ON THE DIMENSION OF VARIETIES OF SPECIAL DIVISORS

BY

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ABSTRACT. Let $T_g$ denote the Teichmüller space and let $V$ denote the universal family of Teichmüller surfaces of genus $g$. Let $V^{(n)}_g$ denote the $n$th symmetric product of $V$ over $T_g$ and let $J$ denote the family of Jacobians over $T_g$. Let $f: V^{(n)}_g \rightarrow J$ be the natural relativization over $T_g$ of the classical map defined by integrating holomorphic differentials. Let

$$u : f^*\Omega_{J/T_g}^1 \rightarrow \Omega_{V^{(n)}_T/T_g}^1$$

be the map induced by $f$. We define $G^r_n$ to be the analytic subspace of $V^{(n)}_T$ defined by the vanishing of $\Lambda^{n-r+1} u$.

Put $\tau = (r + 1)(n - r) - rg$. We show that $G^1_n - G^2_n$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. From this result, we may conclude that, for a generic curve $X$, the fiber of $G^1_n - G^2_n$ over the module point of $X$, if nonempty, is smooth of pure dimension $\tau + 1$, a classical assertion.

Variational formulas due to Schiffer and Spencer and Rauch are employed in the study of $G^r_n$.

0. Introduction. Let $X$ be a complete, nonsingular curve of genus $g$ over an algebraically closed field $K$. Let $X^{(n)}$ denote the $n$th symmetric product of $X$. Let $G^r_n(X)$ denote the subvariety of $X^{(n)}$ of all divisors $D$ of degree $n$ such that $\dim |D| \geq r$. (In the literature, e.g. [12], $G^r_n(X)$ is often used to denote the subvariety of the Jacobian of $X$ consisting of all linear systems of degree $n$ and projective dimension at least $r$.)

Put $\tau$ equal to $(r + 1)(n - r) - rg$. Brill and Noether [2] asserted that if $\tau$ were nonnegative and $X$ were a generic curve, then $G^r_n(X)$ would have dimension $\tau + r$. The recent work of Kleiman and Laksov ([10], [11]) and Kempf [8] shows that for $X$ any curve, if $\tau \geq 0$, then $G^r_n(X)$ has dimension at least $\tau + r$. We will show, in the case $K = \mathbb{C}$, that if $X$ is a generic curve, then $G^1_n(X) - G^2_n(X)$, if nonempty, has dimension $\tau + 1$.
We work in the category of analytic spaces over \( \mathbb{C} \). We do this because we want to consider the Teichmüller space, an analytic, but not algebraic, variety \([5]\).

We take the Séminaire Cartan, 1960–61, as our foundational reference. In particular, we allow the structure sheaf of an analytic space to contain nilpotents.

Let \( Y \) be an analytic space over \( \mathbb{C} \) and let \( E \) and \( F \) be locally free \( \mathcal{O}_Y \)-modules of ranks \( g \) and \( n \) respectively. Suppose we are given a map \( u: E \to F \). In §1, we define the analytic space \( Z^r(u) \) to be given by the vanishing of the map \( \bigwedge^{n-r+1} u \). We then study the infinitesimal structure of \( Z^r(u) \).

Let \( S \) be an analytic space over \( \mathbb{C} \) and let \( X \) be a family of nonsingular curves of genus \( g \) over \( S \). Let \( X^{(n)}_S \) denote the \( n \)th symmetric product of \( X \) over \( S \) and let \( J^S \) denote the family of Jacobians over \( S \) (cf. \([7]\), \([15]\)). Suppose we are given a map \( f: X^{(n)}_S \to J^S \). Let

\[
\begin{align*}
u: f^*\Omega^1_{J^S/S} & \to \Omega^1_{X^{(n)}_S/S}
\end{align*}
\]

be the map induced by \( f \). We study the analytic space \( Z^r(u) \subseteq X^{(n)}_S \) in the following situation: \( S = T_g \), the Teichmüller space, \( X \) is the universal family of Teichmüller surfaces of genus \( g \), and \( f \) is the natural relativization over \( T_g \) of the classical map from the \( n \)th symmetric product of a curve into its Jacobian defined by integrating a basis of homomorphic differentials (cf. §2). We let \( G^r \) denote \( Z^r(u) \) in this situation.

In order to understand explicitly the above map \( f \), we must use certain variational formulas which are similar to those derived by Schiffer and Spencer \([19]\), but much closer in form to those appearing in Rauch \([18]\). We also need a theorem due to Patt \([17]\) concerning local coordinates at a point of \( T_g \).

Our main result is:

**Theorem.** Suppose \( y \in G^1_n - G^2_n \). Then the dimension of the tangent space to \( G^1_n \) at \( y \) is \( 3g - 3 + \tau + 1 \).

From this result, we can conclude that if \( X \) is a generic compact Riemann surface, then \( G^1_n(X) - G^2_n(X) \), if nonempty, is smooth of pure dimension \( \tau + 1 \).

As an application, we show that the subvariety of \( T_g \), for \( g \geq 4 \), of curves with nonempty \( G^1_3 \) (so-called "trigonal" curves), is of dimension \( 2g + 1 \), a result which was known to Severi \([22]\) and B. Segre \([20]\).

In a sequel to this paper, we will show that if \( \tau \geq 0 \) then \( G^2_n \) (resp. \( G^3_n \)) has a component of dimension \( 3g - 3 + \tau + 2 \) (resp. \( 3g - 3 + \tau + 3 \)). The proof involves computations using the examples of Riemann surfaces given by Meis \([16]\).

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1. $Z^r(u)$ and its infinitesimal structure. Let $S$ be an analytic space over $\mathbb{C}$. Denote by $((\mathcal{A}n/S))$ the category of analytic spaces over $S$. Let $Y$ be an analytic space over $S$ and let $E$ and $F$ be locally free $\mathcal{O}_Y$-modules of ranks $g$ and $n$ respectively. Suppose we are given a map $u: E \to F$. Define the functor $Z^r(u): ((\mathcal{A}n/S))^0 \to ((\text{Sets}))$ by

$$Z^r(u)(T) = \left\{ g \in \text{Hom}(T, Y) \right\}_{\land}^{n-r+1} g^*u = 0 \right\}.$$

We wish to show that this functor is represented by an analytic subspace of $Y$.

**Definition 1** [5]. Let $S$ be an analytic space and let $G: ((\mathcal{A}n/S))^0 \to ((\text{Sets}))$ be a functor. We say that $G$ is of a local nature if for every $T$ the presheaf $U \mapsto G(U)$, where $U$ runs through the open sets of $T$, is a sheaf.

