FIXED POINTS OF POINTWISE ALMOST PERIODIC
HOMEOMORPHISMS ON THE TWO SPHERE

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ABSTRACT. A homeomorphism f of the two-sphere $S^2$ onto itself is defined to be pointwise almost periodic (p.a.p.) if the collection of orbit closures forms a decomposition of $S^2$. It is shown that if $f: S^2 \to S^2$ is p.a.p. and orientation-reversing then the set of fixed points of $f$ is either empty or a simple closed curve; if $f: S^2 \to S^2$ is p.a.p. orientation-preserving and has a finite number of fixed points, then $f$ is shown to have exactly two fixed points.

1. Introduction. Every periodic mapping $f$ of the two-sphere $S^2$ to itself is topologically equivalent either to the identity, to a rotation, a reflection, or to a rotation followed by a reflection ([5] and [9]). Thus, the set of fixed points of $f$ is either empty or an $i$-sphere, $0 \leq i \leq 2$. If $f$ satisfies the weaker condition of being almost periodic (equivalent to having equicontinuous iterates) or the still weaker condition of being weakly almost periodic (the collection of orbit closures forms a continuous decomposition of $S^2$), the fixed point set is again either empty or an $i$-sphere, $0 \leq i \leq 2$ ([11] and [12]).

In this paper we study the fixed point sets of pointwise almost periodic (p.a.p.) homeomorphisms on $S^2$ (the collection of orbit closures forms a decomposition of $S^2$). In the orientation-reversing case the set of fixed points must still be either empty or a 1-sphere (Theorem 6). In the orientation-preserving case, on the other hand, there may be a continuum of fixed points together with any finite or countable number of isolated fixed points (see §6). However, if there are only a finite number of fixed points in the orientation-preserving case, then there must be exactly two (Theorem 7).

The main theorems of this paper are contained in §§5 and 6. §3 gives a summary of the results in the theory of prime ends which we use to prove the main lemma in §4.

2. Definitions and notation. If $A$ is a subset of a space $X$, Cl($A$) and Bd($A$) denote, respectively, the closure and boundary of $A$. $A$ is nondegenerate if $A$ is not a single point. If $f: X \to X$ is a map, then $f|A$ denotes the restriction

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of \( f \) to \( A \). Homeomorphisms will always be onto.

A domain is a connected open set. If \( U \) is a domain in \( S^2 \) and \( M \) is a subset of a component \( R \) of \( S^2 - \text{Cl}(U) \), then \( \text{Bd}(R) \) is the outer boundary of \( U \) with respect to \( M \).

If \( f: X \to X \) is a homeomorphism and \( A \subseteq X \), then \( \text{orbit}(A) \), the orbit of \( A \), is the union of the set of iterates \( f^n(A) \), \( n = 0, \pm 1, \pm 2, \ldots \), \( (f^0 = \text{Id}) \). The set of period two points of \( f \) is the set \( \{ x \in X : f^2(x) = x \} \). Thus, the set of period two points includes the fixed points of \( f \).

\( f: X \to X \) is a recurrent homeomorphism if, given any \( x \in X \) and any neighborhood \( U \) of \( x \), there is a positive integer \( n \) and a negative integer \( m \) such that \( \{ f^n(x), f^m(x) \} \subset U \) \[8, 10.18, p. 83\].

\( f: X \to X \) is a pointwise almost periodic (p.a.p.) homeomorphisms if given any \( x \in X \) and any neighborhood \( U \) of \( x \), there is a finite set \( K \) of integers such that the orbit of \( x \) is contained in the union of the sets \( f^n(U) \), \( n \in K \) \[8, 4.02, p. 31\]. If \( X \) is a locally compact \( T^2 \) space, an equivalent definition is: \( f: X \to X \) is p.a.p. if the collection \( \{ \text{Cl}(\text{orbit}(x)) : x \in X \} \) forms a decomposition of \( X \) \[8, 4.10, p. 32\]. Note that if a homeomorphism is p.a.p. then it is recurrent.

If \( U \) is a domain in \( S^2 \) with a nondegenerate boundary, then a crosscut of \( U \) is an open arc in \( U \) whose closure is an arc which intersects \( \text{Bd}(U) \) in two points. An endcut of \( U \) is a half-open arc in \( U \) whose closure is an arc which intersects \( \text{Bd}(U) \) in one point. If \( A \) is a crosscut or an endcut of \( U \), then a subendcut of \( A \) is an endcut of \( U \) which is contained in \( A \).

3. Prime ends. This section contains the results from the theory of prime ends which we use in the next two sections. See also [12, §3] or [15].

Let \( U \) be a simply connected domain in \( S^2 \) with a nondegenerate boundary. A \( C \)-transformation of \( U \) onto the open unit disk \( D \) is a homeomorphism \( T: U \to D \) such that the image of any crosscut of \( U \) is a crosscut of \( D \), and the endpoints of such images of crosscuts of \( U \) are dense in the unit circle. \( C \)-transformations always exist \[15, \text{Appendix 2}\].

A collection of crosscuts \( Q_1, Q_2, \ldots \), of the simply-connected domain \( U \) is a chain if (a) the arcs \( \text{Cl}(Q_1), \text{Cl}(Q_2), \ldots \), are pairwise disjoint; (b) \( Q_n \) separates \( Q_{n-1} \) from \( Q_{n+1} \) in \( U \); (c) there is a point on \( \text{Bd}(U) \) whose greatest distance from \( Q_n \) approaches 0 as \( n \to \infty \). Corresponding to each \( Q_n \) there is a domain \( U_n \) of \( U - Q_n \) containing \( Q_{n+1} \). Note \( U_1 \supset U_2 \supset \cdots \).

If \( \{ Q_i \}, \{ R_i \} \) are chains of crosscuts and \( \{ U_i \}, \{ H_i \} \) are their respective corresponding domains, then \( \{ Q_i \}, \{ R_i \} \) are equivalent chains if for every \( n \) there is an \( m \) such that \( H_m \subset U_n \) and \( U_m \subset H_n \). Equivalent chains are said to define the same prime end. Thus, a prime end of \( U \) is an equivalence class of chains of \( U \).
If \( Q_1, Q_2, \ldots \) is a chain of crosscuts in \( U \), then their images \( T(Q_1), T(Q_2), \ldots \) under the \( C \)-transformation \( T: U \rightarrow D \) is a chain in \( D \). If \( \{Q_i\}, \{R_i\} \) are equivalent chains in \( U \), then \( \{T(Q_i)\}, \{T(R_i)\} \) are equivalent chains in \( D \) and converge to the same point on the boundary of \( D \). Thus, \( T \) sets up a 1-1 correspondence between prime ends of \( U \) and points on the unit circle.

