q L = tL^*$.

Recall Lemmas 3.2 and 3.4: these show that we may realize geometrically a change of base in $H_q(\widetilde{\underline{Q}}, \partial\widetilde{\underline{Q}})$ providing the matrix L satisfies just this condition. Thus by altering the $[\chi_i]_1^{n+m}$ we may arrange for $\underline{B} = B$.

A similar argument shows that we may make $\underline{D} = D$; and as $\xi = Ax + Dy$, Corollary 2.2 shows that the two knots are equivalent. Alternatively we could apply the corollary to $y = (-1)^q A^*\chi - tB^*\xi$. $\square$

6. **Properties of the Blanchfield pairing.** Given a simple $(2q-1)$-knot, we know from §4 that there is a sesquilinear pairing $V: H_q(\widetilde{K}, \partial\widetilde{K}) \times H_q(\widetilde{K}) \longrightarrow R_0/R$.

We say that such a pairing is *nonsingular* if the associated map $H_q(\widetilde{K}, \partial\widetilde{K}) \longrightarrow \overline{\mathrm{Hom}}(H_q(\widetilde{K}), R_0/R)$ is an isomorphism. Here $\overline{\mathrm{Hom}}(A, B)$ denotes the module of conjugate linear maps of A into B.

Lemma 6.1. *V is nonsingular.*

Proof. By Lemma 4.1, $V(\varphi x_i, \psi\alpha_k) = (-1)^q(A^{-1})_{ik}$.

By the results of §1 and Lemma 3.3, the relation matrix associated with φ is A, and that associated with ψ is $(-1)^qA^*$. The work of Blanchfield [1, p. 351] applies to furnish the desired result. □

Recall that $j_*: H_q(\widetilde{K}) \longrightarrow H_q(\widetilde{K}, \partial\widetilde{K})$ is the isomorphism from the long exact homology sequence of the pair $(\widetilde{K}, \partial\widetilde{K})$, and define $[\ ,\ ]: H_q(\widetilde{K}) \times H_q(\widetilde{K}) \longrightarrow R_0/R$ by

$$[u, v] = V(j_*u, v)$$

where $u, v \in H_q(\widetilde{K})$. Then $[\ ,\ ]$ is also a nonsingular sesquilinear pairing, which we call the *modified* Blanchfield pairing of the knot.

Lemma 6.2. $[u, v] = (-1)^{q+1}\overline{[v, u]}$ *for all* $u, v \in H_q(\widetilde{K})$.

Proof. Using the notation in the proof of Theorem 5.1, we have:

$$\begin{aligned}[\psi\alpha_i, \psi\alpha_k] &= V(j_*\psi\alpha_i, \psi\alpha_k) = V(\varphi i_*\alpha_i, \psi\alpha_k)\\ &= V(\varphi(1-t^{-1})\chi_i, \psi\alpha_k) = (1-t^{-1})V(\varphi\chi_i, \psi\alpha_k)\\ &= (1-t^{-1})V(\varphi(B_{ij}x_j + C_{ij}y_j), \psi\alpha_k)\\ &= (1-t^{-1})B_{ij}V(\varphi x_j, \psi\alpha_k) \quad (\text{as } \varphi y_j = 0)\\ &= (1-t^{-1})B_{ij}(-1)^qA^{-1}_{jk} \quad (\text{by Lemma 4.1})\\ &= (-1)^q(1-t^{-1})(BA^{-1})_{ik}.\end{aligned}$$

From Lemma 3.3 we obtain the relation $(-1)^qA^*B - tB^*A = 0$, and thus we deduce:

$$\begin{aligned}[\psi\alpha_k, \psi\alpha_i] &= (-1)^q(1-t^{-1})(BA^{-1})_{ki}\\ &= (-1)^q(1-t^{-1})(-1)^qt(A^{*-1}B^*)_{ki}\\ &= t(1-t^{-1})(\overline{BA^{-1}})_{ik} = (-1)^{q+1}\overline{[\psi\alpha_i, \psi\alpha_k]}.\end{aligned}$$