**Remark.** This is the analog to the notion of a Zariski sheaf in the category of contravariant functors from $((\text{Schemes}))$ to $((\text{Sets}))$.

**Lemma 1.** Let $(S_i)$ be a covering of an analytic space $S$ by open sets. Let $G: ((\mathcal{A}n/S))^0 \to ((\text{Sets}))$ be a functor. Then $G$ is representable iff $G$ is of a local nature and for every $i$, the functor $G/S_i: ((\mathcal{A}n/S_i))^0 \to ((\text{Sets}))$ is representable.

**Proof.** [5, Corollary 5.7 of Exposé 7].

Our functor $Z^r(u)$ is clearly of a local nature. Hence, by the lemma, its representability is a local question.

Let $y$ be a point of $Y$. Since $E$ and $F$ are locally free of ranks $g$ and $n$ respectively, the map $u$ is given locally at $y$ by an $n \times g$ matrix $[f_{jk}]$ of functions regular at $y$. The functor $Z^r(u)$ is then locally represented by the closed analytic subspace defined by the vanishing of the minors of order $n-r+1$ of the matrix $[f_{jk}]$. Thus we have

**Proposition 1.** $Z^r(u)$ is represented by a closed analytic subspace of $Y$.

We will also use $Z^r(u)$ to denote this analytic subspace.

Put $\rho = \text{rank}(u \otimes \kappa(y))$. Locally at $y$, both $E$ and $F$ split off a direct summand of rank $\rho$, and $u$ maps one summand isomorphically onto the other. The map that $u$ induces on the other two summands is given by an $(n-\rho) \times (g-\rho)$ matrix $[e_{jk}]$ of functions regular at $y$. The analytic space $Z^r(u)$ is also defined locally at $y$ by the vanishing of the minors of order $(n-r+1-\rho)$ of the matrix $[e_{jk}]$ (cf. [10]).

**Proposition 2.** Assume $r > 0$. Then the points of $Z^{r+1}(u)$ are singular points of $Z^r(u)$.
Proof. Suppose $y \in Z^{r+1}(u)$. Then we have $\rho < n - r$. By construction, the $e_{jk}$ above vanish at $y$, hence are in the maximal ideal $m$ of $\mathcal{O}_{Y,y}$. The analytic space $Z'(u)$ is defined locally at $y$ by the vanishing of the minors of order $(n - r + 1 - \rho)$ of the matrix $[e_{jk}]$ and, since $\rho < n - r$, all these minors are of order at least 2, hence are in $m^2$. Thus $y$ cannot be a smooth point of $Z'(u)$.

We want now to study the infinitesimal structure of $Z'(u)$. Let $\xi$ denote a tangent vector to $Y$ at $y$. We will also use $\xi$ to denote the comorphism, which is a $C$-homomorphism of local rings $\xi: \mathcal{O}_{Y,y} \to C[e]/(e^2)$.

We are interested in seeing when $\xi$ is a tangent vector to $Z'(u)$ at $y$. By definition, this will be true if $\wedge^{n-r+1} \xi^*u = 0$.

Proposition 3. $\xi$ is a tangent vector to $Z'(u)$ at $y$ iff the minors of order $n - r + 1$ of the matrix $[\xi f_{jk}]$ are all zero.

Proof. It is easy to see that the map $\xi^*u$ is given by the matrix $[\xi f_{jk}]$. Thus we have $\wedge^{n-r+1} \xi^*u = 0$ iff the minors of order $n - r + 1$ of $[\xi f_{jk}]$ all vanish.

We now assume that $Y$ is smooth of dimension $m$ over $C$. Let $y \in Y$ and let $\sigma_1, \cdots, \sigma_m$ be local parameters on $Y$ at $y$. Let $s_i$ in $C$ be given by

$$\xi(\sigma_l) = s_l \epsilon, \quad l = 1, 2, \cdots, m.$$ 

Then, by Taylor's Theorem, we have

$$\xi(f_{jk}) = f_{jk}(y) + \epsilon \sum_{l=1}^{m} s_l \frac{\partial f_{jk}}{\partial \sigma_l}(y).$$

The vanishing of the minors of order $n - r + 1$ of the matrix $[\xi f_{jk}]$ gives rise to linear equations in the $s_l$. These equations must be satisfied for $\xi$ to be a tangent vector to $Z'(u)$ at $y$. If we view $s_1, \cdots, s_m$ as being unknowns, then the dimension of the solution space of this system of equations is the dimension of the tangent space to $Z'(u)$ at $y$.

If $y \in Z'(u) - Z^{r+1}(u)$, we will want to use the following lemma.

Lemma 2. Let $A$ be a commutative ring (with unit). Let $M = [a_{jk}]$ be an $m \times n$ matrix over $A$. Suppose that a minor $\mu$ of order $r$ is a unit, and that every minor of order $r + 1$ containing $\mu$ vanishes. Then every minor of order $r + 1$ vanishes.

Proof. Without loss of generality, we may assume that $\mu$ is the leading (i.e., upper left) minor of order $r$. Since $\mu$ is a unit, we may perform column operations using the first $r$ columns to change $M$ to the matrix
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\[ M' = \begin{bmatrix} \mu & \cdots & \mu \\ a_{r+1,1} & \cdots & a_{r+1,r} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,r} \end{bmatrix} \]

where \( N \) is an \((m - r) \times (n - r)\) matrix.

Then by row operations, using the first \( r \) rows, we may change \( M' \) to the matrix

\[ M'' = \begin{bmatrix} \mu & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & N \end{bmatrix} \]

where \( N \) is the same matrix as before.

Now, no minor containing \( \mu \) is affected by performing these row and column operations. Hence, the minors of order \( r + 1 \) of \( M'' \) which contain \( \mu \) are all zero. Thus \( N \) is the zero matrix.

But this implies that every column of \( M \) is a linear combination of the first \( r \) columns of \( M \). Hence, every minor of order \( r + 1 \) of \( M \) is zero. \( \square \)

Suppose now that \( y \in Z'(u) - Z^{r+1}(u) \). Then the matrix \([f_{jk}]\) has rank \( n - r \). We may thus assume that the leading minor of order \( n - r \) of \([f_{jk}]\), call it \( \mu \), is nonzero. Let \( \mu' \) denote the leading minor of order \( n - r \) of \([\xi(f_{jk})]\).