If \( Q_1, Q_2, \ldots \) is a chain defining the prime end \( E \) and \( U_1, U_2, \ldots \) are the corresponding domains of the chain, then the impression of \( E \) is the set \( \bigcap_{i=1}^{\infty} \text{Cl}(U_i) \). The impression of \( E \) is easily seen to be independent of which defining chain is used. Note that Impression (\( E \)) \( \subset \text{Bd}(U) \) and that distinct prime ends may have identical impressions.

Suppose \( U \) is a simply connected domain, \( T: U \rightarrow D \) is a \( C \)-transformation, \( E \) is a prime end of \( U \), and \( e \) is the point on the unit circle corresponding to \( E \) under \( T \). A half-open arc \( B \) in \( U \) defines the prime end \( E \) if \( T(B) \) is an endcut in \( D \) with \( e \) as a limit point. Among the half-open arcs defining the prime end \( E \) there is one \( A \) such that \( \text{Cl}(A) - A \) is minimal (contained in \( \text{Cl}(B) - B \) for every half-open arc \( B \) defining \( E \)). This minimal set is the set of principal points of \( E \) (for an alternate definition of principal point see [12, §3]). Note that a prime end defined by an endcut has just one principal point.

Given a homeomorphism \( f: \text{Cl}(U) \rightarrow \text{Cl}(U) \), with \( f(U) = U \), and a \( C \)-transformation \( T: U \rightarrow D \), it follows that \( TfgT^{-1}: D \rightarrow D \) is a \( C \)-transformation which may be extended to a homeomorphism \( h \) of the closed unit disk onto itself [15, 4.10, p. 6; A1.7, p. 27]. If \( E \) is a prime end of \( U \), \( e \) is the point on the unit circle corresponding to \( E \) under \( T \), and \( h(e) = e \), then \( E \) is a fixed prime end of \( f \). If \( G \) is another prime end of \( U \), \( p \) is the point on the unit circle corresponding to \( G \) under \( T \), and \( p \) converges to \( e \) under positive iterates of \( h \), then we say the prime end \( G \) converges to the prime end \( E \) under positive iterates of \( f \). These last two definitions are independent of the choice of the \( C \)-transformation \( T \).

The reader unfamiliar with prime ends might attempt to show as an exercise that if \( K \) is a pseudo-arc [1] and \( E \) is a prime end of \( S^2 - K \) then Impression (\( E \)) = \( K \); also, there exists a prime end \( E \) of \( S^2 - K \) such that every point of \( K \) is a principal point of \( E \). Completing this exercise, however, is not necessary for understanding the rest of the paper.

4. The Main lemma. Preliminary remark: In the proof of Lemma 1 below, various crosscuts and endcuts are constructed in a domain \( U \). Our diagrams, however, will always show the images in the open unit disk of these crosscuts and endcuts under a \( C \)-transformation. To avoid clumsy notation the crosscuts and endcuts will be denoted with the same letters as their \( C \)-images in the diagrams.
**Lemma 1.** Suppose \( f : S^2 \rightarrow S^2 \) is an orientation-preserving homeomorphism and \( U \) is an invariant, simply-connected domain with nondegenerate boundary. Suppose \( A \) and \( B \) are endcuts of \( U \) such that (1) the prime end \( E \) of \( U \) which is defined by \( A \) is fixed under \( f \), and (2) the prime end \( F \) of \( U \) which is defined by \( B \) is distinct from \( E \) but converges to \( E \) under positive iterates of \( f \). Then \( f \) is not recurrent on \( \text{Bd}(U) \).

**Proof.** Let \( b \) be the point \( \text{Cl}(B) \cap \text{Bd}(U) \).

**Case 1.** \( b \) is not in the impression of the prime end \( E \). Then let \( V \) be a neighborhood (in \( S^2 \)) of \( b \) whose closure misses Impression \( (E) \). Let \( U_1 \supset U_2 \supset \cdots \) be a sequence of subdomains of \( U \) such that \( \text{Impression}(E) = \bigcap_{i=1}^{\infty} \text{Cl}(U_i) \). Then for some \( n \), \( U_n \cap V = \emptyset \). If \( f \) were recurrent at \( b \), then subendcuts of infinitely many positive iterates of \( B \) would be contained in \( V \), and thus would miss \( U_n \). Then the prime end \( F \) would not converge to the prime end \( E \).

**Case 2.** \( f \) is periodic at \( b \), with least period \( n \). If \( f(b) = b \), \( n = 1 \), let \( Y_1 \) be an open arc in \( U \) such that \( \text{Cl}(Y_1) \) is a simple closed curve and \( Y_1 \) contains a subendcut of \( B \) and a subendcut of \( f(B) \). If \( f^n(b) = b \), \( n > 1 \), let \( Y_1, \cdots, Y_n \) be a set of pairwise disjoint crosscuts of \( U \) such that each \( Y_i \) contains a subendcut defining the same prime end as \( f^{i-1}(B) \) and a subendcut defining the same prime end as \( f^i(B) \), \( 1 \leq i \leq n \). Note that \( \text{Cl}(Y_n) \cap \text{Cl}(Y_1) = \{ b \} = \{ f^n(b) \} \). In both cases, \( \text{Cl}(Y_1) \cup \cdots \cup \text{Cl}(Y_n) \) forms a simple closed curve \( J \) such that \( J \) separates a subendcut of \( A \) from the closure of some endcut \( N \) of \( U \) (see Figure 1 for a sketch of the \( C \)-images of \( Y_1, \cdots, Y_n, A, N \)). Note that \( \text{Bd}(U) \cap \text{Cl}(N) \) cannot lie in Impression \( (E) \).

Now, if \( J' \) is the arc in the unit circle bounded by the endpoints of the \( C \)-images of \( A \) and \( B \) and containing the endpoint of the \( C \)-image of \( f(B) \), and if \( h \) is the (orientation-preserving) homeomorphism of the closed unit disk associated with \( f \) (see definition of convergent prime end, §3), then \( h(J') \subset J' \). The endpoint of the \( C \)-image of \( N \) is in \( J' \) and thus converges to the fixed endpoint of \( J' \) under positive iterates of \( h \).