This is true for all i, k, and so it is true for a generating set; therefore it is true for all $u, v \in H_q(\widetilde{K})$. □

We take the opportunity to list some properties of $H_q(\widetilde{K})$. The work of Kervaire [5] shows that the map $J: H_q(\widetilde{K}) \longrightarrow H_q(\widetilde{K})$ defined by $Jx = (1-t)x$ is an isomorphism. For a particularly neat argument see the proof of Assertion 5 by Milnor in [9]. The work of Crowell [2] shows that $H_q(\widetilde{K})$ is $\mathbf{Z}$-torsion-free.

Let $M = H_q(\widetilde{K})$; then M has the following properties:

(i) M is a finitely generated R-torsion-module.

(ii) $J: M \longrightarrow M$ is an isomorphism, where $Jx = (1 - t)x$.

(iii) M is **Z**-torsion-free.

(iv) There is a nonsingular sesquilinear pairing, $[\ ,\ ]: M \times M \longrightarrow R_0/R$, which satisfies $[u, v] = (-1)^{q+1}\overline{[v, u]}$ for all $u, v \in M$.

The work of the next section will show that these properties characterize the modules and pairings which can arise.

I should like to thank the referee for pointing out that (iii) is a consequence of the other properties. The proof which follows is his.

By (i) and (ii), there exists $\Delta(t) \in R$ such that $\Delta(t)M = 0$ and $\Delta(1) = 1$ (for if $\{\alpha_i\}$ generate M, then write $\alpha_i = (t-1)\Sigma_j \lambda_{ij}\alpha_j$; so $X = I - (t-1)(\lambda_{ij})$ is a relation matrix for M and we set $\Delta(t) = \det X$). Suppose $\alpha \in M$ is **Z**-torsion, i.e. $k\alpha = 0$ for some $k \in Z$. If $\beta \in M$, the $[\alpha, \beta] = \lambda/k \bmod R$ for some $\lambda \in R$: since $\Delta(t)\alpha = 0$, $\Delta(t)\lambda/k \in R$ and so $k|\lambda$. By (iv), $\alpha = 0$.

7. Finding a knot for the algebra.

LEMMA 7.1. *A finitely generated R-module M which is* **Z**-*torsion-free and such that J is an isomorphism, has a presentation by a square matrix A over R. Moreover, the matrix obtained from A by putting $t = 1$ is unimodular.*

PROOF. This is essentially Lemma II.12 of Kervaire [5]; see also the remark halfway down p. 256 of [5]. The second assertion comes from the fact that the matrix obtained by putting $t = 1$ presents the **Z**-module obtained from M by identifying x and tx, for all $x \in M$. As J is an isomorphism, this is the zero module, and so the matrix presenting it is unimodular. □

THEOREM 7.2. *If M is an R-module satisfying properties* (i)–(iv) *of* §6, *then for $q > 3$ there is a simple $(2q-1)$-knot having M as its homology module $H_q(\widetilde{K})$ and the given pairing as its modified Blanchfield pairing.*

PROOF. Let $(-1)^q A^*$ be the $n \times n$ presentation matrix of M guaranteed by Lemma 7.1, with generators $[\alpha_i]_1^n$. Then M^*, the dual module of M, is presented by A with generators $[x_i]_1^n$ related to the generators of M by $(\varphi x_i, \psi\alpha_j) = (-1)^q A_{ij}^{-1}$ where $\varphi: [x_1, \cdots, x_n:] \longrightarrow M^*$ and $\psi: [\alpha_1, \cdots, \alpha_n:] \longrightarrow M$ are the quotient maps. This is essentially the result of Blanchfield [1, p. 351]; note that in our case M^* denotes the module of conjugate linear maps of M to R_0/R. By property (iv) of §6, there is an isomorphism $j_*: M \longrightarrow M^*$ given by the formula $(\varphi x_i, \psi\alpha_j) = [j_*^{-1}\varphi x_i, \psi\alpha_j]$. Let this isomorphism be described by the matrix B, so that

$$j_*\psi\alpha_i = (1 - t^{-1})B_{ij}\varphi x_j,$$

and let its inverse be described by $-t^{-1}D^*$, so that

$$(1-t^{-1})\varphi x_i = -t^{-1}D^*_{ij}j_*\psi\alpha_j.$$

Note that this uses property (ii) of M.