Then \( \mu' = \mu + ce \), for some \( c \in \mathbb{C} \). Since \( \mu \) is nonzero, \( \mu' \) does not lie in the maximal ideal of \( \mathbb{C}[e]/(e^2) \), hence is a unit. We then have, by Proposition 3 and Lemma 2, that \( \xi \) is a tangent vector to \( Z'(u) \) at \( y \) iff the minors of order \( n - r + 1 \) of \([\xi(f_{jk})]\) which contain \( \mu' \) all vanish. Obviously, there are \( r(g - n + r) \) such minors. If the equations in the \( s_i \) given by the vanishing of these minors are linearly independent (over \( \mathbb{C} \)), then the dimension of the tangent space to \( Z'(u) \) at \( y \) is \( m - r(g - n + r) \). We could then conclude that \( y \) is a smooth point of \( Z'(u) \) by virtue of the following proposition.

**Proposition 4.** Either \( Z'(u) \) is empty, or each component has codimension at most \( r(g - n + r) \) in \( Y \).

**Proof.** This is proved in [9] for \( Y \) a scheme. With the obvious modifications, the proof is valid for \( Y \) an analytic space.

2. The universal family of Teichmüller surfaces. In [5], Grothendieck proved the following

**Theorem 1.** There exist an analytic space \( T_g \) and a family \( V \) of Teichmüller surfaces of genus \( g \) over \( T_g \) which is universal in the following sense: for every family \( X \) of Teichmüller surfaces of genus \( g \) over an analytic space \( S \), there exists a unique map \( \Phi \colon S \to T_g \), such that \( X \) is isomorphic (as a family of Teichmüller surfaces) to the pullback via \( \Phi \) of \( V/T_g \).
The Teichmüller space (for Teichmüller surfaces of genus \(g\)).
The Teichmüller space is a smooth, irreducible, and simply connected analytic
space [5].

Let \(h: V \to T_g\) denote the structural morphism. By well-known topological
facts, since \(T_g\) is simply connected, the fiber bundle \(R^1h_*\mathbb{Z}\) is trivial. Thus,
there are sections of this bundle which give rise to cycles \(\gamma_i(s), \delta_i(s), i = 1, \cdots, g,\)
which form a canonical homology basis for \(H_1(V_s, \mathbb{Z}), s \in T_g\) [15].

Consider the sheaf \(\Omega_{V/T_g}^1\). For all \(s \in T_g\), we have

\[
\dim H^0(V_s, \Omega_{V/T_g}^1 \otimes \kappa(s)) = \dim H^0(V_s, \Omega_{V_s}^1) = g.
\]

Hence, \(h_*\Omega_{V/T_g}^1\) is a vector bundle of rank \(g\) over \(T_g\) and we have

\[
h_*\Omega_{V/T_g}^1 \otimes \kappa(s) \cong H^0(V_s, \Omega_{V_s}^1)
\]
by [4].

Choose holomorphic sections \(d_{\gamma_i}^*, i = 1, \cdots, g,\) of \(h_*\Omega_{V/T_g}^1\) such that
\(\{d_{\gamma_i}^*(s)\}_{i=1}^g\) is a basis for \(H^0(V_s, \Omega_{V_s}^1), s \in T_g\) (cf. [15]). Put

\[
a_{ij}(s) = \int_{\gamma_i(s)} d_{\gamma_j}^*(s), \quad b_{ij}(s) = \int_{\delta_i(s)} d_{\gamma_j}^*(s), \quad i, j = 1, \cdots, g.
\]

For each \(s \in T_g\), the matrix \([a_{ij}(s), b_{ij}(s)]\) is the period matrix of \(V_s\).

Recall that the columns of this matrix generate a maximal lattice subgroup of \(\mathbb{C}^g\).

Let \(J\) be the quotient of \(T_g \times \mathbb{C}^g\) by this family of lattices. The induced pro-
jection \(J \to T_g\) gives a complex analytic family of complex tori, the fiber \(J_s\)
being the Jacobian variety of the Teichmüller surface \(V_s\) [15].

Since our concern will only be local, we assume that there exist sections of
\(V \to T_g\). Let \(P_0(s)\) be such a section. As in [15], define a map \(\psi: V \to J\)
by

\[
\psi(s, P) = \left( s, \int_{P_0^*(s)} d_{\gamma_1}^*(s), \cdots, \int_{P_0^*(s)} d_{\gamma_g}^*(s) \right) \mod \text{periods}
\]
for \(P \in V_s\).

Denote by \(V_{T_g}^{(n)}\) the \(n\)th symmetric product of \(V\) over \(T_g\) (cf. [7]).

Extend \(\psi\) to a map \(f: V_{T_g}^{(n)} \to J\) as follows. If \(s \in T_g\) and \(D \in (V_{T_g}^{(n)})_s\) is
the divisor \(\sum_{i=1}^n P_i\) on \(V_s\), then

\[
f(s, D) = \left( s, \sum_{i=1}^n \int_{P_0^*(s)} d_{\gamma_i}^*(s), \cdots, \sum_{i=1}^n \int_{P_0^*(s)} d_{\gamma_g}^*(s) \right) \mod \text{periods}.
\]
be the map induced by \( f \). Since \( J \) and \( V^{(n)}_{T_g} \) are smooth over \( T_g \) of relative dimensions \( g \) and \( n \) respectively, the sheaves
\[
f^*\Omega^1_{J/T_g} \quad \text{and} \quad \Omega^1_{V^{(n)}_{T_g}/T_g}
\]
are locally free of ranks \( g \) and \( n \) respectively. Thus, we may consider the analytic subspace \( Z'(u) \subseteq V^{(n)}_{T_g} \) of \( \S 1 \). We will denote by \( G'_n \) the analytic space \( Z'(u) \) which arises in this situation. We will see in \( \S 4 \) that \( (G'_n)_s \) is what was denoted by \( G'_n(V_s) \) in \( \S 0 \).

We wish to study the infinitesimal structure of \( G'_n \). To do this, we need explicit knowledge of the above map \( f \). And to obtain this knowledge, we need certain variational formulas which are contained in the next section.

3. The variational formula. For a detailed treatment of the material in this section, the reader is referred to Rauch [18] or Patt [17].

Let \( X \) be a compact Riemann surface of genus \( g > 0 \). Let \( \Gamma = (\gamma_1, \cdots, \gamma_g) \) and \( \Delta = (\delta_1, \cdots, \delta_g) \) be a canonical homotopy basis and let \( \Pi \) be the simply connected surface obtained by the canonical dissection of \( X \) determined by \( \Gamma \) and \( \Delta \) (cf. [23]).