But then the prime end determined by \( N \) converges to the prime end \( E \), and \( \text{Bd}(U) \cap \text{Cl}(N) \) cannot lie in Impression \( (E) \). Hence, by Case 1, \( \text{Bd}(U) \cap \text{Cl}(N) \) is not a recurrent point of \( f \).

**Case 3.** \( b \) is in the impression of \( E \), but is not a periodic point. We suppose \( f \) recurrent on \( \text{Bd}(U) \) and derive a contradiction.

Our plan is to construct a simple closed curve \( J \), made up of crosscuts of \( U \) plus an arc \( Y \), such that \( J \) separates the endcut \( A \) and some point of Impression \( (E) \), (to construct \( J \) we may have to modify \( f \) on some subdisks of \( U \)), then to obtain a certain subcontinuum \( L \) of Impression \( (E) \) such that \( L \cap Y \neq \emptyset \), but \( L \) misses one component of \( S^2 - J \), and finally to obtain
the contradiction that \( f(L) = L \) but \( f(L) \) must intersect both components of \( S^2 - J \). We now proceed with this plan.

Since \( E \) is a fixed prime end we may assume \( f = \text{Id} \) on \( A \). For, there is an open disk \( Z \subset U \) and a subendcut \( A' \) of \( A \) such that \( \text{Cl}(Z) \cap \text{Bd}(U) = \text{Cl}(A) \cap \text{Bd}(U) \), and \( A' \cup f(A') \subset Z \). Then replace \( A \) by \( A' \) and \( f \) by \( f \) followed by a homeomorphism which is the identity outside \( Z \) and which is equal to \( f^{-1} \) on \( f(A') \).

Choose a crosscut \( Q \) of \( U - A \) such that (a) \( Q \) has one endpoint on \( A \) and the other on \( \text{Bd}(U) \), (b) there is a positive integer \( n \) such that \( Q \) separates \( f^{-1}(B) \cup B \cup f(B) \) from some subendcut of \( f^n(B) \) in \( U - A \), (c) \( \text{Cl}(Q) \) is disjoint from \( \text{Cl}(f^{-1}(B) \cup B \cup f(B)) \) and from \( f^n(b) \). The existence of \( Q \) follows from the fact that the prime end defined by \( B \) converges to the prime end defined by \( A \).

Next, choose a crosscut \( X \) in \( U \) such that (d) the endpoints of \( X \) are \( b \) and \( f(b) \), (e) \( X \) contains a subendcut defining the prime end, \( F \), (f) the crosscuts \( f^{-1}(X), X, f(X), \ldots, f^n(X) \) form a pairwise disjoint collection (here, \( n \) is the integer mentioned in (b) of the preceding paragraph), and (g) \( Q \) separates \( f^{-1}(X) \) from \( f^n(X) \) in \( U - A \). See Figure 2 for a sketch of \( C \)-images.

The existence of \( X \) follows from the facts that \( f(b) \neq b \) and that \( F \) converges to \( E \).

Next, choose an open (in \( S^2 \)) neighborhood \( O \) of \( b \) such that
- \( \text{Cl}(O) \) is a 2-cell;
- \( \text{Cl}(O) \cap f(\text{Cl}(O)) = \emptyset ; \)
- \( [A \cup \text{Cl}(Q)] \cap [\text{Cl}(O) \cup f(\text{Cl}(O))] = \emptyset ; \)
- \( \text{Cl}(O) \cap [\text{Cl}(X) \cup f^{-1}(\text{Cl}(X))] \) is an arc; and
- \( \text{Cl}(O) \cap f^i(X) = \emptyset \) for \( i = 1, 2, \ldots, n \).
Claim. There is a homeomorphism $k : S^2 \to S^2$ such that:

1. $k = f$ on some neighborhood of $\text{Bd}(U) \cup \text{Cl}(O) \cup A$,
2. $k^i(X) = f^i(X)$ for $i = -1, 0, 1, \cdots, n$,
3. for some integer $m > n$, $k^{-1}(X), X, k(X), \cdots, k^m(X)$ is a pairwise disjoint collection, $k^i(X) \cap \text{Cl}(O) = \emptyset$, \hspace{1cm} $n \leq i < m$, $k^m(X) \cap \text{Cl}(O) \neq \emptyset$, and $Q$ separates $k^{-1}(X)$ and $k^1(X)$ in $U - A$, $n \leq i \leq m$.

Proof of claim. We construct $k$ by modifying $f$ on various subdisks of $U$.

Suppose $f^{n+1}(X) \cap Q \neq \emptyset$. Let $W$ be the component of $U - (f^m(X) \cup f^{-1}(Q))$ such that $\text{Bd}(W)$ contains two disjoint subendcuts of $f^m(X)$. Let $N$ be a crosscut of $W$ such that $\text{Cl}(N) \cap \text{Bd}(U) = \emptyset$, the endpoints of $N$ separate $f^n(b)$ and $f^{n+1}(b)$ from $f^{-1}(Q) \cap f^n(X)$ in $\text{Cl}(f^n(X))$, and $N$ is contained in a small neighborhood of $f^n(X) \cup f^{-1}(Q)$. Since $f^n(X) \cap \text{Cl}(O) = \emptyset$ and $\text{Cl}(Q) \cap f(\text{Cl}(O)) = \emptyset$, we may choose $N$ so that $\text{Cl}(N) \cap \text{Cl}(O) = \emptyset$, (see Figure 3 for $C$-images).
Let $Z \subset U$ be the disk bounded by $N$ and the subarc of $f^m(X)$ cut off by the endpoints of $N$. Then $Z$ misses $\text{Cl}(O)$ since $\text{Bd}(Z)$ misses $\text{Cl}(O)$.

Let $Z_i$ be a small neighborhood of $Z$ such that $\text{Cl}(Z_i)$ is a subdisk of $U$, $\text{Cl}(Z_i) \cap \text{Cl}(O) = \emptyset$, $\text{Cl}(Z_i) \cap A = \emptyset$, and $Z_i \cap f^i(X) = \emptyset$, $i = -1, 0, 1, \ldots, n - 1$. Let $g: S^2 \rightarrow S^2$ be a homeomorphism such that $g = \text{Id}$ outside $Z_i$, $g = \text{Id}$ on $f^m(X) - Z$, and $g(\text{Bd}(Z) - N) = \text{Cl}(N)$. Finally, let $h = fg$.