We deduce that

$$(1-t^{-1})\varphi x_i = -t^{-1}D^*_{ij}j_*\psi\alpha_j = -t^{-1}D^*_{ij}(1-t^{-1})B_{jk}\varphi x_k$$

and so

$$\varphi[x_i + t^{-1}(D^*B)_{ik}x_k] = 0$$

by property (ii). Therefore, as A presents M^* with generators $[x_i]^n_1$, there exists an $n \times n$ matrix C^* with entries in R such that

(*) $$I + t^{-1}D^*B = (-1)^qC^*A, \quad \text{i.e.} \quad -t^{-1}D^*B + (-1)^qC^*A = I.$$

From the equation

$$\begin{pmatrix} -t^{-1}D^* & (-1)^qC^* \\ (-1)^qA^* & -tB^* \end{pmatrix}\begin{pmatrix} B & C \\ A & D \end{pmatrix}$$
$$= \begin{pmatrix} -t^{-1}D^*B + (-1)^qC^*A & -t^{-1}D^*C + (-1)^qC^*D \\ (-1)^qA^*B - tB^*A & (-1)^qA^*C - tB^*D \end{pmatrix}$$

we remark that (*) is one of the equations needed to prove that the matrix on the right is the identity. The dual of (*) is another of the equations needed.

ASSUMPTION. $(-1)^qA^*B - tB^*A = 0$.

Now consider the remaining expression, $-t^{-1}D^*C + (-1)^qC^*D$; this may not be zero, but we shall show how to make it zero by permissible alterations of C and D.

Recall that $-t^{-1}D^*$ was defined as the matrix representing an isomorphism; as $(-1)^qA^*$ is the relevant relation matrix, we may alter $-t^{-1}D^*$ to $-t^{-1}D^* + L(-1)^qA^*$, where L is an $n \times n$ matrix over R.

We shall need to alter C^* if the equation (*) is to retain its form: let $\underline{C}^*$ be the new value of C^*, so that

$$[-t^{-1}D^* + (-1)^qLA^*]B + (-1)^q\underline{C}^*A = I.$$

By subtracting, we obtain

$$\begin{aligned}(-1)^q(\underline{C}^* - C^*)A &= -(-1)^qLA^*B \\ &= -tLB^*A \quad \text{by the assumption.}\end{aligned}$$

Therefore, as $\det A \neq 0$,

$$(-1)^q\underline{C}^* = (-1)^qC^* - tLB^*.$$

Let us consider the effect of such a change on $-t^{-1}D^*C + (-1)^qC^*D$: we obtain:

$$[-t^{-1}D^* + (-1)^q LA^*][C - (-1)^q t^{-1}BL^*] + [(-1)^q C^* - tLB^*][D - (-1)^q t^{-1}AL^*]$$
$$= -t^{-1}D^*C + (-1)^q C^*D + L[(-1)^q A^*C - tB^*D]$$
$$+ t^{-1}[(-1)^q t^{-1}D^*B - C^*A]L^* + (-1)^q L[(-1)^q t^{-1}A^*B + B^*A]L^*$$
$$= -t^{-1}D^*C + (-1)^q C^*D + L + (-1)^q t^{-1}L^*$$

by (*) and the Assumption above. Putting $L = -(-1)^q C^*D$, we obtain zero as desired.