Let \( w \) be a point in the interior of \( \Pi \) and let \( \tau_{w,v}(x) \) denote the (normalized) elementary integral of the second kind with pole of order \( v + 1 \) at \( w \) and zero \( \Gamma \)-periods.

Let \( \xi \) be an Abelian integral of the first kind. Let \( a_i, i = 1, \cdots, g, \) denote the \( \Gamma \)-periods of \( d\xi \); that is,
\[
a_i = \int_{\gamma_i} d\xi, \quad i = 1, \cdots, g.
\]
The value of the derivatives of a determination of \( \xi \) at \( w \) and the periods of the differentials \( d\tau_{w,v} \) are related by
\[
\xi^{(v+1)}(w) = \frac{v!}{2\pi i} \sum_{j=1}^g a_j \int_{\delta_j} d\tau_{w,v}(x)
\]
which follows from the bilinear relation for differentials of the first and second kinds [23, p. 260].

Let \( Q_1, \cdots, Q_n \) be distinct points in the interior of \( \Pi \) such that all the \( Q_j \) are different from \( w \) and none of the \( Q_j \) is a zero of \( d\xi \). Let \( t_j, j = 1, \cdots, n, \) be a local parameter at \( Q_j \). Let \( D_1, \cdots, D_n \) be disjoint disks about \( Q_1, \cdots, Q_n \) respectively, such that \( D_j \) lies in the domain of \( t_j \), is completely
contained in the interior of $\Pi$ and such that no $D_j$ contains either $w$ or any zero of $d\xi$.

Inside $D_j$, we can vary the local parameter $t_j$ to a new parameter $t_j^*$ given by

$$t_j^* = t_j + c_j/t_j, \quad j = 1, \ldots, n,$$

where $c_j$ is sufficiently small. This defines a new Riemann surface $X^*$, having the same canonical homotopy basis as $X$ has (since all variations take place in the interior of $\Pi$).

Let $\xi^*$ be the Abelian integral of first kind on $X^*$ with the same $\Gamma$-periods as $\xi$. We wish to compute

$$\Delta\xi^{(\nu+1)}(w) = \xi^*^{(\nu+1)}(w) - \xi^{(\nu+1)}(w).$$

**Notation.** $d\tau_{w,v}d\xi$ is a (not necessarily finite) quadratic differential on $X$. Locally at $Q_j$, we may write $d\tau_{w,v}d\xi = h(t_j)d\tau_j^2$. We now introduce the notation $\tau'_{w,v}(Q_j)\xi'(Q_j)$ for $h(0)$.

Utilizing the techniques and formulas in [18] and [17], one can obtain the following proposition:

**Proposition 5.**

$$\Delta\xi^{(\nu+1)}(w) = \nu! \sum_{m=1}^{n} c_m \tau'_{w,v}(Q_m)\xi'(Q_m) + O(c^2)$$

where $c = \max_{1 \leq m \leq n} |c_m|$.

We will also want to use the following theorem, due to Patt [17]:

**Theorem 2.** One may choose $3g - 3$ points $Q_1, \ldots, Q_{3g-3}$ on $X$ such that, if $c_m$ is the variation parameter at $Q_m$, then a neighborhood of the origin in the $c_1, \ldots, c_{3g-3}$ space describes a complex-analytic structure for a neighborhood of $X$ in the Teichmüller space. Moreover, the set of collections of $3g - 3$ points with this property is open in $X^{3g-3}$.

**Proof.** The first assertion follows from Theorems 2 and 4 of [17]. Although Patt does not state the second assertion, his proofs demonstrate it, as was noted by Farkas [3, p. 885].

4. The equations which define the tangent space. Let $X$ be a compact Riemann surface of genus $g > 1$. Let $\{\gamma_j, \delta_j\}_{j=1}^g$ be a canonical homotopy basis and let $\{d\xi_k\}_{k=1}^g$ be a basis of the holomorphic differentials. Put

$$A_{jk} = \int_{\gamma_j} d\xi_k, \quad j, k = 1, \ldots, g.$$
Let $P$ be a point of $X$ and let $t$ be a local parameter on $X$ at $P$.

Write

$$d\xi_k = \sum_{l=0}^{\infty} a_{k,l} t^l dt.$$ 

Fix a point $P_0$ different from $P$. Choose a point $(Q_1, \ldots, Q_{3g-3})$ from the open subset of $X^{3g-3}$ in Theorem 2 such that all the $Q_m$ are different from $P$ and $P_0$ and such that none of the $Q_m$ is a zero of any $d\xi_k$. Perform the variation described in §3, taking the disk about each $Q_m$ sufficiently small so that no two disks intersect and no disk contains $P, P_0$, or any zero of any $d\xi_k$.

Let $c_m$ denote the variation parameter at $Q_m$, $m = 1, \ldots, 3g-3$, as in §3.

Let $s_0 \in T_g$ be the module point of $X$ (i.e., $V_{s_0} = X$). By definition of the variation, there exists a complex-analytic neighborhood $U$ of $s_0$ in $T_g$ such that, for all $s' \in U$, the curves $\{\gamma_j, \delta_j\}_{j=1}^{3g}$ are a canonical homotopy basis on $V_{s'}$, the points $P_0$ and $P$ are on $V_{s'}$, and $t$ is a local parameter on $V_{s'}$ at $P$. Choose holomorphic sections $d\xi_k$, $k = 1, \ldots, g$, of $h_*\Omega_{V/T}^1$ such that

$$\int_{\gamma_j} d\xi_k(s') = A_{jk}, \quad s' \in U, \quad j, k = 1, \ldots, g$$

(cf. [15, §3]).

**Proposition 6.** With notation as in §3 and above, if we define $a_{k,l}^*$ by $d\xi_k = \sum_{l=0}^{\infty} a_{k,l}^* t^l dt$, then we have

$$a_{k,l}^* = a_{k,l} + \sum_{m=1}^{3g-3} c_m r_{P,l}(Q_m) \xi_k(Q_m) + O(c^2).$$

**Proof.** The variational formula (Proposition 5) shows that this equality holds in a complex-analytic neighborhood of $(s_0, P)$ on $V$. This is the main import of the variational formula.

In order to study the map

$$u: f^*\Omega_{V/T}^1 \longrightarrow \Omega_{V/T}^1(n)$$

of §2, we first consider the divisor $nP$ on $X$. Let $t_1, \ldots, t_n$ be $n$ copies of $t$, and let $\sigma_1, \ldots, \sigma_n$ denote the $n$ elementary symmetric functions in $t_1, \ldots, t_n$.