Note that (1) $h = f$ on some neighborhood of $\text{Bd}(U) \cup \text{Cl}(O) \cup A$, (2) $h^{n+1}(X) \cap Q = \emptyset$, and (3) $\{h^{-1}(X), X, h(X), \ldots, h^{n+1}(X)\} = \{f^{-1}(X), X, f(X), \ldots, f^n(X)\} \cup \{h^{n+1}(X)\}$ is a pairwise disjoint collection. (For, $h^{n+1}(X)$ can intersect the preceding images of $X$ only in $f^{-1}(X)$; otherwise, the preceding images would not be disjoint. But $Q$ separates $h^{n+1}(X)$ and $f^{-1}(X)$.)

If $\text{Cl}(h^{n+1}(X)) \cap \text{Cl}(O) \neq \emptyset$, then $h$ is the homeomorphism we seek. If $\text{Cl}(h^{n+1}(X)) \cap \text{Cl}(O) = \emptyset$, we repeat the above process, modifying $h$ to add another image of $X$ to our collection. This process must finally terminate, however, because we are assuming that $f$ is recurrent at $b$, and $b \in O$, so eventually some image of $X$ will intersect $\text{Cl}(O)$. This completes the proof of our claim.

To simplify notation let us assume that $f$ requires no modification, that $f^{n+1}(X) \cap Q = \emptyset$ and $\text{Cl}(f^{n+1}(X)) \cap \text{Cl}(O) \neq \emptyset$.

Let $P$ be the arc $[\text{Cl}(X) \cup f^{-1}(\text{Cl}(X))] \cap \text{Cl}(O)$. We may assume that $O$ was chosen small enough so that one of the components $V$ of $\text{Cl}(O) - P$ is contained in $U - (A \cup Q)$. Then we must have $\text{Cl}(f^{n+1}(X)) \cap \text{Cl}(O)$ contained in $\text{Cl}(O) - V$. Let $Y$ be an arc in $\text{Cl}(O) - V$ from $b$ to a point of $\text{Cl}(f^{n+1}(X))$ such that, except for its endpoints, $Y$ misses $\text{Cl}(f^{-1}(X) \cup X \cup f(X) \cup \cdots \cup f^{n+1}(X))$. Then from the set $Y \cup \text{Cl}(X \cup f(X) \cup \cdots \cup f^{n+1}(X))$ we may form a simple closed curve $J$.

Note that $f(Y - \{b\})$ does not intersect $J$, because $Y \subset \text{Cl}(O)$ and $f(\text{Cl}(O)) \cap \text{Cl}(O) = \emptyset$ (this is the reason for never modifying $f$ on $\text{Cl}(O)$).

Also, note that $\text{Cl}(A)$ and $f(Y - \{b\})$ are in different components of $S^2 - J$. For, we may obtain an arc $C$ from $A$ to $f(b)$, which does not intersect $J - \{f(b)\}$, by starting at $A$ and traveling along close to $Q$ and then close to $f^{n-1}(X) \cup f^{n-2}(X) \cup \cdots \cup f(X)$ until we hit $f(V)$, and then traveling through $f(V)$ up to $f(b)$. Then $C \cup f(Y)$ is an arc which intersects $J$ in the piercing point $f(b)$, and thus its endpoints must lie in different components of $S^2 - J$.

Let $H$ be the component of $S^2 - J$ containing the endcut $A$. We want to construct a subcontinuum $L$ of $\text{Bd}(U)$ such that $L \subset \text{Cl}(H)$, $L \cap Y \neq \emptyset$, and $f(L) = L$. This will yield a contradiction.

Choose a chain $R_1, R_2, \cdots$ of crosscuts of $U$ defining $E$, with corresponding domains $U_1 \supset U_2 \supset \cdots$, such that $U_1 \cap f^{-1}(X) \cup X \cup \cdots \cup f^{n+1}(X) = \emptyset$ and $U_1 \cap f(V) = \emptyset$.
Note that \( f(b) \in \text{Cl}(U_i), \ i = 1, 2, \cdots, \) since \( b \in \text{Impression}(E) \), and \( E \) is a fixed prime end. Let \( A_1 \) be an arc in \( U_1 \) from \( A \) to a point very close to \( f(b) \). Since \( U_1 \cap f(V) = \emptyset \), we must have \( A_1 \cap f(V) = \emptyset \), hence \( A_1 \cap Y = \emptyset \). Let \( B_1 \) be the subarc of \( A_1 \) from \( A \cap A_1 \) to the first point at which \( A_1 \) intersects \( Y \). Then \( B_1 \subset \text{Cl}(H) \).

Repeating this procedure, we may construct a sequence \( B_1, B_2, \cdots \) of disjoint arcs of \( U \) such that:

1. \( B_i \) intersects \( A \) and \( Y \),
2. \( B_i \subset \text{Cl}(H) \),
3. \( B_i \subset U_{m(i)} \), for some \( m(i) > i \).

We may assume all \( B_i \)'s lie on the "same side" of \( A \) (\( A \cup Q \) does not separate any \( B_i - A \) from any \( B_j - A \) in \( U \)) and that the \( B_i \)'s converge to a subcontinuum \( L \) of \( \text{Bd}(U) \). (See Figure 4 for \( C \)-images.)

But then we may choose a half-open arc \( T \subset U \) in a small neighborhood of \( A \cup B_1 \cup B_2 \cup \cdots \) such that \( T \cap A = \emptyset \), \( T \) also defines the prime end \( E \), and \( \text{Cl}(T) - T = L \). Since \( f \) is orientation-preserving, both \( T \) and \( f(T) \) lie on the same side of \( A \). Thus, by [15, 2.2, p. 2] or [14, 3.38, p. 321], either \( L \subset f(L) \) or \( f(L) \subset L \). But we are assuming that \( f \) is recurrent on \( \text{Bd}(U) \) so we must have \( f(L) = L \) [16, 4.12, p. 247].

Note also that \( L \subset \text{Cl}(H) \) and \( L \cap Y = \emptyset \).