PROOF OF ASSUMPTION. Let $Y = (-1)^q BA^{-1} - t(A^*)^{-1}B^*$. Then Y is a matrix over R, by the following argument.

$$[\psi\alpha_i, \psi\alpha_k] = (j_*\psi\alpha_i, \psi\alpha_k) = ((1 - t^{-1})B_{ij}\varphi x_j, \psi\alpha_k)$$
$$= (1 - t^{-1})B_{ij}(\varphi x_j, \psi\alpha_k) = (1 - t^{-1})B_{ij}(-1)^q A_{jk}^{-1}$$
$$= (-1)^q(1 - t^{-1})(BA^{-1})_{ik},$$

and from property (iv) we obtain

$$[\psi\alpha_i, \psi\alpha_k] \equiv (-1)^{q+1}[\overline{\psi\alpha_k, \psi\alpha_i}] \mod R.$$

Thus

$$(-1)^q(1 - t^{-1})(BA^{-1}) \equiv (-1)^{q+1}(-1)^q(1 - t)(BA^{-1})^* \mod R,$$

from which it follows that

$$(1 - t)Y \equiv 0 \mod R.$$

Now $A^*YA = (-1)^q A^*B - tB^*A = X$, a matrix over R. We may rewrite this as

$$Y = (A^*)^{-1}XA^{-1} = \frac{\operatorname{adj} A^*}{\det A^*} \cdot X \cdot \frac{\operatorname{adj} A}{\det A}.$$

Now $(1 - t)$ is not a factor of $\det A$ (or $\det A^*$), by Lemma 7.1. It follows from this, and the fact that $(1 - t)Y$ is a matrix over R, that Y is a matrix over R.

Just as we were able to change $-t^{-1}D^*$ to $-t^{-1}D^* + (-1)^q LA^*$, so we may alter B to $B + (-1)^q LA$. We then obtain, in place of A^*YA:

$$(-1)^q A^*[B + (-1)^q LA] - t[B^* + (-1)^q A^*L^*]A = A^*[Y + L - (-1)^q tL^*]A.$$

Put

$$L_{ij} = \begin{cases} -Y_{ij}, & i < j, \\ 0, & i \geqslant j. \end{cases}$$

Then $Y + L - (-1)^q tL^*$ has elements only on the diagonal, because $Y = -(-1)^q tY^*$.

Thus we may arrange for Y to be a diagonal matrix; suppose that $Y_{11} = a_{-m}t^{-m} + \cdots + a_0 + \cdots + a_k t^k$. Then

$$tY^*_{11} = a_k t^{-k+1} + \cdots + a_1 + a_0 t + \cdots + a_{-m}t^{m+1},$$

and from $Y = -(-1)^q tY^*$ we obtain

$$Y_{11} = -(-1)^q a_k t^{-(k-1)} - \cdots - (-1)^q a_1 + a_1 t + \cdots + a_k t^k.$$

Now let L be the diagonal matrix with $L_{11} = -(a_1 t + \cdots + a_k t^k)$ and with the other diagonal terms similarly defined.

Then $Y + L - (-1)^q tL^* = 0$, and so we have justified the assumption.

Now it is a simple matter, using Lemmas 3.3 and 3.4, to construct a knot corresponding to the matrix $\begin{pmatrix} B & C \\ A & D \end{pmatrix}$, and hence to the given module and pairing.□

8. **The Seifert matrix.** We conclude by examining the relationship between the work of the preceding sections and some results of Kervaire and Levine. A simple $(2q-1)$-knot has associated with it an integer matrix A, called the Seifert matrix, such that $A + (-1)^q A'$ is unimodular. A is not unique, but is determined up to S-equivalence (see [8], [10], [12]).

The matrix A is defined in terms of linking numbers as follows (see [7], [11]). If U is a spanning surface of the knot, let $i_*: H_q(U) \longrightarrow H_q(S^{2q+1} - U)$ be the map induced by translating q-cycles off U in the positive direction. If $B_j(S)$ denotes the torsion-free part of $H_j(S)$ for any space S, then there is a bilinear map $L: B_q(U) \times B_q(S^{2q+1} - U) \longrightarrow \mathbf{Z}$ which is defined by linking numbers. By Alexander duality, L is a completely dual pairing, i.e. given a basis $\alpha_1, \cdots, \alpha_r$ of $B_r(U)$, there is a dual basis $\beta_1, \cdots, \beta_r$ of $B_q(S^{2q+1} - U)$ such that $L(\alpha_i, \beta_j) = \delta_{ij}$.