**Proposition 7.** Local parameters on $V_{T_g}^1(n)$ at $(s_0, nP)$ are given by

$$c_1, \ldots, c_{3g-3}, \sigma_1, \ldots, \sigma_n.$$
Proof. By Theorem 2, local parameters on $T_g$ at $s_0$ are given by $c_1$, 
$\cdots$, $c_{3g-3}$. By [1], local parameters on $X^{(n)}$ at $nP$ are given by $\sigma_1$, 
$\cdots$, $\sigma_n$. By the definition of the variation in §3, local parameters on $(V^{(n)}_{Tg})_{s'}$ at $nP$, 
for $s' \in U$, are also given by $\sigma_1$, $\cdots$, $\sigma_n$. Thus, local parameters on $V^{(n)}_{Tg}$ at 
$(s_0, nP)$ are given by $c_1$, $\cdots$, $c_{3g-3}$, $\sigma_1$, $\cdots$, $\sigma_n$.

Put
\[ \tau_j = t_1^j \, dt_1 + \cdots + t_n^j \, dt_n, \quad j = 0, 1, 2, \cdots. \]

We have

Proposition 8. The space of holomorphic 1-forms on $X$ is naturally isomorphic to the space of holomorphic 1-forms on $X^{(n)}$. Both these spaces are isomorphic to the space of symmetric holomorphic 1-forms on the Cartesian product $X^n$. If $d\xi = \sum_{\ell=0}^{\infty} a_\ell \, dt$ is a holomorphic 1-form on $X$ and $d\tilde{\xi}$ is the corresponding symmetric holomorphic 1-form on $X^n$, then $d\tilde{\xi} = \sum_{\ell=0}^{\infty} a_\ell \tau_\ell$.

Proof. [14, pp. 226–227].

This result is easily seen to relativize to the following proposition.

Proposition 9. The space of relative holomorphic 1-forms on $V^{(n)}_{Tg}$ over $T_g$ and the space of relative holomorphic 1-forms on $V$ over $T_g$ are naturally isomorphic. Both spaces are isomorphic to the space of relative symmetric holomorphic 1-forms on $V^n_{Tg}$, the product over $T_g$ of $n$ copies of $V$, over $T_g$. If $d\tilde{\xi}_k^*$ is the relative symmetric holomorphic 1-form on $V^n_{Tg}$ over $T_g$ corresponding to $d\tilde{\xi}_k^*$ (cf. Proposition 6), then

\[ d\tilde{\xi}_k^* = \sum_{i=0}^{\infty} a_{k,i}^* \tau_i. \]

We will identify relative symmetric holomorphic 1-forms on $V^n_{Tg}$ over $T_g$ and relative holomorphic 1-forms on $V^{(n)}_{Tg}$ over $T_g$.

Now, we can express $d\tilde{\xi}_k^*$ in terms of $d\sigma_1, \cdots, d\sigma_n$ by using the following identities [14]:

\[ \sigma_k \tau_0 - \sigma_{k-1} \tau_1 + \cdots + (-1)^k \tau_k = d\sigma_{k+1}. \]

(By convention, $\sigma_k = 0$ and $d\sigma_k = 0$ if $k > n$.) Inverting these identities, and writing out only the linear terms, we obtain

\[ \tau_k = (-1)^k (d\sigma_{k+1} - \sigma_1 d\sigma_k - \cdots - \sigma_k d\sigma_1) \]

+ higher order terms.

Thus we may write

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\[ d\tilde{\xi}_k^* = \sum_{l=0}^{\infty} (-1)^l \left[ \left( a_{k,l} + \sum_{m=1}^{3g-3} c_m \tau_{P,l}(Q_m) \right) \right] \]

\[ (d\sigma_{l+1} - \sigma_1 d\sigma_l - \cdots - \sigma_l d\sigma_1) \]

\[ + O(\sigma^2, c^2) \]

where \( O(\sigma^2, c^2) \) denotes higher order terms in the \( \sigma_j \) and the \( c_m \).

By definition of the map \( f: V^{(n)}_{T_g} \to J \) in §2, it is easy to see that \( f \) is given at \((s_0, nP)\) by

\[ f(s_0, nP) = \left( s_0, \int_{P_0} d\tilde{\xi}_1^*(s_0), \cdots, \int_{P_0} d\tilde{\xi}_g^*(s_0) \right) \mod \text{periods} \]

where the integrals \( \int_{P_0} d\tilde{\xi}_k^*(s_0) \) are evaluated by recalling that \( t_1, \cdots, t_n \) are just copies of \( t \). Let \( \frac{\partial \tilde{\xi}_k^*}{\partial \sigma_j} \) be given by

\[ d\tilde{\xi}_k^* = \sum_{j=1}^{n} \frac{\partial \tilde{\xi}_k^*}{\partial \sigma_j} d\sigma_j. \]

Then we have

**Proposition 10.** The map

\[ u: f^*\Omega^1_{J/T_g} \to \Omega^1_{V^{(n)}_{T_g}/T_g} \]

is given locally at \((s_0, nP)\) by the matrix

\[ \left[ \frac{\partial \tilde{\xi}_k^*}{\partial \sigma_j} \right], \quad j = 1, \cdots, n, \quad k = 1, \cdots, g. \]

**Proof.** This follows easily from the definitions of \( f \) and \( \frac{\partial \tilde{\xi}_k^*}{\partial \sigma_j} \). (Compare with [3] and [6].)

**Remark.** Let \( J \) denote the Jacobian variety of \( X \) and let \( f_0: X^{(n)} \to J \) be the classical map (i.e., the map \( f \otimes \kappa(s_0) \)). Then the matrix \( M = [(\frac{\partial \tilde{\xi}_k^*}{\partial \sigma_j})(s_0, nP)] \) is the matrix of the map \( u_0: f_0^*\Omega^1_j \to \Omega^1_{X^{(n)}} \) at \( nP \) (cf. [3], [6]). It is then easy to see that \( (\mathcal{G}_n)^{s_0} \) is what was denoted by \( \mathcal{G}_n^r(X) \) in §0.