This easily yields a contradiction, for if \( L \cap (Y - \{b\}) \neq \emptyset \), then, since \( f(Y - \{b\}) \subset S^2 - \text{Cl}(H) \), we cannot have \( f(L) = L \); and, if \( b \in L \), then \( f(L) = L \) is a continuum containing \( f(b) \), but every nondegenerate subcontinuum of \( \text{Bd}(U) \) containing \( f(b) \) must intersect \( S^2 - \text{Cl}(H) \), since \( f(V) \cap \text{Bd}(U) = \emptyset \).

![Figure 4](https://www.ams.org/journal-terms-of-use)

**Lemma 2.** Let \( f: X \rightarrow X \) be a p.a.p. homeomorphism of the complete metric space \( X \), \( K \) a compact invariant subset of \( X \), and \( U \) an open subset of \( X \) such that \( U - K \neq \emptyset \). Then there is an open subset \( V \) of \( U \) such that the orbit of \( V \) misses some neighborhood of \( K \).

**Proof.** For each positive integer \( n \), let \( R(n) = \{ x \in U - K: \text{for some integer } m, \text{dist}(f^m(x), K) < 1/n \} \). \( U - K \) is complete and each \( R(n) \) is open in \( U - K \). Therefore, some \( R(n) \) is not dense in \( U - K \); otherwise, \( \bigcap_{n=1}^{\infty} R(n) \) is not empty by the Baire category theorem, but \( f \) cannot be p.a.p. at points of \( \bigcap_{n=1}^{\infty} R(n) \). Hence, for some positive integer \( m \), there is an open set \( V \) in \( U - K \) which misses \( R(m) \). \( V \) is the required open set, and the proof is complete.

**Lemma 3.** Suppose \( f: S^2 \rightarrow S^2 \) is an orientation-reversing p.a.p. homeomorphism, \( p \) is a fixed point of \( f \), and \( Y \) is the component of the set of period two points such that \( p \in Y \). Then \( Y \) is nondegenerate.

**Proof.** Assume \( Y = \{ p \} \). We shall establish a contradiction.

Denote by \( P(2, f) \) the set of period two points of \( f \).

**Claim 1.** There is an orientation-reversing, p.a.p. self-homeomorphism of \( S^2 \) whose set of period two points is totally disconnected and contains a fixed point.

**Proof of Claim 1.** Let \( G \) be the decomposition of \( S^2 \) whose elements are the points of \( S^2 - P(2, f) \) and the components of \( P(2, f) \). \( G \) is upper semicontinuous [7, p. 137], hence the decomposition space \( S^2/G \) is a cactoid [16, (2.2)', p. 172]. It is easily seen that the induced map \( g = \pi f \pi^{-1}: S^2/G \rightarrow S^2/G \) (where \( \pi: S^2 \rightarrow S^2/G \) is the decomposition map) is a p.a.p. homeomorphism, and \( P(2, g) = \pi(P(2, f)) \). Thus, \( P(2, g) \) is totally disconnected. Now, \( \pi(p) \) is not a cut point of \( S^2/G \), since \( \pi^{-1}(\pi(p)) = p \), so \( \pi(p) \) is either an endpoint of \( S^2/G \) or is contained in a true cyclic element of \( S^2/G \) [16, p. 66]. If \( M \) is a true cyclic element (2-sphere) containing \( \pi(p) \), then \( g(\pi(p)) = \pi(p) \), so \( g(M) = M \), and \( g|_M \) is the required homeomorphism (it is clear that \( g \) must be orientation-reversing).

If \( \pi(p) \) is an endpoint, then there is a cutpoint \( \pi(q) \) fixed by \( g \) [16, 4.22, p. 247]. If \( C(\pi(q), \pi(p)) \) is the cyclic chain [16, p. 71] of \( S^2/G \) from \( \pi(q) \) to \( \pi(p) \), then every cyclic element of \( C(\pi(q), \pi(p)) \) is invariant under \( g \) [16, 4.3, p. 248]. Let \( M \) be any true cyclic element of \( C(\pi(q), \pi(p)) \). Then \( M \) is a 2-sphere and \( g(M) = M \). \( M \) contains a point which separates \( \pi(q) \) and \( \pi(p) \) in \( S^2/G \) [16, 5.2, p. 71], and this point must be a fixed point of \( g \) [16, 4.21, p. 247]. Thus, \( g|_M \) is the required orientation-reversing homeomorphism.

Claim 1 has been established.
By Claim 1 we may assume without loss of generality that \( P(2, f) \) is totally disconnected.

Let \( x, y \) be points of \( S^2 - \{p\} \) such that \( f(x) = y, \) and \( f(y) = x \) (\( x = y \) is allowed). The existence of \( x, y \) follows from the fact that if \( f^2|S^2 - \{p\} \) were fixed point free, then \( S^2 - \{p\} \) would contain a point converging to \( p \) under positive iterates of \( f^2 \) \([2, \text{Theorem } 8, \text{p. } 45]\) (or see Theorem 7 of the present paper). Let \( K \) be a continuum in \( S^2 \) which is invariant under \( f \), which contains \( \{p, x, y\} \) and which is minimal with respect to containing \( \{p, x, y\} \) and being closed, connected, and invariant. By Lemma 2, there is a set \( U_1 \subset K \) open in \( K \), such that the orbit of \( U_1 \) misses a neighborhood of \( P(2, f) \cap K \). Let \( A_1 \) be the component of \( K - \text{orbit}(U_1) \) containing \( p \). Note that \( A_1 \) is invariant and nondegenerate. Since \( K \) is minimal, \( A_1 \) cannot contain \( x \) or \( y \).

Let \( D_1 \) be the component of \( S^2 - A_1 \) containing \( x \).

Claim 2. \( f(D_1) \cap D_1 = \emptyset \).

Proof of Claim 2. Suppose \( f(D_1) \cap D_1 \neq \emptyset \). Then \( D_1 \) is a simply-connected, invariant domain with a nondegenerate boundary. We note that \( D_1 \) has a prime end which is fixed under \( f \). For, let \( T \) be a \( C \)-transformation of \( D_1 \) onto the open unit disk, and extend \( T f T^{-1} \) to an orientation-reversing homeomorphism \( h \) of the closed unit disk onto itself. Then \( h \) must have two fixed points on the unit circle, and these two fixed points correspond to fixed prime ends of \( f \).

But then the orientation-preserving homeomorphism \( f^2 \) must also have a fixed prime end in \( D_1 \). Hence, every prime end of \( D_1 \) is either fixed under \( f^2 \) or converges to a fixed prime end under positive iterates of \( f^2 \) \([3, \text{Lemma } 14]\).