Suppose that $i_*(\alpha_i) = A_{ij}\beta_j$ (using the summation convention); this defines a Seifert matrix A of the knot. Then from [7], [11] we know that $tA + (-1)^q A'$ presents the R-module $H_q(\widetilde{K})$, with generators $\beta_1, \cdots, \beta_r$; let the quotient map of this presentation be φ.

Following Levine [7], we consider a $(q+1)$-chain $I \times \alpha_i$ of $\widetilde{K}$, where $I \times \alpha_i$ is transverse to U along $\frac{1}{2} \times \alpha_i$. We have

$$\partial(I \times \alpha_i) = (tA + (-1)^q A')_{ij}\beta_j.$$

In terms of intersections, this gives us

$$S(I \times \alpha_i, \beta_j) = (1-t)\delta_{ij},$$

and so we obtain a formula for the modified Blanchfield duality pairing as follows:

$$(1-t)\delta_{ik} = S(I \times \alpha_i, \beta_k) = V(\varphi\partial(I \times \alpha_i), \varphi\beta_k)$$
$$= [tA + (-1)^q A']_{ij} V(\varphi\beta_j, \varphi\beta_k),$$

and so

$$(*) \qquad V(\varphi\beta_j, \varphi\beta_k) = (1 - t)([tA + (-1)^q A']^{-1})_{jk}.$$

We may use this connection between the Seifert matrix and the duality pairing to extend our earlier theorems. First we quote three theorems due to Kervaire [5] and Levine [8]: note that although they work with homotopy spheres in the differential category, their proofs use handlebody techniques and apply in the PL category.

THEOREM A. *Seifert matrices of equivalent knots of any (odd) dimension are S-equivalent.*

THEOREM B. *Let q be a positive integer and A a square integral matrix such that $A + (-1)^q A'$ is unimodular. If $q \neq 2$, there is a simple $(2q - 1)$-knot with Seifert matrix A. If $q = 2$, and* signature$(A + A')$ *is a multiple of* 16, *there is a simple 3-knot with Seifert matrix S-equivalent to A.*

THEOREM C. *Let $q \geq 2$ and k_1, k_2 be simple $(2q - 1)$-knots with S-equivalent Seifert matrices. Then k_1 is equivalent to k_2.*

From Theorems 5.1, 7.2, A, B, and C, we deduce the following theorems.

THEOREM 8.1. *Let q be a positive integer and A a square integral matrix such that $A + (-1)^q A'$ is unimodular. Then the S-equivalence class of A determines, and is determined by, the duality pairing defined by* (*).

THEOREM 8.2. *If two simple $(2q - 1)$-knots, $q \geq 2$, have isomorphic Blanchfield pairings, then they are equivalent knots.*

THEOREM 8.3. *If M is an R-module satisfying properties* (i)–(iv) *of* §6, *then for $q \neq 2$ there is a simple $(2q - 1)$-knot having M as its homology module $H_q(\widetilde{K})$ and the given pairing as its modified Blanchfield pairing.*

The referee has pointed out that Theorem 8.3 is true for $q = 2$, subject to the following remarks. By the theorem, a module M satisfying properties (i)–(iv) is the module of some knot; M is therefore presented by $tA + (-1)^q A'$ for some Seifert matrix A. Define $\sigma(M, [\ ,\ ]) = \text{signature}(A + A')$; σ is well defined by virtue of Theorem 8.1.

ADDENDUM 8.3. If M is an R-module satisfying properties (i)–(iv) of §6, and $16 | \sigma(M, [\ ,\ ])$, then there is a simple 3-knot having M as its homology module $H_2(\widetilde{K})$ and the given pairing as its modified Blanchfield pairing.

Finally, we remark that Theorem 8.1 has been proved by Trotter [12] using entirely algebraic techniques.

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