Now let \( \xi \) be a tangent vector to \( V^{(n)}_{T_g} \) at \((s_0, nP)\). Let \( s_j \) and \( b_m \) in \( \mathbb{C} \) be given by

\[ \xi(s_j) = s_j e, \quad j = 1, \cdots, n, \]

\[ \xi(c_m) = b_m e, \quad m = 1, \cdots, 3g - 3. \]

Then, using Taylor’s Theorem as in §1, we have
We will now use (15) to compute the partial derivatives of \( \frac{\partial^2 \xi_k^*}{\partial \sigma_i \partial \sigma_j} \) with respect to \( \sigma_i \) and with respect to \( c_m \). (We remind the reader that the functions \( \sigma_i \) and \( c_m \) vanish at \((s_0, nP)\).) We obtain

\[
\frac{\partial^2 \xi_k^*}{\partial \sigma_i \partial \sigma_j} (s_0, nP) = (-1)^{j+1} a_{k, j, l-1}
\]

and

\[
\frac{\partial^2 \xi_k^*}{\partial c_m \partial \sigma_j} (s_0, nP) = \tau'_{P, j-1}(Q_m) \xi_k' (Q_m).
\]

Proposition 11.

Now on to the general case. Consider a divisor \( D \) on \( X \) of the form \( D = m_1P_1 + \cdots + m_dP_d \). Assume \( D \) is in \( G^r_n(X) \) and choose a basis \( \{d_{k, j}^r\}_{k=1}^g \) of the holomorphic differentials on \( X \) such that the last \( i = \dim H^1(X, \mathcal{O}_X(D)) \) of them vanish on \( D \).

In performing the variation in §3, choose a point \((Q_1, \cdots, Q_{3g-3})\) from the open set in \( X^{3g-3} \) in Theorem 2 so that each \( Q_m \) is different from \( P_0, P_1, \cdots, P_d \) and any other zero of any \( d_{k, j}^r \). (The choice of this point will be further modified later.) Take the disk about each \( Q_m \) sufficiently small so that no two disks intersect and such that no disk contains \( P_0, P_1, \cdots, P_d \) or any other zero of any \( d_{k, j}^r \).

Let \( f_j: V_{T_g}^T(\xi_j) \rightarrow J \) be the map defined in §2 and let

\[
u_j: f_j^* \Omega^1_{J/T_g} \rightarrow \Omega^1_{V_{T_g}^T/(m_j)}.
\]

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be the map induced by $f_j$. The obvious map

$$V_{Tg}^{(m_1)} 	imes_{Tg} V_{Tg}^{(m_2)} \times_{Tg} \cdots \times_{Tg} V_{Tg}^{(m_d)} \rightarrow V_{Tg}^{(n)}$$

is a local analytic isomorphism by an argument analogous to that given in [14] in the case of a curve over a field. Locally, the map $f$ is the one induced by the $f_j$ and the map $u$ is the one induced by the $u_j$. Thus, the matrix of $u$ locally at $(s_0, D)$ is obtained by “stacking” the matrices of the $u_j$ locally at $(s_0, m_jP_j)$.

Let $\xi$ be a tangent vector to $V_{Tg}^{(n)}$ at $(s_0, D)$ and let $\xi_j$ be the tangent vector to $V_{Tg}^{(m_j)}$ at $(s_0, m_jP_j)$ induced by $\xi$, for $j = 1, \cdots, d$. Then the matrix of $\xi^*u$ is obtained by “stacking” the matrices of the $\xi_j^*u_j$, for $j = 1, \cdots, d$.

Let $M'$ denote the matrix of $\xi^*u$. Let $\mu$ denote the leading minor of order $n - r$ of $M$, the matrix $[(\partial s^g_k/\partial \sigma_j)(s_0, D)]$, and let $\mu'$ denote the leading minor of order $n - r$ of $M'$. Then we have $\mu' = \mu + ce$ for some $c$ in $C$. Now, by our choice of a basis of the holomorphic differentials on $X$, the last $i$ columns of $M$ are identically zero, hence the last $i$ columns of $M'$ contain “pure” $e$ terms (i.e., members of the maximal ideal of $C[e]/(e^2)$). Thus, in computing a minor of order $n - r + 1$ containing $\mu'$, any $e$‘s in the first $n - r$ columns will be “killed” by the $e$ in the last column of the minor of order $n - r + 1$. Hence, we have established

**Lemma 3.** For purposes of computing the minors of order $n - r + 1$ of $M'$, we may replace the first $n - r$ columns of $M'$ by the first $n - r$ columns of $M$.

Let $M$ denote the resulting matrix.

$M$ has a particularly nice form in the case that $D = P_1 + P_2 + \cdots + P_n$, with all points distinct. Let $t_j$ be a local parameter at $P_j$ and write $d\xi_k = \varphi_{j,k}dt_j$. Then we have

$$M = \begin{bmatrix}
\varphi_{j,k}(P_j) \\
j = 1, \cdots, n \\
k = 1, \cdots, g - i
\end{bmatrix} = e \begin{bmatrix}
(s_j\varphi_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_{m,\tau_{P_j,0}(Q_m)}\xi_{k}(Q_m)) \\
j = 1, \cdots, n \\
k = g - i + 1, \cdots, g
\end{bmatrix}$$

Going back to the general case, recall that, by Proposition 1, $\xi$ will be a tangent vector to $G_n^r$ at $(s_0, D)$ iff the minors of order $n - r + 1$ of the matrix $M$ all vanish. Assume $D = m_1P_1 + \cdots + m_dP_d$ is in $G_n^r(X) - G_n^{r+1}(X)$. Then the matrix $M$ has rank precisely $n - r$. Hence, by permuting the rows of
If necessary, we end up with a matrix whose leading minor of order \( n - r \), which we will denote by \( \mu \), is nonzero. We will continue to denote this matrix by \( M \), although its form may differ slightly from that specified earlier.

Perform the same row permutations as above on the matrix \( M \) and denote the resulting matrix also by \( M \). Then \( \mu \) is also the leading minor of order \( n - r \) of \( M \), so we may apply Lemma 2. Thus, for all the minors of order \( n - r + 1 \) of \( M \) to vanish, it is sufficient that every minor of order \( n - r + 1 \) which contains \( \mu \) vanishes. The vanishing of each of these minors gives rise to a linear equation in the \( s_j \) and the \( b_m \).

Let \( \mu_{j,k} \) denote the minor of order \( n - r + 1 \) of \( M \) obtained by adjoining to \( \mu \) the first \( n - r \) elements of the \((n - r + j)\)th row of \( M \) and the first \( n - r \) elements and the \((n - r + k)\)th element of the \((n - r + i)\)th column of \( M \) (thus \( j \) runs from 1 through \( r \) and \( k \) runs from 1 through \( i \)). The equation \( \mu_{j,k} = 0 \) is of the form \( \epsilon E_{j,k} = 0 \) where \( E_{j,k} \) is a linear equation in the \( s_j \) and the \( b_m \) with coefficients in \( C \).