Since \( P(2, f) \) is totally disconnected and closed, there is an endcut \( B \) in \( D_1 \) such that \( \text{Cl}(B) \cap \text{Bd}(D_1) \) is not in \( P(2, f) \). Thus, if \( F \) is the prime end defined by \( B \), the principal point of \( F \) is not fixed under \( f^2 \), and so \( F \) is not a fixed prime end of \( f^2 \) \([12, \text{Lemma } 1]\). Thus, \( F \) converges under positive iterates of \( f^2 \) to a fixed prime end \( E \). Since every principal point of \( E \) is fixed under \( f^2 \) \([12, \text{Lemma } 1]\) and \( P(2, f) \) is totally disconnected, \( E \) has only one principal point, \([14, \text{Corollary, p. } 275]\). Thus, there is an endcut \( A \) of \( D_1 \) which defines \( E \). But then, by Lemma 1, \( f^2 \) is not recurrent on \( \text{Bd}(D_1) \). This contradicts the fact that, since \( f \) is p.a.p. on \( \text{Bd}(D_1) \), \( f^2 \) is p.a.p. on \( \text{Bd}(D_1) \) \([8, \text{p. } 31]\).

Claim 2 is established.

Since \( x \in D_1 \) and \( A_1 \) is invariant, \( f^2(D_1) = D_1 \), hence \( \text{Bd}(D_1) \cup f(\text{Bd}(D_1)) \) is an invariant subset of \( f \).

Claim 3. There is a nondegenerate, invariant sub continuum \( L \) of \( A_1 \) such that if \( O \) is the component of \( S^2 - L \) containing \( x \), then \( p \in \text{Bd}(O) \).

Proof of Claim 3. If \( p \in \text{Bd}(D_1) \), then \( \text{Bd}(D_1) \cup f(\text{Bd}(D_1)) \) is the required sub continuum. Suppose \( p \notin \text{Bd}(D_1) \). Let \( B_1 \) be an invariant sub continuum of
$A_1$ containing $\{p\} \cup \text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ and minimal with respect to these properties. Let $V_2$ be a ball of radius less than $\frac{1}{2}$, centered at $p$. By Lemma 2, there is a set $U_2 \subset B_1 \cap V_2$, open in $B_1$, such that the orbit of $U_2$ misses a neighborhood of $[B_1 \cap P(2, f)] \cup \text{Bd}(D_1) \cup f(\text{Bd}(D_1))$. Let $A_2$ be the component of $B_1 - \text{orbit}(U_2)$ containing $p$. Then $A_2$ is invariant, nondegenerate; and, by the minimality of $B_1$, $A_2$ does not intersect $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$. If $D_2$ is the component of $S^2 - A_2$ containing $\text{Bd}(D_1)$ (and thus containing $\text{Cl}(D_1)$), then by the same proof as in Claim 2, $f(D_2) \cap D_2 = \emptyset$. Also, $f^2(D_2) = D_2$, since $x \in D_2$. And clearly, we must have $\text{orbit}(U_2) \cap D_2 \neq \emptyset$. But then, for some integer $n$, $f^n(D_2) \cap U_2 \neq \emptyset$. Thus, either $D_2 \cap U_2 \neq \emptyset$ or $f(D_2) \cap U_2 \neq \emptyset$. Hence, $D_2 \cup f(D_2)$ intersects a neighborhood of $p$ of radius less than $\frac{1}{2}$.

If $p \in \text{Bd}(D_2)$, then $\text{Bd}(D_2) \cup f(\text{Bd}(D_2))$ is the required subcontinuum. If $p \notin \text{Bd}(D_2)$, we continue the above process. Either we terminate at a finite stage or else we obtain a sequence $D_1, D_2, \cdots$ of simply-connected domains such that:

1. $D_1 \subset \text{Cl}(D_1) \subset D_2 \subset \text{Cl}(D_2) \subset D_3 \subset \cdots$;
2. $\text{Bd}(D_i) \subset A_1$, $f(D_i) \cap D_i = \emptyset$, $f^2(D_i) = D_i$, for each $i$;
3. $f(D_i) \cup D_i$ intersects a neighborhood of $p$ of radius less than $1/i$, for each $i$.

If we let $O = \bigcup_{i=1}^{\infty} D_i$, and $L = \text{Bd}(O) \cup f(\text{Bd}(O))$, then $L$ is the required subcontinuum, with $p \in \text{Bd}(O) \cap f(\text{Bd}(O))$.

Claim 3 is established.

As in Claim 2, we must have $f(O) \cap O = \emptyset$. And since $x \in O$, $f^2(O) = O$. Now let $L_1$ be the outer boundary of $O$ with respect to $f(O)$. Since $p \in \text{Bd}(O) \cap f(\text{Bd}(O))$, we must have $p \in L_1$. By Lemma 2, there is a set $V \subset L_1 \cup f(L_1)$, open in $L_1 \cup f(L_1)$, such that $\text{orbit}(V)$ misses a neighborhood of $P(2, f) \cap [L_1 \cup f(L_1)]$. Let $X$ be the component of $[L_1 \cup f(L_1)] - \text{orbit}(V)$ containing $p$. Then $X$ is nondegenerate and invariant. Let $W$ be the component of $S^2 - X$ containing $x$. As before, $f(W) \cap W = \emptyset$. Since $L_1 - \text{orbit}(V)$ does not separate $x$ and $f(x)$ [13, p. 176], $\text{Bd}(W)$ must contain points of $f(L_1) - L_1$. Let $q$ be a point of $\text{Bd}(W) \cap f(L_1)$, $q \neq p$. Since $f(L_1) \subset \text{Bd}(f(O))$, we note that $f(O) \cup \{q\} \cup W$ is a connected set containing $x$ and $f(x)$. Now let $X_1$ be an invariant subcontinuum of $X$ containing $p$ and $\text{Cl}(\text{orbit}(q))$ and minimal with respect to these properties. By Lemma 2, there is a set $Z \subset X_1$, open in $X_1$, such that orbit(Z) misses a neighborhood of $[P(2, f) \cap X_1] \cup \text{Cl}(\text{orbit}(q))$. Let $X_2$ be the component of $X_1 - \text{orbit}(Z)$ containing $p$. Then $X_2$ is nondegenerate, invariant, and $X_2$ misses the set $f(O) \cup \{q\} \cup W$. But then the component of $S^2 - X_2$ containing $f(O) \cup \{q\} \cup W$ is invariant, and we arrive at exactly the same contradiction as in the proof of Claim 2.
This final contradiction establishes Lemma 3.