We will now view the \( s_j \) and the \( b_m \) as being unknowns (as in §1). Thus, \( E_{j,k} \) is an equation in \( 3g - 3 + n \) unknowns. By the discussion after Proposition 1, the dimension of the tangent space to \( G^r_\lambda \) at \((s_0, D)\) is

\[
3g - 3 + n - \text{(the number of } E_{j,k} \text{ which are linearly independent)}.
\]

Consider the coefficient of \( b_m \) in \( E_{j,k} \). This coefficient will be a linear combination of certain of the \( \tau_{P_{j',\nu}}(Q_m)\zeta_k(Q_m) \). That is, the coefficient of \( b_m \) will be a certain quadratic differential (the above linear combination of certain of the \( d\tau_{P_{j',\nu}}d\zeta_k \)) evaluated at the point \( Q_m \). It should be noted that, by the symmetry of the matrix \( M \) in the \( b_m \), this quadratic differential does not depend on \( m \), but only on \( j \) and \( k \). The coefficient of \( b_1 \) in \( E_{j,k} \) is the value of this quadratic differential at \( Q_1 \), the coefficient of \( b_2 \) in \( E_{j,k} \) the value at \( Q_2 \), etc. Put \( \alpha_{j,k} \) equal to the above linear combination of certain of the \( d\tau_{P_{j',\nu}}d\zeta_k \). Then \( \alpha_{j,k} \) is a (not necessarily finite) quadratic differential.

**Notation.** Choose a local parameter \( u_m \) on \( X \) at \( Q_m \) and write \( \alpha_{j,k} = g(u_m)du_m^2 \). Then we will write \( \alpha_{j,k}(Q_m) \) for \( g(0) \). Hence, by the above discussion, \( \alpha_{j,k}(Q_m) \) is the coefficient of \( b_m \) in \( E_{j,k} \).

Our aim now is to show that, in certain situations, by suitably choosing the point \( (Q_1, \cdots, Q_{3g-3}) \), we may conclude that the \( E_{j,k} \) are linearly independent. Assume that \( ri \leq 3g - 3 \). By elementary linear algebra, to conclude that the \( E_{j,k} \) are linearly independent, it is sufficient to show that the matrix of coefficients

\[
A = [\alpha_{j,k}(Q_m)], \quad j = 1, \cdots, r; k = 1, \cdots, i; m = 1, \cdots, ri,
\]

is nonsingular.
Lemma 4. Assume that the $\alpha_{j,k}$ for $j = 1, \cdots, r$ and $k = 1, \cdots, i$, are linearly independent and that $r_i \leq 3g - 3$. Then we may choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open set in $X^{3g-3}$ in Theorem 2 such that each $Q_m$ is different from $P_0$ and no $Q_m$ is a zero of $d\xi_1, \cdots, d\xi_g$ and such that the above matrix $A$ is nonsingular.

Proof. The lemma will follow readily from the following

Sublemma. Let $\beta_1, \cdots, \beta_n$ be $n$ linearly independent quadratic differentials on $X$. Let $U$ be an open set contained in $X^n$. Then we may choose a point $(P_1, \cdots, P_n) \in U$ such that each $P_m$ is different from a finite set of points of $X$ and such that the matrix $[\beta_j(P_k)]$ $(j = 1, \cdots, n, k = 1, \cdots, n)$ is nonsingular.

Proof. By induction on $n$. If $n = 1$, then $\beta_1$ is a nontrivial quadratic differential. Hence, $\beta_1$ is nonzero and finite on a dense open set of $X$. So, given any open set in $X$, there exists a point in that set satisfying the requirements of the Sublemma.

Now suppose $U$ is an open set contained in $X^n$. Let $V$ be the projection of $U$ onto $X^{n-1}$. Then $V$ is open and, by induction, we may choose a point $(P_0, \cdots, P_{n-1}) \in V$ such that each $P_m$ is different from a finite set of points of $X$ and such that the leading subdeterminant of order $n - 1$ of the determinant

$$\begin{vmatrix}
\beta_1(P_1) & \cdots & \beta_1(P_{n-1}) & \beta_1 \\
\vdots & & \vdots & \vdots \\
\vdots & & \vdots & \vdots \\
\beta_n(P_1) & \cdots & \beta_n(P_{n-1}) & \beta_n
\end{vmatrix}$$

is nonzero. Expanding the full determinant by the last column, we obtain a nontrivial linear combination of $\beta_1, \cdots, \beta_n$. By the linear independence of these quadratic differentials, this linear combination is a nontrivial quadratic differential, hence is nonzero and finite on an open dense set $W$ contained in $X$. Since $U$ is open in $X^n$ and $W$ is dense in $X$, we may choose a point in the intersection of $U$ and $\{(P_0, \cdots, P_{n-1})\} \times W$ which satisfies the requirements of our Sublemma.

Now, since the set of points in $X^{3g-3}$ in Theorem 2 is open, it is easy to see that we may choose a point $(Q_1, \cdots, Q_{3g-3})$ in this set such that each $Q_m$ is different from $P_0$ and the zeros of $d\xi_1, \cdots, d\xi_g$ and so that $Q_1, \cdots, Q_{ri}$ make the matrix $A$ nonsingular. This completes the proof of the lemma.

We then have
Proposition 12. Suppose $D$ is in $G_n^r(X) - G_n^{r+1}(X)$ and that $r_i \leq 3g - 3$. Then if all the $\alpha_{j,k}$ are linearly independent, the dimension of the tangent space to $G_n^r$ at $(s_0, D)$ is $3g - 3 + r + r$.

Proof. By Lemma 4, we may choose a point $(Q_1, \cdots, Q_{3g-3})$ from the open set in Theorem 2 such that each of the $Q_m$ is different from $P_0$ and the zeros of $d\xi_1, \cdots, d\xi_g$ (note that this latter set includes the points of $D$), and such that the equations $E_{j,k}$ are linearly independent. Thus the dimension of the tangent space to $G_n^r$ at $(s_0, D)$ is $3g - 3 + n - i = 3g - 3 + r + r$.

In the next section, we show that if $D$ is in $G_n^1(X) - G_n^2(X)$, then the $\alpha_{j,k}$ are linearly independent. (Note that we have $i < 3g - 3$ if $g > 1$.)

5. The dimension of $G_n^1 - G_n^2$. For simplicity, we will first treat a divisor consisting of $n$ distinct points. So assume $D = P_1 + P_2 + \cdots + P_n$, all points distinct, is in $G_n^1(X) - G_n^2(X)$. Recall that the matrix $M$ is

$$M = \begin{bmatrix} \varphi_{j,k}(P_j) & \cdots & \left( s_j \varphi'_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_m \tau'_{P_j,0}(Q_m) \xi'_{k}(Q_m) \right) \\ j = 1, \cdots, n \\ k = 1, \cdots, g - i \end{bmatrix}.$$

Let $|\hat{1}|$ denote the minor of order $n - 1$ obtained by omitting the $j$th row from the matrix $[\varphi_{j,k}(P_j)]$ ($j = 1, \cdots, n$, $k = 1, \cdots, g - i$). Then we have

$$\alpha_{1,k}(Q) = \sum_{j=1}^{n} (-1)^{j-1} |\hat{1}| \tau'_{P_j,0}(Q) \xi'_{n+k-1}(Q)$$

for $k = 1, 2, \cdots, i$. Suppose we had a linear relation of the form $\sum_{k=1}^{l} a_k \alpha_{1,k} = 0$ with some $a_l$ nonzero. Then this would imply that

$$\left( \sum_{j=1}^{n} (-1)^{j-1} |\hat{1}| \tau'_{P_j,0}(Q) \right) \left( \sum_{k=1}^{l} a_k \xi'_{n+k-1}(Q) \right) = 0. \tag{*}$$

But the $d\tau_{P_j,0}$, $j = 1, \cdots, n$, are linearly independent, since they have poles at different points. This, together with the fact that $|\hat{1}| \neq 0$, implies that there is a dense open set of points of $X$ where the expression $\sum_{j=1}^{n} (-1)^{j-1} |\hat{1}| \tau'_{P_j,0}(Q)$ is nonzero.

And the linear independence of $d\xi_n, \cdots, d\xi_g$, together with the fact that some $a_l$ is nonzero, implies that the expression $\sum_{k=1}^{l} a_k \xi'_{n+k-1}(Q)$ is nonzero on a dense open set of points of $X$. Hence, we may choose a point $Q$ such that $\alpha_{1,1}, \cdots, \alpha_{1,i}$ are linearly dependent.
Now suppose $D = m_1P_1 + \cdots + m_dP_d$ is in $G^1_n(X) - G^2_n(X)$. Then we have

$$\alpha_{1, k}(Q) = \tau'_{n+k-1}(Q)((1 - 1)^{n-1} |\tau'_{P_1, 0}(Q) + \cdots + (- 1)^{n-1} |\tau'_{P_d, m_d-1}(Q)).$$

Hence, if there existed a linear relation $\sum_{k=1}^l a_k \alpha_{1, k} = 0$, we would have

$$\left(\sum_{k=1}^l a_k \tau'_{n+k-1}(Q)\right)\left(1 - 1\right)^{n-1} |\tau'_{P_1, 0}(Q) + \cdots + (- 1)^{n-1} |\tau'_{P_d, m_d-1}(Q) = 0.\right.$$ (1)

The same reasoning as in the case of simple points applies, since $d\tau_{P_1, 0}, \ldots, d\tau_{P_d, m_d-1}$ are easily seen to be linearly independent (they either have poles at different points or have poles of differing orders at the same point).

**Remark.** The above reasoning shows that if $D \in G^r_n - G^{r+1}_n$, then the $\alpha_{j, k}$ for a fixed $j$ are linearly independent.

**Theorem 3.** $G^1_n - G^2_n$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$.

**Proof.** Let $(s_0, D)$ be any point of $G^1_n - G^2_n$. By Proposition 12 and the work of this section, we may conclude that the dimension of the tangent space to $G^1_n$ at $(s_0, D)$ is $3g - 3 + \tau + 1$. By Proposition 4, the dimension of $G^2_n$ at $(s_0, D)$ is at least $3g - 3 + \tau + 1$, hence $G^1_n$ is smooth at $(s_0, D)$ and has dimension precisely $3g - 3 + \tau + 1$.

**Remark.** Theorem 3 does not depend upon $\tau$ being nonnegative.

**Theorem 4.** Suppose that $G^1_n(X) - G^2_n(X)$ is nonempty for a generic curve $X$. Then $G^1_n(X) - G^2_n(X)$, for a generic $X$, is smooth of pure dimension $\tau + 1$.

**Proof.** Under our assumption, the image of $G^1_n - G^2_n$ in $T_g$ would be a dense open subspace $U$. By Sard’s Theorem, since $G^1_n - G^2_n$ is smooth, the generic fiber of the map $G^1_n - G^2_n \to U$ is smooth. And since $U$ has dimension $3g - 3$ and $G^1_n - G^2_n$ has dimension $3g - 3 + \tau + 1$, the generic fiber has dimension $\tau + 1$. Thus, for a generic curve, $G^1_n(X) - G^2_n(X)$ is smooth of dimension $\tau + 1$.

**Remark.** If $\tau \geq 0$, then by [10] we know that $G^r_n(X)$ is nonempty. If we knew that $G^r_n(X)$ were reduced for a generic $X$, then, since the points of $G^{r+1}_n$ are singular points of $G^r_n$, we could conclude that $G^r_n(X) - G^{r+1}_n(X)$ is nonempty for generic $X$ if $\tau \geq 0$.

6. Moduli of trigonal curves. A trigonal curve is a curve $X$ such that $G^1_3(X)$ is nonempty. We can use Theorem 3 to compute the moduli of trigonal curves. By Clifford’s Theorem, $G^1_3$ is empty hence, by Theorem 3, $G^1_3$, if
nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. Now $\tau = 2(3 - 1) - g = 4 - g$, so $G_3^1$, if nonempty, has dimension $2g + 2$.

By Theorem 1 of [12], we have that, for $g \geq 4$, if $G_3^1(X)$ is nonempty, then every component has dimension at least $5 - g$ and at most 2, with the upper bound occurring if and only if $X$ is hyperelliptic. So, if there exists a nonhyperelliptic trigonal curve of genus $g$, then we must have that the dimension of the generic fiber of the map $G_3^1 \to T_g$ is 1. Examples of such curves (for every $g \geq 3$) are given in [1a, p. 196].(1) Hence, the dimension of the subvariety of $T_g$, for $g \geq 4$, of trigonal curves is $2g + 2 - 1 = 2g + 1$. This agrees with the number which appears in Segre [20] and Severi [22].

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(1) The author wishes to thank the referee for pointing out this reference.


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