**Lemma 4.** Suppose \( f: S^2 \to S^2 \) is a homeomorphism and \( U \) is a simply-connected domain such that \( f(U) \cap U = \emptyset \) and \( f = \text{Id} \) on \( \text{Bd}(U) \). Then \( \text{Bd}(U) \) is a simple closed curve.

**Proof.** Fix a point \( x \in \text{Bd}(U) \).

**Claim.** There is an endcut \( A \) of \( U \) such that \( \text{Cl}(A) \cap \text{Bd}(U) = \{x\} \).

**Proof of Claim.** Let \( R_1, R_2, \ldots \) be a chain of crosscuts defining a prime end whose impression contains \( x \), and let \( U_1 \supset U_2 \supset \cdots \) be the corresponding subdomains of \( U \) (so \( x \in \bigcap_{i=1}^{\infty} \text{Cl}(U_i) \)). For each \( i \), let \( J_i \) be the simple closed curve \( \text{Cl}(R_i) \cup f(\text{Cl}(R_i)) \), and let \( D_i \) be the component of \( S^2 - J_i \) containing \( U_i \). Then \( \text{Cl}(D_1) \supset \text{Cl}(D_2) \supset \cdots \). Since the diameters of the \( R_i \)'s converge to zero, the diameters of the \( J_i \)'s converge to zero, and so \( \bigcap_{i=1}^{\infty} \text{Cl}(U_i) \subset \bigcap_{i=1}^{\infty} \text{Cl}(D_i) \) is the single point \( x \). Thus, if \( A \) is a half-open arc in \( U \) such that each \( R_i \) separates the endpoint of \( A \) from some terminal portion of \( A \), we see that \( \text{Cl}(A) - A \subset \bigcap_{i=1}^{\infty} \text{Cl}(U_i) = \{x\} \). Thus, \( A \) is the required endcut and the claim is proved.

By [16, p. 58], \( \text{Bd}(U) \) is a simple closed curve provided any two points of \( \text{Bd}(U) \) separate \( \text{Bd}(U) \). Let \( x, y \) be any two points of \( \text{Bd}(U) \). By the claim there is a crosscut \( A \) of \( U \) such that \( \text{Cl}(A) \cap \text{Bd}(U) = \{x, y\} \). Then \( J = \text{Cl}(A) \cup f(\text{Cl}(A)) \) is a simple closed curve, and \( \text{Bd}(U) \) must intersect both components of \( S^2 - J \) (otherwise \( A \) and \( f(A) \) would lie in the same component of \( S^2 - \text{Bd}(U) \)). But then \( J \) separates \( \text{Bd}(U) - \{x, y\} \) in \( S^2 \), hence \( \{x, y\} \) separates \( \text{Bd}(U) \). The proof of Lemma 4 is complete.

**Lemma 5.** Suppose \( f: S^2 \to S^2 \) is an orientation-reversing, p.a.p. homeomorphism, \( K \) is a component of the set of period two points of \( f \) such that \( K \) contains a fixed point, and \( U \) is a component of \( S^2 - K \). Then \( f(U) \cap U = \emptyset \), and \( f^2(U) = U \).

**Proof.** First we show that \( f(U) \cap U = \emptyset \). Suppose not. Then, since \( K \) is invariant, \( f(U) = U \). Let \( G \) be the (upper semicontinuous) decomposition of \( S^2 \) whose only nondegenerate element is \( S^2 - U \). Then the decomposition space \( S^2/G \) is homeomorphic to \( S^2 \) [16, (2.1)', p. 171], and the induced map \( g = \pi f \pi^{-1}: S^2/G \to S^2/G \) (where \( \pi: S^2 \to S^2/G \) is the decomposition map) is easily seen to be a p.a.p. orientation-reversing homeomorphism. If we denote the set of period two points of \( f \) by \( P(2, f) \), then \( \pi(P(2, f)) = P(2, g) \), and \( \pi(S^2 - U) \) is a degenerate component of \( P(2, g) \). This contradicts Lemma 3. Hence \( f(U) \cap U = \emptyset \).

Now suppose \( f^2(U) \neq U \). Then \( f^2(U) \cap U = \emptyset \). And \( f^2 = \text{Id} \) on \( \text{Bd}(U) \), since \( K \subset P(2, f) \). Hence, by Lemma 4, \( \text{Bd}(U) \) is a simple closed
curve. But then $U$ must be one component of $S^2 - \text{Bd}(U)$ and $f^2(U)$ must be the other component. This is impossible because $f^2$ is orientation-preserving. Hence $f^2(U) = U$, and the proof of Lemma 5 is complete.

**Theorem 6.** Suppose $f: S^2 \rightarrow S^2$ is a p.a.p. orientation-reversing homeomorphism which has a fixed point. Then the set of fixed points of $f$ is a simple closed curve.

**Proof.** Let $K$ be a component of the set of period two points of $f$ such that $K$ contains a fixed point of $f$. Let $V_1, V_2, \cdots$ be a list of the components of $S^2 - K$. By Lemma 5, $f(V_i) \cap V_i = \emptyset$ and $f^2(V_i) = V_i$, for each $i$. Let $A_1 = V_1$ and $B_1 = f(V_1)$, and suppose we have defined sets $A_n, B_n$ which are unions of components of $S^2 - K$ such that:

1. $A_n \cap B_n = \emptyset$;
2. $V_1 \cup V_2 \cup \cdots \cup V_n \subset A_n \cup B_n$;
3. for each $i$, $V_i$ intersects $A_n$ if and only if $f(V_i)$ intersects $B_n$.

Form $A_{n+1}, B_{n+1}$ as follows: if $V_{n+1}$ intersects $A_n \cup B_n$, let $A_{n+1} = A_n$ and $B_{n+1} = B_n$; if $V_{n+1}$ does not intersect $A_n \cup B_n$, let $A_{n+1} = A_n \cup V_{n+1}$, and $B_{n+1} = B_n \cup f(V_{n+1})$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $A \cap B = \emptyset$, $f(A) = B$, $f(B) = A$, and $S^2 = A \cup B \cup K$.

Define a map $g: S^2 \rightarrow S^2$ by

$$g(x) = \begin{cases} f(x) (= f^{-1}(x)), & \text{if } x \in K, \\ f(x), & \text{if } x \in A, \\ f^{-1}(x), & \text{if } x \in B. \end{cases}$$

It is easily checked that $g$ is a periodic, orientation-reversing homeomorphism, and that the set of fixed points of $g$ is identical with the set of fixed points of $f$. The set of fixed points of $g$ is a simple closed curve by [5]. The proof of Theorem 6 is complete.

6. Orientation-preserving homeomorphisms. In the orientation-preserving case, the similarity between the fixed point sets of p.a.p. homeomorphisms and the fixed point sets of periodic (or weakly almost periodic, see [12]) homeomorphisms no longer holds.

**Example.** Let $D_1, D_2, \cdots$ be a (finite or infinite) collection of closed disks in $S^2$ such that the union of the $D_i$'s is compact and locally connected, and if $i \neq j$, then $D_i \cap D_j$ is the south pole $p_0 \in S^2$. For each $i$, let $g_i$ be a homeomorphism of $D_i$ onto the disk $\{(r, \theta) \in R^2 : r < 1 \}$ ($\theta$ polar coordinates). Define $g: S^2 \rightarrow S^2$ by setting $g = \text{Id}$ outside the union of the $D_i$'s and setting $g|D_i = g_i^{-1} f g_i$ where $f(r, \theta) = (r, \theta + 1 - r)$. Then $g$ is
orientation-preserving and p.a.p., and the number of isolated fixed points of \( g \) is equal to the number of disks \( D_i \).

We do, however, have the following partial result.

**Theorem 7.** Suppose \( f: \mathbb{S}^2 \to \mathbb{S}^2 \) is a recurrent, orientation-preserving homeomorphism with a finite number of fixed points. Then \( f \) has exactly two fixed points.

**Proof.** The proof consists mostly of combining known results. Let \( p \) be a fixed point of \( f \), and let \( U \) be a neighborhood of \( p \) which contains no other fixed points.

**Claim.** The fixed point index \( i(f, U) \) of \( f \) on \( U \) is equal to \( +1 \) (see [12, §4] for a short discussion of the local fixed point index \( i(f, U) \) or [4] for a more comprehensive treatment).

**Proof of Claim.** We may assume \( U \neq \mathbb{S}^2 \) so that \( U \) may be identified with a subset of the plane \( \mathbb{R}^2 \). Using the construction in the proof of [6, Lemma 1], we obtain an orientation-preserving homeomorphism \( h: \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( p \) is the only fixed point of \( h \), and \( h = f \) on some neighborhood \( V \subset U \), with \( p \in V \). (In the proof of Lemma 1 of [6], choose the set \( D_1 \) of that proof to be any neighborhood of \( p \) such that \( \text{Cl}(D_1) \) is a disk contained in \( U \). The construction then easily yields the required homeomorphism \( h \).) Since no point of \( V \) converges to \( p \) under positive or negative iterates of \( h \), there is a point \( x \in V - \{p\} \) whose orbit under \( h \) is contained in \( V \) [8, 10.28, p. 85]. If \( x \) is not a period two point, the construction given in [10, p. 89] or [2, p. 45], yields an arc \( A \) in \( \mathbb{R}^2 \) (a so-called translation arc) such that \( x \in A \), one of the endpoints of \( A \) is the image under \( h \) of the other endpoint, and \( A \cap h(A) \) is this common endpoint. Since \( h \) is recurrent at \( x \) it is easy to see that \( A \) can be chosen so that \( h^n(x) \in A \) for some \( n \geq 1 \). If \( x \) has period two then the construction of [10, p. 89] yields either a translation arc as in the previous sentence or an arc \( A \) joining \( x \) and \( h(x) \) such that \( A \cap h(A) = \{x, h(x)\} \). If \( A \cap h(A) = \{x, h(x)\} \), let \( J \) be the simple closed curve \( A \cup h(A) \). In the case \( A \cap h(A) \) is a single point, let \( J \) be a simple closed curve constructed from \( A \cup h(A) \cup \cdots \cup h^n(A) \), where \( n \) is the least integer greater than one such that \( A \cap h^n(A) \neq \emptyset \).

Then if \( D \) is the bounded component of \( \mathbb{R}^2 - J \), \( D \) must contain a fixed point of \( h \) [10, Lemma 1.1, p. 89 and Proof of Theorem 1, p. 90]. Thus, \( p \in D \). And in fact, the fixed point index \( i(h, D) \) is equal to \( 1 \), for, Lemma 1.1 of [10, p. 89], shows that as a point \( t \) makes one positive circuit of \( J \), the vector from \( t \) to \( h(t) \) turns through an angle of \( 2\pi \); thus, the map from \( J \) to the unit circle which takes \( t \) to \( (h(t) - t)/|h(t) - t| \) has degree \( 1 \); thus, the fundamental class of \( H_2(D, D - \{p\}) \) is mapped to the fundamental
class of $H_2(R^2, R^2 - \{0\})$ by the homomorphism induced by $Id - h$, and thus $i(h, D) = 1$ [4]. But $D - V$, $V - D$ contain no fixed point of $h$, and $U - V$ contains no fixed point of $f$, hence:

$$1 = i(h, D) = i(h, V) = i(f, V) = i(f, U).$$

The claim is established.

Thus, if $U_1, \ldots, U_m$ is a collection of pairwise disjoint open subsets of $S^2$ whose union contains all fixed points of $f$ and such that each $U_i$, $1 \leq i \leq m$, contains exactly one fixed point of $f$, then:

$$\sum_{j=1}^{m} i(f, U_j) = L(f),$$

where $L(f)$ is the Lefschetz number of $f$ [12, §4]. But $L(f) = 2$, since $f$ is an orientation-preserving homeomorphism. By our claim, $i(f, U_j) = 1$, $1 \leq j \leq m$, hence $m = 2$, and the proof of Theorem 7 is complete.

We conclude with two questions.

**Question 1.** Is there an example of a homeomorphism $f: S^2 \rightarrow S^2$ such that $f$ is recurrent but not p.a.p.?

**Question 2.** Suppose $f: S^2 \rightarrow S^2$ is an orientation-preserving p.a.p. homeomorphism such that no component of the set of fixed points separates $S^2$. Must the set of fixed points have exactly two components?

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